Cardinal characteristics and projective wellorders

Vera Fischer[a,1,*], Sy David Friedman[a,1]

[a] *Kurt Gödel Research Center, University of Vienna, Währinger Strasse 25, A-1090 Vienna, Austria*

Abstract

Using countable support iterations of S-proper posets, we show that the existence of a Δ^1_3 definable wellorder of the reals is consistent with each of the following: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$.

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1. Introduction

If $V = L$ then there exists a Σ^1_2 well-ordering of the reals. Furthermore, by Mansfield's Theorem (see (11), Theorem 25.39) the existence of a Σ^1_2 well-ordering of the reals, implies that every real is constructible. Using a finite support iteration of ccc posets, L. Harrington showed that the existence of a $\boldsymbol{\Delta^1_3}$ wellordering of the reals is consistent with the continuum being arbitrarily large (see (12), Theorem A). S. D. Friedman showed that Martin's Axiom (and not CH) is consistent with the existence of a Δ^1_3 definable wellordering of the reals (see (8) and see (12) for the corresponding boldface result). As shown in (5) BPFA is consistent with the existence of a Δ^1_3 wellorder of the reals. Note that since in the last two models MA holds, all cardinal characteristics of the continuum in these models are equal to $\mathfrak{c}$. On the other hand large cardinals imply projective determinacy and so they imply that there are no projective wellorders of the reals (see (11)). In this paper, using a countable support iteration of S-proper posets, we show that the existence of a parameter free Δ^1_3-definable wellorder of the reals is consistent with each of the following: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, $\mathfrak{b} < \mathfrak{g}$.

Throughout the paper, except if it is explicitly stated otherwise, we work over the constructible universe L. In section 2, we introduce a particular instance of the method of

*Corresponding author; Phone: (43-1) 43 1 4277 50523; Fax: (43-1) 43 1 4277 50599
Email addresses: `vfischer@logic.univie.ac.at` (Vera Fischer), `sdf@logic.univie.ac.at` (Sy David Friedman)

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localization, which originates in the work of R. David on Π^1_2-singletons (see (6) or (8)). We show that this instance of the method is proper (see Lemma 3) and does not add reals (see Lemma 4). In section 3, we define coding with perfect trees, establish its properness and show that the poset is ${}^\omega\omega$-bounding. In section 4, we discuss some preservation theorems for iterations of S-proper posets. In section 5, using a countable support iteration of length ω_2 of S-proper posets, we obtain a model in which there is a Δ^1_3 definable wellorder of the reals and the continuum is ω_2. At each stage of this iteration, first we force with an arbitrary proper poset of size at most $\aleph_1$ and then introduce the definable wellorder of the reals in three successive steps: we destroy countably many stationary sets from some fixed sequence of stationary, co-stationary sets in the ground model, we localize this information to a certain class of countable models and finally add a real coding this same information. The freedom, given by forcing with an arbitrary proper poset, as well as the combinatorial properties of the posets used to introduce the definable wellorder of the reals, allow us to modify some of the known cardinal characteristics of the real line. Thus in section 6, we obtain that the existence of a Δ^1_3-definable wellordering of the reals is consistent with each of the following: $\mathfrak{d} < \mathfrak{c}$ (see Theorem 2), $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ (see Theorem 3) and $\mathfrak{b} < \mathfrak{g}$ (see Theorem 4). To the best knowledge of the authors, this is the first work on projective wellorders and cardinal characteristics of the continuum. We conclude with some open questions.

Following standard notation ${}^\omega\omega$ denotes the set of functions from ω to ω, $[\omega]^\omega$ the set of infinite subsets of ω. Whenever f, g are in ${}^\omega\omega$, f is *dominated by* g, denoted $f \leq^* g$, if there is $k \in \omega$ such that for all $n \geq k$, $f(n) \leq g(n)$. A family $\mathcal{B} \subseteq {}^\omega\omega$ is *unbounded*, if there is no single real g which dominates all elements of $\mathcal{B}$. A family $\mathcal{D} \subseteq {}^\omega\omega$ is *dominating* if every real is dominated by an element of the family $\mathcal{D}$.

2. Localization

Say that a transitive ZF$^-$ model $\mathcal{M}$ is *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$. Throughout this section assume that the ground model is some generic extension $L[G^*]$ of the constructible universe L, in which cofinalities (and so cardinals) have not been changed. Let $X \subseteq \omega_1$ and let $\phi(\omega_1, X)$ be a Σ_1-sentence with parameters ω_1, X, which is true in all suitable models containing ω_1 and X as elements.

Definition 1. Let $\mathcal{L}(\phi)$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal, such that

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(2\gamma) = 1$
2. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable, suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$.

The extension relation is end-extension.

Remark 1. If $r \in \mathcal{L}(\phi)$, then the even part of r codes $X \cap |r|$.

Lemma 1. Let $r \in \mathcal{L}(\phi)$ and let γ be a countable limit ordinal greater than $|r|$. Then there is $r^* \in \mathcal{L}(\phi)$ such that $|r^*| = \gamma$ and $r^* \leq r$.

Proof. Take the odd part of r^* on the interval $[|r|, |r| + \omega)$ to code γ and to consist only of 0's on $[|r| + \omega, \gamma)$. Then there are no new instances of requirement (2) for being a condition to check, because no ZF^- model containing $r^* \upharpoonright |r| + \omega$ can have its ω_1 in the interval $(|r|, \gamma]$. $\square$

Lemma 2. Let G be $\mathcal{L}(\phi)$-generic and let $Y = \bigcup G$. Let $\mathcal{M}$ be a countable suitable model containing $Y \upharpoonright \gamma$ as an element, where $\gamma = \omega_1^{\mathcal{M}}$. Then $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$.

Proof. Note that if $r \in \mathcal{L}(\phi)$ and δ is a limit ordinal, $\delta < |r|$, then $r \upharpoonright \delta \in \mathcal{L}(\phi)$. Then by Lemma 1, the set $D_\delta = \{r \in \mathcal{L}(\phi) : |r| = \delta\}$ is predense. Let $\mathcal{M}$ be a countable suitable model, $\gamma = \omega_1^{\mathcal{M}}$ and $Y \upharpoonright \gamma \in \mathcal{M}$. Then $r = Y \upharpoonright \gamma$ is a condition and so by definition the formula $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$. $\square$

Lemma 3. $\mathcal{L}(\phi)$ is proper.

Proof. Let $\mathcal{M}$ be a countable elementary submodel of $L_\Theta[G^*]$ for some sufficiently large Θ, such that $\mathcal{L}(\phi), X$ are elements of $\mathcal{M}$ and let $p \in \mathcal{M} \cap \mathcal{L}(\phi)$. Let $i = \mathcal{M} \cap \omega_1$, $\{i_k\}_{k\in\omega} \subseteq \mathcal{M} \cap \omega_1$ a sequence cofinal in i. Let $\{D_k\}_{k\in\omega}$ enumerate the dense subsets of $\mathcal{L}(\phi)$ which belong to $\mathcal{M}$. Define a sequence $\{p_k\}_{k\in\omega} \subseteq \mathcal{M} \cap \mathcal{L}(\phi)$ such that $p_0 \leq p$ and for all $k \in \omega$, $p_{k+1} \leq p_k$, $|p_k| \geq i_k$, $p_k \in D_k \cap \mathcal{M}$. Suppose p_k has been defined (to obtain p_0 consider $p_{-1} = p$). Since $D'_{i_{k+1}} = \{s \in \mathcal{L}(\phi) : |s| \geq i_{k+1}\}$ is dense in $\mathcal{L}(\phi)$ and belongs to $\mathcal{M}$ (as it is definable from parameters in $\mathcal{M}$), there is $r \in D'_{i_{k+1}} \cap \mathcal{M}$ extending p_{k+1}. Then (by elementarity) there is $p_{k+1} \in D_{k+1} \cap \mathcal{M}$ extending r. Since $p_k \in \mathcal{M}$, $i_k \leq |p_k| < i$ for all $k \in \omega$. We will show that $q = \bigcup_{k\in\omega} p_k$ is a condition in $\mathcal{L}(\phi)$ and thus is an $(\mathcal{M}, \mathcal{L}(\phi))$-generic extension of p.

Let $\mathcal{N}_0$ be a countable, suitable model containing $q \upharpoonright \gamma$ as an element, where $\gamma = \omega_1^{\mathcal{N}_0} \leq |q| = i$. *Case 1.* If $\omega_1^{\mathcal{N}_0} = \gamma < i$, then $\gamma \in |p_k|$ for some $k \in \omega$. However $p_k \upharpoonright \gamma \in \mathcal{L}(\phi)$ and so $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{N}_0$ by definition of $p_k \upharpoonright \gamma$. *Case 2.* Let $\omega_1^{\mathcal{N}_0} = \gamma = i$. Since $\mathcal{M}$ is a countable elementary submodel of $L_\Theta[G^*]$, $\mathcal{M}$ satisfies that $\phi(\omega_1, X)$ holds in all suitable models containing ω_1 and X as elements. Then the transitive collapse $\bar{\mathcal{M}}$ of $\mathcal{M}$ satisfies that $\phi(\gamma, X \cap \gamma)$ holds in all suitable models containing γ and $X \cap \gamma$ as elements. In particular $\phi(\gamma, X \cap \gamma)$ holds in the least suitable model containing γ and $X \cap \gamma$ as elements, and as ϕ is Σ_1 it holds in all suitable models containing γ and $X \cap \gamma$ as elements. As $\mathcal{N}_0$ is suitable and contains γ, $X \cap \gamma$ as elements, we conclude that $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{N}_0$ as desired. $\square$

Lemma 4. $\mathcal{L}(\phi)$ does not add new reals.

Proof. Let $\dot{f}$ be a $\mathcal{L}(\phi)$-name for a real, let $\mathcal{M}$ be a countable elementary submodel of $L_\Theta[G^*]$, where Θ is a sufficiently large cardinal such that $\dot{f}, \mathcal{L}(\phi), X$ are elements of $\mathcal{M}$

and let $p \in \mathcal{L}(\phi) \cap \mathcal{M}$. Let $i = \omega_1^{\mathcal{M}}$ and let $\{i_k\}_{k\in\omega} \subseteq \mathcal{M} \cap \omega_1^{\mathcal{M}}$ be a sequence cofinal in i. Recursively define a sequence $\{p_k\}_{k\in\omega}$ of conditions in $\mathcal{M} \cap \mathcal{L}(\phi)$ such that $p_0 \leq p$, $p_{k+1} \leq p_k$, $p_k \Vdash \dot{f}(k) = \check{m}_k$ where $m_k \in \omega$ and $|p_k| \geq i_k$ for all $k \in \omega$. Let $q = \bigcup_{k\in\omega} p_k$. Just as in the proof of Lemma 3 one can show that q is a condition. Then q is a common extension of the p_k's, and so $q \Vdash \dot{f} = \check{g}$, where $g(k) = m_k$ for all $k \in \omega$. □

Remark 2. In fact, the above arguments show that $\mathcal{L}(\phi)$ has a countably closed dense subset. Let $\mathcal{M}_0$ be the least suitable model containing ω_1 and X as elements, and let C be the closed unbounded subset of ω_1 consisting of the intersections with ω_1 of countable elementary submodels of $\mathcal{M}_0$. Then the set D of conditions r such that $|r| \in C$ is dense and the union of a countable sequence of elements of D is also a condition in D.

3. Coding with perfect trees

Let $Y \subseteq \omega_1$ be generic over L such that in $L[Y]$ cofinalities have not been changed. Inductively define a sequence $\bar{\mu} = \{\mu_i\}_{i\in\omega_1}$ of L-countable ordinals as follows: μ_i is the least $\mu > \bigcup\{\mu_j : j < i\}$ (this condition is vacuous if i is equal to 0) such that $L_\mu[Y \cap i] \models ZF^-$ and $L_\mu \models \omega$ is the largest cardinal. There are many μ's with these properties, for example any μ such that $L_\mu[Y \cap i]$ is an elementary submodel of $L_{\omega_1}[Y \cap i]$. We say that a real R *codes* Y below i if for all $j < i$, $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \models ZF^-$. For $T \subseteq 2^{<\omega}$ a perfect tree, let $|T|$ be the least i such that $T \in L_{\mu_i}[Y \cap i]$.

Definition 2. Let $C(Y)$ be the poset of all perfect trees T such that R codes Y below $|T|$, whenever R is a branch through T. For T_0, T_1 conditions in $C(Y)$ let $T_0 \leq T_1$ if and only if T_0 is a subtree of T_1.

Remark 3. Note that $T_0 \leq T_1$ if and only if $[T_0] \subseteq [T_1]$ where $[T]$ denotes the set of infinite branches through T. Define $T_0 \leq_n T_1$ if and only if $T_0 \leq T_1$ and T_0, T_1 have the same first n splitting levels. For T a perfect tree, $m \in \omega$, let $S_m(T)$ be the set of m-splitting nodes of T (and so $|S_m(T)| = 2^m$), and for $t \in T$ let $T(t) = \{\eta \in T : t \subseteq \eta \text{ or } \eta \subseteq t\}$.

Remark 4. By absoluteness, if T is a condition then R codes Y below $|T|$ even for branches R through T in the generic extension. In particular this holds for the generic branch.

Lemma 5. Let $T \in C(Y)$ and $|T| \leq i < \omega_1$. Then there is $T^* \leq T$ such that $|T^*| = i$.

Proof. By induction on i. We may assume that $|T|$ is less than i. If $i = j + 1$ then we may also assume by induction that $|T| = j$ and hence that $T \in \mathcal{A}_j = L_{\mu_j}[Y \cap j]$. If $j \in Y$, then we take $T^* \leq T$ to have the property that R is P_T-generic over $\mathcal{A}_j$ for $R \in [T^*]$, where P_T is the forcing (isomorphic to Cohen forcing) whose conditions are elements of T, ordered by extension. Note that T^* can be chosen in $\mathcal{A}_i = L_{\mu_i}[Y \cap i]$, as $\mathcal{A}_j$ is a countable element of $\mathcal{A}_i$. Also $L_{\mu_j}[Y \cap j, R] \models ZF^-$ for $R \in [T^*]$, by the P_T-genericity of $R \in [T^*]$. So T^* is a condition and $|T^*| = i$. If j does not belong to Y then choose a real R_0 coding a well ordering of ω of order type μ_j, $R_0 \in \mathcal{A}_i$ and take $T^* \leq T$ to be the tree whose branches R are exactly the branches through T such that for all n, $n \in R_0$ if and only if R goes right at the $2n$-th splitting level of T. Then T^* belongs to $\mathcal{A}_i$ and for $R \in [T^*]$, (R, T) computes R_0 and hence $L_{\mu_j}[Y \cap j, R]$ is not a model of ZF^-, since it contains R_0 as an element.

If i is a limit ordinal then choose $\{i_n\}_{n\in\omega}$, where $|T| = i_0$, to be an ω-sequence cofinal in i which belongs to $\mathcal{A}_i = L_{\mu_i}[Y \cap i]$. Let $T_0 = T$, and for each n let $T_{n+1} \in \mathbb{Q}(\bar{\mu}, Y)$ be least in $\mathcal{A}_{i_{n+1}}$ such that $|T_{n+1}| = i_{n+1}$ and $T_{n+1} \leq_n T_n$. Such T_n's exist by induction. If $T^* = \bigcap_{n\in\omega} T_n$ then $T^* \leq T$ belongs to $\mathcal{A}_i$ and satisfies the requirement for belonging to $C(Y)$. So $T^* \leq T$, $|T^*| = i$ as desired. □

Lemma 6. Let G be $C(Y)$-generic. Then $R = \bigcap G$ codes Y. That is for all $j < \omega_1 (j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \models ZF^-)$.

Proof. Let $j < \omega_1$. Then by Lemma 5 the set $D_j = \{T \in C(Y) : |T| > j\}$ is dense. Thus there is some $T \in D_j \cap G$ and so $R \in [T]$. Then since $j < |T|$ by Remark 4 we have that $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \models ZF^-$. □

Lemma 7. $C(Y)$ is proper.

Proof. Let $\mathcal{M}$ be a countable elementary submodel of $L_\Theta[Y]$, for some sufficiently large Θ, which contains $C(Y), \bar{\mu}, Y$ as elements. Let $T \in \mathcal{M} \cap C(Y)$ and let $i = \mathcal{M} \cap \omega_1$. The transitive collapse isomorphism $\bar{\mathcal{M}}$ of $\mathcal{M}$ is of the form $L_{\bar{i}}[Y \cap i]$. However $\bar{\mathcal{M}} \models$ (i is uncountable) and so $\bar{\mathcal{M}} \in L_{\mu_i}[Y \cap i]$. Since $L_{\mu_i}[Y \cap i] \models$ (i is countable), we can fix a sequence $\bar{i} = \{i_k\}_{k\in\omega}$ which is cofinal in i and belongs to $L_{\mu_i}[Y \cap i]$ as an element.

Let $\{D_k\}_{k\in\omega} \subseteq \mathcal{M}$ enumerate the dense subsets of $C(Y)$ in $\mathcal{M}$. Inductively define a sequence $\{T_k\}_{k\in\omega}$ of conditions in $C(Y) \cap \mathcal{M}$ such that $T_0 \leq T$ and for all $k \in \omega$, $T_{k+1} \leq_{k+1} T_k$, $|T_k| \geq i_k$ (and since $T_k \in \mathcal{M}$, also $|T_k| < i$) and $T_k \Vdash D_k \cap \mathcal{M} \cap \dot{G} \neq \emptyset$, where $\dot{G}$ is the canonical name for the $C(Y)$-generic filter. Suppose we have defined $T_k \in \mathcal{M}$. Let $D'_{i_{k+1}} = \{S \in C(Y) : |S| \geq i_{k+1}\}$. By elementarity $D'_{i_{k+1}} \in \mathcal{M}$ and $(D'_{i_{k+1}}$ is dense$)^{\mathcal{M}}$. Let $t \in S_{k+1}(T_k)$. Then there is $\tilde{T}(t) \in D'_{i_{k+1}} \cap \mathcal{M}$ such that $\tilde{T}(t) \leq T_k(t)$. Furthermore there is $\hat{T}(t) \in D_{k+1} \cap \mathcal{M}$ such that $\hat{T}(t) \leq \tilde{T}(t)$. Then let $T_{k+1} = \bigcup\{\hat{T}(t) : t \in S_{k+1}(T_k)\}$. Note that since $S_{k+1}(T_k) \in \mathcal{M}$, also $T_{k+1} \in \mathcal{M}$.

Claim. $T_{k+1} \Vdash D_{k+1} \cap \mathcal{M} \cap \dot{G} \neq \emptyset$.

Proof. The set $\{X \in C(Y) : \exists t \in S_{k+1}(T_{k+1})(X \leq T_{k+1}(T))\}$ is dense below T_{k+1}. Let G be $C(Y)$-generic filter such that $T_{k+1} \in G$ and let $X \in G$ such that $X \leq T_{k+1}(t)$ for some $t \in S_{k+1}(T_{k+1})$. However $T_{k+1}(t) = \hat{T}(t)$ and so $\hat{T}(t) \in D_{k+1} \cap \mathcal{M} \cap G$. □*Claim*

Note that we could have chosen $\{D_k\}_{k\in\omega}$ so that $\{\bar{D}_k\}_{k\in\omega}$, where $\bar{D}_k$ is the image of D_k under the transitive collapse isomorphism, belongs to $L_{\mu_i}[Y \cap i]$. Therefore we could have also chosen $\bar{\tau} = \{T_k\}_{k\in\omega}$ to belong to $L_{\mu_i}[Y \cap i]$. Then $T^* = \bigcap \bar{\tau} = \bigcap_{k\in\omega} T_k \in L_{\mu_i}[Y \cap i]$ and so T^* is a condition in $C(Y)$. Indeed, it is clear that every branch of T^* codes Y below i: let $R \in [T^*]$ and $j < i$. Then $j < i_k$ for some k. However $R \in [T_k]$ and so

$j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \models ZF^-$. Then for every dense subset D of $C(Y)$ in $\mathcal{M}$, $T^* \Vdash D \cap \dot{G} \cap M \neq \emptyset$. Thus $T^* \leq T$ and T^* is $(\mathcal{M}, C(Y))$-generic. □

Recall that a poset $\mathbb{Q}$ which preserves the ground model reals as a dominating family is called ${}^{\omega}\omega$-bounding (see (1)).

Lemma 8. $C(Y)$ is ${}^{\omega}\omega$-bounding.

Proof. Let $\dot{f}$ be a $C(Y)$-name for a real, $T \in C(Y)$ and let $\mathcal{M}$ be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ such that $\dot{f}, C(Y), T, \bar{\mu}, Y$ are elements of $\mathcal{M}$. Let $i = \mathcal{M} \cap \omega_1$ and let $\bar{i} = \{i_k\}_{k\in\omega}$ be a sequence (which belongs as an element) in $L_{\mu_i}[Y \cap i]$ cofinal in i. Using the elementarity of $\mathcal{M}$ recursively define a sequence $\{T_k\}_{k\in\omega}$ in $\mathcal{M} \cap C(Y)$ such that $T_0 \leq T$ and for all $k \in \omega$, $T_{k+1} \leq_{k+1} T_k$, $|T_k| \geq i_k$, $T_k \Vdash \dot{f}(k) \in \check{d}_k$ for some $d_k \in [\omega]^{<\omega}$. Just as in the proof of Lemma 7, one can argue that $\{T_k\}_{k\in\omega} \in L_{\mu_i}[Y \cap i]$ and so $T^* = \bigcap_{k\in\omega} T_k \in L_{\mu_i}[Y \cap i]$. Therefore T^* is a condition in $C(Y)$, which extends T and $T^* \Vdash \dot{f} \leq \check{g}$ where $g(k) = \max d_k$ for all $k \in \omega$. □

4. *S*-properness and shooting clubs

The poset which we will use for adding a closed unbounded subset to the complement of a stationary, co-stationary set is well known (see (11)).

Definition 3. Let $S \subseteq \omega_1$ be a stationary, co-stationary set. Then $Q(S)$ is the poset of all countable closed subsets of $\omega_1 \backslash S$, with the end-extension as the extension relation.

If G is $Q(S)$-generic, then $\bigcup G$ is a closed unbounded subset of ω_1 disjoint from S. Thus $Q(S)$ destroys the stationarity of S.

Lemma 9. $Q(S)$ is ω-distributive and so $Q(S)$ does not add new reals.

Proof. See (11). □

Since $Q(S)$ destroys the stationarity of S, it is not proper. However $Q(S)$ is almost proper in the following sense (see (10)).

Definition 4. Let $T \subseteq \omega_1$ be a stationary set. A poset $\mathbb{Q}$ is *T-proper*, if for every countable elementary submodel $\mathcal{M}$ of $H(\Theta)$, where Θ is a sufficiently large cardinal, such that $\mathcal{M} \cap \omega_1 \in T$, every condition $p \in \mathbb{Q} \cap \mathcal{M}$ has an $(\mathcal{M}, \mathbb{Q})$-generic extension q.

If S is a stationary, co-stationary subset of ω_1 and $Q(S)$ is the poset defined above (see Definition 3), then $Q(S)$ is $\omega_1 \backslash S$-proper (see (10)). The proofs of the following two Lemmas can be found in (10).

Lemma 10. If $\mathbb{Q}$ is S-proper, then $\mathbb{Q}$ preserves ω_1. Also $\mathbb{Q}$ preserves the stationarity of every stationary subset S' of ω_1 which is contained in S.

Lemma 11. If $\langle\langle \mathbb{P}_\alpha : \alpha \leq \delta\rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle\rangle$ is a countable support iteration of S-proper posets, then $\mathbb{P}_\delta$ is S-proper.

The proofs of the next two Lemmas follow almost identically the corresponding statements for proper posets (see (1), Theorems 2.10 and 2.12).

Lemma 12. Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper posets of size ω_1. Then $\mathbb{P}_\delta$ is $\aleph_2$-c.c.

Lemma 13. Assume CH. Let $\langle \mathbb{P}_\alpha : \alpha \leq \delta \rangle$ be a countable support iteration of length $\delta < \omega_2$ of S-proper posets of size ω_1. Then CH holds in $V^{\mathbb{P}_\delta}$.

5. Forcing a Δ^1_3 well-order of the reals and not CH

Lemma 14. Let $V = L$. There is a function $F : \omega_2 \to L_{\omega_2}$, which is Σ_1 definable over L_{ω_2} and a sequence $\bar{S} = (S_\beta : \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1, which is Σ_1 definable over L_{ω_2} with parameter ω_1 such that $F^{-1}(a)$ is unbounded in ω_2 for every $a \in L_{\omega_2}$, and whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ agree with $F^{\mathcal{N}}$, $\bar{S}^{\mathcal{N}}$ on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. In addition if $\mathcal{M}$ is suitable and $\omega_1^{\mathcal{M}} = \omega_1$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ equal the restrictions of $F, \bar{S}$ to the ω_2 of $\mathcal{M}$.

Proof. Define $F(\alpha) = a$ iff via Gödel pairing α codes a pair (α_0, α_1) where a has rank α_0 in the natural wellorder of the sets in L. For the almost disjoint stationary sets, let $(D_\gamma : \gamma < \omega_1)$ be the canonical L_{ω_1} definable $\Diamond$ sequence (see (7)), for each $\alpha < \omega_2$ let A_α be the L-least subset of ω_1 coding α and define S_α to be the set of all $i < \omega_1$ such that $D_i = A_\alpha \cap i$. □

Let F and $\bar{S} = (S_\beta : \beta < \omega_2)$ be as above. Let S be a stationary subset of ω_1 almost disjoint from every element of $\bar{S}$. Note that we may assume that such an S exists. The function F will be used as a bookkeeping function. Recursively, we will define a countable support iteration $\langle\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle\rangle$ such that $\mathbb{P} = \mathbb{P}_{\omega_2}$ will be a poset adding a Δ^1_3-definable wellorder of the reals. We can assume that all names for reals are nice in the following sense. If $\dot{f}$ is an $\mathbb{H}$-name for a real, for some poset $\mathbb{H}$, then $\dot{f}$ is a nice $\mathbb{H}$-name for a real if $\dot{f} = \bigcup_{i \in \omega} \{\langle\langle i, j^i_p \rangle, p\rangle : p \in \mathcal{A}_i(\dot{f})\}$ where for all $i \in \omega$, $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $\mathbb{H}$, $j^i_p \in \omega$ and for all $p \in \mathcal{A}_i(\dot{f})$, $p \Vdash \dot{f}(i) = j^i_p$. Then for $\alpha < \beta < \omega_2$ we can assume that all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals which are not $\mathbb{P}_\alpha$ names. For each $\alpha < \omega_2$, define a wellorder $<_\alpha$ on the reals of $L[G_\alpha]$, where G_α is a $\mathbb{P}_\alpha$-generic as follows. If x is a real in $L[G_\alpha]$ let σ^α_x be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \leq \alpha$ is least so that x has a $\mathbb{P}_\gamma$-name. For x, y reals in $L[G_\alpha]$ define $x <_\alpha y$ if and only if $\sigma^\alpha_x <_L \sigma^\alpha_y$. Abusing notation, we will identify $<_\alpha$ with its $\mathbb{P}_\alpha$-name. Since for $\alpha < \beta$, $\sigma^\alpha_x = \sigma^\beta_x$ we have that $<_\alpha$ is an initial segment of $<_\beta$. Then if G is a $\mathbb{P}$-generic filter, $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ will be the desired wellorder of the reals. If x, y are reals in $L[G_\alpha]$ and $x <_\alpha y$ let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$.

We proceed with the recursive definition of $\mathbb{P}_{\omega_2}$. Let $\mathbb{P}_0$ be the trivial poset. Suppose $\mathbb{P}_\alpha$ has been defined. Let $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha$-name for a poset such that $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for a proper forcing notion of cardinality at most $\aleph_1$ and $\dot{\mathbb{Q}}^1_\alpha$ is defined as follows. If $F(\alpha)$ is not of the form $\{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals $x <_\alpha y$ in $L[G_\alpha]$ then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for the trivial poset. Otherwise $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ for some reals $x <_\alpha y$ in $L[G_\alpha]$. Set $x_\alpha = x$, $y_\alpha = y$. Then let $\dot{\mathbb{Q}}^1_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-name for $\mathbb{K}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$ where:

(1) In $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha}$, $\mathbb{K}^0_\alpha$ is the direct limit $\langle \mathbb{P}^0_{\alpha,n}, \dot{\mathbb{K}}^0_{\alpha,n} : n \in \omega \rangle$, where $\dot{\mathbb{K}}^0_{\alpha,n}$ is a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n})$ for $n \in x_\alpha * y_\alpha$, and $\dot{\mathbb{K}}^0_{\alpha,n}$ is a $\mathbb{P}^0_{\alpha,n}$-name for $Q(S_{\alpha+2n+1})$ for $n \notin x_\alpha * y_\alpha$.

(2) Let G^0_α be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-generic filter and let H_α be a $\mathbb{K}^0_\alpha$-generic over $L[G^0_\alpha]$. In $L[G^0_\alpha * H_\alpha]$ let X_α be a subset of ω_1 coding α, coding the pair (x_α, y_α), coding a level of L in which α has size at most ω_1 and coding the generic $G^0_\alpha * H_\alpha$, which we can regard as a subset of an element of L_{ω_2}. Let $\mathbb{K}^1_\alpha = \mathcal{L}(\phi_\alpha)$ where $\phi_\alpha = \phi_\alpha(\omega_1, X_\alpha)$ is the Σ_1-sentence which holds if and only if X_α codes an ordinal $\bar{\alpha} < \omega_2$ and a pair (x, y) such that $S_{\bar{\alpha}+2n}$ is nonstationary for $n \in x * y$ and $S_{\bar{\alpha}+2n+1}$ is nonstationary for $n \notin x * y$. Let $\dot{X}_\alpha$ be a $\mathbb{P}^0_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha$-name for X_α and let $\dot{\mathbb{K}}^1_\alpha$ be a $\mathbb{P}^0_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha$-name for $\mathbb{K}^1_\alpha$.

(3) Let Y_α be $\mathbb{K}^1_\alpha$-generic over $L[G^0_\alpha * H_\alpha]$. Note that the even part of Y_α-codes X_α and so codes the generic $G^0_\alpha * H_\alpha$. Then in $L[Y_\alpha] = L[G^0_\alpha * H_\alpha * Y_\alpha]$, let $\mathbb{K}^2_\alpha = C(Y_\alpha)$. Finally, let $\dot{\mathbb{K}}^2_\alpha$ be a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{K}}^0_\alpha * \dot{\mathbb{K}}^1_\alpha$-name for $\mathbb{K}^2_\alpha$.

With this the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$ is complete.

Lemma 15. $\mathbb{P}$ is S-proper and ω_2-c.c.

Proof. By Lemma 11 and Lemma 12. □

Lemma 16. Let G be a $\mathbb{P}$-generic filter and let x, y be reals in $L[G]$. If $x <^G y$, then there is a real R such that for every countable suitable $\mathcal{M}$ containing R as an element, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$.

Proof. Let γ_1, γ_2 be minimal such that x has a $\mathbb{P}_{\gamma_1}$-name, y has a $\mathbb{P}_{\gamma_2}$-name. Thus for every $\alpha \geq \max\{\gamma_1, \gamma_2\}$, $\sigma^\alpha_x = \sigma^{\gamma_1}_x$ and $\sigma^\alpha_y = \sigma^{\gamma_2}_y$. Since $F^{-1}(\{\sigma^{\gamma_1}_x, \sigma^{\gamma_2}_y\})$ is unbounded in ω_2, there is α such that $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ and so $x_\alpha = x$, $y_\alpha = y$. Let G^0_α be $\mathbb{P}_\alpha * \dot{\mathbb{Q}}^0_\alpha$-generic, let H_α be $\mathbb{K}^0_\alpha$-generic over $L[G^0_\alpha]$, Y_α be the $\mathbb{K}^1_\alpha$-generic over $L[G^0_\alpha * H_\alpha]$ and let R_α be the $\mathbb{K}^2_\alpha$-generic over $L[Y_\alpha]$. By Lemma 6, R_α codes Y_α and Y_α codes X_α which in turn codes the pair $(x_\alpha, y_\alpha) = (x, y)$. Let $\mathcal{M}$ be a countable suitable model containing R_α as an element. Then using $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$, $\mathcal{M}$ contains $Y_\alpha \restriction \gamma$ and therefore $X_\alpha \cap \gamma$ as an element, where $\gamma = \omega_1^{\mathcal{M}}$. By Lemma 2, $\phi_\alpha(\gamma, X_\alpha \cap \gamma)$ holds in $\mathcal{M}$ and therefore there is an ordinal $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for $n \in x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for $n \notin x * y$ as desired. □

Lemma 17. Let G be $\mathbb{P}$-generic. Then for β not of the form $\alpha + 2n$, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha + 2n + 1$, $n \notin x^G_\alpha * y^G_\alpha$, the set S_β is stationary in $L[G]$.

Proof. Let $p \in \mathbb{P}$ be a condition forcing that $\beta < \omega_2$ is not of the form $\alpha + 2n$, $n \in x^G_\alpha * y^G_\alpha$ and not of the form $\alpha + 2n + 1$, $n \notin x^G_\alpha * y^G_\alpha$. Now consider the forcing notion $\mathbb{P} \restriction p$, consisting of all conditions in $\mathbb{P}$ which extend p. This is also an iteration, where at stage α one forces with $\mathbb{Q}_\alpha \restriction p(\alpha)$. Note that G is also $\mathbb{P} \restriction p$ generic. However $\mathbb{P} \restriction p$ is S_β-proper and so S_β remains stationary in $L[G]$. □

Theorem 1. It is consistent with the negation of CH that there is a projective (indeed Δ^1_3-definable) wellorder of the reals.

Proof. Let $\mathbb{P} = \mathbb{P}_{\omega_2}$ be the partial order constructed in this section and let G be $\mathbb{P}$-generic. Then $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ is a wellorder on the reals of $L[G]$. By Lemma 17 for every pair of reals x, y in $L[G]$ we have that

(1) $x < y$ iff for some $\alpha < \omega_2$, $S_{\alpha+2n}$ is nonstationary for n in $x * y$ and $S_{\alpha+2n+1}$ is nonstationary for n not in $x * y$.

However by Lemma 16, $L[G]$ also satisfies:

(2) If $x < y$ then there exists a real R such that for every suitable, countable model $\mathcal{M}$ containing R there is an ordinal $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for n in $x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for n not in $x * y$.

However (1) implies the converse of (2). Indeed, assume (1) and let R be a real such that for every countable suitable ZF$^-$ model $\mathcal{M}$ containing R there is an ordinal $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+2n}$ is nonstationary in $\mathcal{M}$ for n in $x * y$ and $S^{\mathcal{M}}_{\bar{\alpha}+2n+1}$ is nonstationary in $\mathcal{M}$ for n not in $x * y$. By Löwenheim-Skolem this holds for arbitrary suitable ZF$^-$ models $\mathcal{M}$ containing R. Note that as our forcing preserves cardinals, $L_\Theta[R]$ is suitable for a large regular Θ. Thus let $\mathcal{M} = L_\Theta[R]$ and let $\alpha < \omega_2^{\mathcal{M}} = \omega_2$ be the ordinal guaranteed by the conclusion of (2) for $\mathcal{M}$. As $(S_\beta : \beta < \omega_2)$ is definable over L_{ω_2} and Θ is greater than ω_2, it follows that $S^{\mathcal{M}}_\beta$ equals S_β for each $\beta < \omega_2$. Thus $S_{\alpha+2n}$ is nonstationary in $\mathcal{M}$ for n in $x * y$ and $S_{\alpha+2n+1}$ is nonstationary in $\mathcal{M}$ for n not in $x * y$. It follows that these sets are nonstationary in the larger model $L[G]$ and therefore by (1), we have $x < y$.

Therefore in $L[G]$, the union $<^G$ of the wellorders $<^G_\alpha$, $\alpha < \omega_2$, has a Σ^1_3 definition:

$x <^G y$ iff there exists a real R such that for all countable, suitable $\mathcal{M}$ containing R as an element there is an $\alpha < \omega_2^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\alpha+2n}$ is nonstationary in $\mathcal{M}$ for n in $x * y$ and $S^{\mathcal{M}}_{\alpha+2n+1}$ is nonstationary in $\mathcal{M}$ for n not in $x * y$.

It remains to observe that since $x \not<^G y$ is Π^1_3 and $<^G$ is a linear order, $<^G$ indeed has a Δ^1_3 definition. □

6. Cardinal characteristics and projective wellorders

We will use the combinatorial properties of the forcing notions for destroying a stationary set, "localization", and coding with perfect trees defined in section 4, 2 and 3 respectively, to show that the existence of a Δ^1_3 definable wellorder of the reals is consistent with certain inequalities between some of the known cardinal characteristics of the continuum. We will need the following preservation theorems.

Lemma 18. Let $S \subseteq \omega_1$ be a stationary set and let $\langle\langle \mathbb{P}_i : i \leq \delta\rangle, \langle \dot{\mathbb{Q}}_i : i < \delta\rangle\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is ${}^\omega\omega$-bounding and S-proper". Then $\mathbb{P}_\delta$ is ${}^\omega\omega$-bounding and S-proper.

Proof. The proof follows almost identically the proof of the corresponding theorem for proper posets (see (10) and (1)). □

A forcing notion $\mathbb{P}$ is *almost ${}^\omega\omega$-bounding* if for every $\mathbb{P}$-name for a real $\dot{f}$ and condition $p \in \mathbb{P}$, there is a ground model real g such that for every infinite ground model subset A of ω, there is an extension q_A of p such that $q_A \Vdash_{\mathbb{P}} \exists^\infty i \in A(\dot{f}(i) \leq \check{g}(i))$ (see (14)). A poset which preserves the ground model reals as an unbounded family is called *weakly bounding*. Note that even finite iterations of weakly bounding posets may add a real dominating the ground model reals (see (1), section 4.1).

Lemma 19. Let $S \subseteq \omega_1$ be stationary set and let $\langle\langle \mathbb{P}_i : i \leq \delta\rangle, \langle \dot{\mathbb{Q}}_i : i < \delta\rangle\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, almost ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is almost ${}^\omega\omega$-bounding and S-proper". Then $\mathbb{P}_\delta$ is weakly bounding and S-proper.

Proof. The proof follows almost identically the proof of the corresponding theorem for proper posets (see (10) and (1)). □

Recall that the *bounding number* $\mathfrak{b}$ is the minimal size of an unbounded family and that the *dominating number* $\mathfrak{d}$ is the minimal size of a dominating family (see (2)).

Theorem 2. It is consistent with $\mathfrak{d} < \mathfrak{c}$ that there is a Δ^1_3 definable wellorder of the reals.

Proof. In the definition of $\mathbb{P} = \mathbb{P}_{\omega_2}$ from Section 5, for every $\alpha < \omega_2$ we defined $\dot{\mathbb{Q}}^0_\alpha$ to be a $\mathbb{P}_\alpha$-name for an arbitrary proper poset of size at most $\aleph_1$. Now define $\mathbb{P}_{\mathbb{S}}$ to be a countable support iteration of length ω_2, defined just as $\mathbb{P}_{\omega_2}$ with the additional requirement that for every $\alpha < \omega_2$, $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial forcing notion. By Lemma 18 (as well as Lemmas 4, 8, and 9) the poset $\mathbb{P}_{\mathbb{S}}$ is ${}^\omega\omega$-bounding (and S-proper). Thus if G is $\mathbb{P}_{\mathbb{S}}$-generic, in $L[G]$ the dominating number $\mathfrak{d}$ is ω_1 while $\mathfrak{c} = \omega_2$. □

For convenience of the reader, we will state the definitions of $\mathfrak{a}$ and $\mathfrak{s}$ (see also (2)). A family $\mathcal{A} \subseteq [\omega]^\omega$ is almost disjoint, if every two distinct elements of $\mathcal{A}$ have finite intersection. An infinite almost disjoint family $\mathcal{A}$ is maximal (abbreviated mad family) if for every $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $|A \cap B| = \omega$. The *almost disjointness number* $\mathfrak{a}$ is the minimal size of a maximal almost disjoint family. Whenever A, B are infinite subsets of ω, A is *split by* B if $|A \cap B| = |A \cap B^c| = \omega$. A family $W \subseteq [\omega]^\omega$ is *splitting* if for every $A \in [\omega]^\omega$ there is $B \in W$ such that A is split by B. The *splitting number* $\mathfrak{s}$ is the minimal cardinality of a splitting family.

Theorem 3. It is consistent with $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ that there is a Δ^1_3 definable wellorder of the reals.

Proof. In (14), S. Shelah obtains a proper almost ${}^\omega\omega$-bounding poset Q of size $\mathfrak{c}$, which adds a real not split by the ground model reals (see (14), Definition 2.8, Lemma 1.14). In addition he shows that if V is a model of CH and $\mathcal{A}$ is a mad family of size ω_1 then in $V_1 = V^{\mathbb{C}(\omega_1)}$, where $\mathbb{C}(\omega_1)$ is the poset for adding ω_1 Cohen reals, there is an almost ${}^\omega\omega$-bounding, proper poset which destroys the maximality of $\mathcal{A}$, i.e. forces over V_1 that $\mathcal{A}$ is not maximal (see (14), Definition 2.10 and Claim 2.16).

Let F_0 be a function with domain ω_2, such that for every $\mathbb{H}$-name $\dot{\mathcal{A}}$ for a mad family of size ω_1, where $\mathbb{H}$ is a poset of size ω_1, the set $F_0^{-1}(\dot{\mathcal{A}})$ is unbounded. Note that we can consider only normalized posets $\mathbb{H}$, i.e. posets which can be realized as subsets of ω_1 and also we can assume that all names for reals are nice. Let $\mathbb{P}_Q$ be a countable support iteration of length ω_2 defined as $\mathbb{P}_{\omega_2}$ from section 5 with the additional requirement that for every $\alpha < \omega_2$, in $L^{\mathbb{P}_\alpha}$ we have that $\mathbb{Q}^0_\alpha = \mathbb{H}^0_\alpha * \dot{\mathbb{H}}^1_\alpha * \dot{\mathbb{H}}^2_\alpha$ where $\mathbb{H}^0_\alpha$, $\dot{\mathbb{H}}^1_\alpha$ and $\dot{\mathbb{H}}^2_\alpha$ are defined as follows. Let $\mathbb{H}^0_\alpha$ be the poset for adding ω_1 Cohen reals. If $F_0(\alpha)$ is not a $\mathbb{P}_\alpha$-name for a mad family of size ω_1, then let $\dot{\mathbb{H}}^1_\alpha$ be an $\mathbb{H}^0_\alpha$-name for the trivial poset. If $F_0(\alpha)$ is a $\mathbb{P}_\alpha$-name for a mad family of size ω_1, then let $\dot{\mathbb{H}}^1_\alpha$ be a $\mathbb{H}^0_\alpha$-name for an almost ${}^\omega\omega$-bounding poset which destroys the maximality of $\mathcal{A}$ (by the remark in the previous paragraph such forcing notion exists in $L^{\mathbb{P}_\alpha * \mathbb{H}^0_\alpha}$). Let $\dot{\mathbb{H}}^2_\alpha$ be $\mathbb{H}^0_\alpha * \dot{\mathbb{H}}^1_\alpha$-name for Shelah's poset Q. With this the definition of $\mathbb{P}_Q$ is complete.

Let G be $\mathbb{P}_Q$-generic filter over L. Since Cohen forcing is almost ${}^\omega\omega$-bounding, by Lemma 19 (as well as Lemmas 4, 8, and 9) $\mathbb{P}_Q$ preserves the ground model reals as an unbounded family, and so $L[G] \vDash \mathfrak{b} = \omega_1$. To see that $\mathfrak{s} = \omega_2$ in $L[G]$ consider an arbitrary family $W \subseteq [\omega^\omega] \cap L[G]$ of cardinality ω_1. Then for some $\alpha < \omega_2$, $W \subseteq L[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$. By definition of the poset, $\mathbb{H}^2_\alpha$ adds a real which is not split by the reals of $L[G_\alpha]$ and so not split by W. Therefore W is not splitting and so $(\mathfrak{s} = \omega_2)^{L[G]}$. Finally suppose that $L[G] \vDash \mathfrak{a} = \omega_1$ and let $\mathcal{A}$ be a maximal almost disjoint family in $L[G]$ of size ω_1. Then for some $\alpha < \omega_2$, $\mathcal{A} \subseteq L[G \cap \mathbb{P}_\alpha]$ and so $\mathcal{A}$ has a $\mathbb{P}_\alpha$-name $\dot{\mathcal{A}}$. Since $F_0^{-1}(\dot{\mathcal{A}})$ is unbounded in ω_2 for some $\beta \geq \alpha$ we have $F(\beta) = \dot{\mathcal{A}}$. By definition of $\mathbb{H}^1_\beta$, $L[G_{\beta+1}] \vDash \mathcal{A}$ is not mad, which is a contradiction and so $(\mathfrak{a} = \omega_2)^{L[G]}$. $\square$

Recall that a family $D \subseteq [\omega]^\omega$ is *groupwise dense* if D is closed with respect to the "almost subset" relation (i.e. whenever $X \in D$ and $Y \backslash X$ is finite, $Y \in D$) and for every family Π of infinitely many pairwise disjoint finite subsets of ω, the union of some subfamily of Π is in D. *The groupwise density number* $\mathfrak{g}$ is the minimal cardinal κ such that for some family $\mathcal{D}$ of κ-many groupwise dense families, $\bigcap \mathcal{D} = \emptyset$ (see (2)).

Theorem 4. It is consistent with $\mathfrak{b} < \mathfrak{g}$ that there is a Δ^1_3 definable wellorder of the reals.

Proof. Let $\mathbb{P}_{\mathbb{M}}$ be the countable support iteration of length ω_2 defined as the poset $\mathbb{P}$ from section 5, with the additional requirement that for every $\alpha < \omega_2$, $\dot{\mathbb{Q}}^0_\alpha$ is a $\mathbb{P}_\alpha$-name for Miller forcing (for definition see (2), 11.9). Let G be $\mathbb{P}_{\mathbb{M}}$-generic over L. Since Miller forcing is almost ${}^\omega\omega$-bounding (see (4), Theorem 8.13), by Lemma 19 (as well as Lemmas 4, 8, and 9) $\mathbb{P}_{\mathbb{M}}$ is weakly bounding and so $L[G] \vDash \mathfrak{b} = \omega_1$. To see that $(\mathfrak{g} = \omega_2)^{L[G]}$ consider an arbitrary family $\mathcal{D} = \{D_i : i \in \omega_1\}$ of groupwise dense sets in $L[G]$. For every $\alpha < \omega_2$ let $D_{i,\alpha} = D_i \cap L[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$. Note that for every i the set of α's such that $D_{i,\alpha}$ is a groupwise dense family in $L[G_\alpha]$ forms an $\aleph_1$-closed unbounded subset of ω_2. Therefore there is a limit $\alpha < \omega_2$ such that $D_{i,\alpha}$ is a groupwise dense family in $L[G_\alpha]$ for all $i < \omega_1$. Then the Miller real added by $\mathbb{Q}^0_\alpha$ has supersets in all $D_{i,\alpha}$'s (see (3), Lemma 1) and so $\bigcap \mathcal{D} \neq \emptyset$. □

7. Questions

1. Which other inequalities between the standard cardinal characteristics of the real line are consistent with the existence of a projective wellorder of the reals?

2. What is the complexity in the projective hierarchy of the witnesses of the corresponding cardinal characteristics in these models?

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