Projective wellorders and mad families with large continuum

Vera Fischer[a,1,*], Sy David Friedman[a,1], Lyubomyr Zdomskyy[a,1]

[a]*Kurt Gödel Research Center, University of Vienna, Währinger Strasse 25, A-1090 Vienna, Austria*

Abstract

We show that $\mathfrak{b} = \mathfrak{c} = \omega_3$ is consistent with the existence of a Δ^1_3-definable wellorder of the reals and a Π^1_2-definable ω-mad subfamily of $[\omega]^\omega$ (resp. ω^ω).

Keywords: coding, projective wellorders, projective mad families, large continuum
2000 MSC: 03E15, 03E20, 03E35, 03E45

1. Introduction

The existence of a projective, in fact Δ^1_3-definable wellorder of the reals in the presence of large continuum, i.e. $\mathfrak{c} \geq \omega_3$, was established by Harrington in [8]. In the present paper, we develop an iteration technique which allows one not only to obtain the consistency of the existence of a Δ^1_3-definable wellorder of the reals with large continuum (see Theorem 1), but in addition the existence of a Π^1_2-definable ω-mad family with $\mathfrak{b} = \mathfrak{c} = \omega_3$ (see Theorem 2). The method is a natural generalization to models with large continuum of the iteration technique developed in [5]. We expect that an application of Jensen's coding techniques will lead to the same result with essentially arbitrary values for $\mathfrak{c}$.

For a more detailed introduction to the subject of projective wellorders of the reals and projective mad families, see [5] and [7]. Recall that a family $\mathcal{A}$ of infinite subsets of ω is almost disjoint if any two of its elements have finite intersection. An infinite almost disjoint family $\mathcal{A}$ is maximal (abbreviated mad family), if for every infinite subset b of ω, there is an element $a \in \mathcal{A}$ such that $|a \cap b| = \omega$. If $\mathcal{A}$ is an almost disjoint family, let $\mathcal{L}(\mathcal{A}) = \{b \in [\omega]^\omega : b$ is not covered by finitely many elements of $\mathcal{A}\}$. A mad family $\mathcal{A}$

*Corresponding author; Phone: (43-1) 43 1 4277 50523; Fax: (43-1) 43 1 4277 50599
Email addresses: `vfischer@logic.univie.ac.at` (Vera Fischer), `sdf@logic.univie.ac.at` (Sy David Friedman), `lzdomsky@logic.univie.ac.at` (Lyubomyr Zdomskyy)

[1]The authors would like to thank the Austrian Science Fund FWF for the generous support through grants no P. 20835-N13 (Fischer, Friedman), P. 19898-N18 and M1244-N13(Zdomskyy).

is ω-mad if for every $B \in [\mathcal{L}(\mathcal{A})]^\omega$, there is $a \in \mathcal{A}$ such that $|a \cap b| = \omega$ for all $b \in B$. For the definition of $\mathfrak{b}$, as well as an introduction to the subject of cardinal characteristics of the continuum we refer the reader to [1].

In section 2 we introduce a model in which $\mathfrak{b} = \mathfrak{c} = \omega_3$ and there is a Δ^1_3-definable wellorder of the reals. In section 3 we show how to modify the argument to obtain in addition the existence of a Π^1_2-definable ω-mad family. We begin by fixing an appropriate sequence $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3 \rangle$ of stationary subsets of ω_3 and explicitly destroying the stationarity of each S_α by adding a closed unbounded subset of ω_3 disjoint from it. The wellorder is produced by introducing reals (see Steps 1 through 3 in section 2) which code this stationary kill for certain stationary sets from $\vec{S}$. For this purpose, we use almost disjoint coding as well as a modified version of the method of localization (see [4] and [5, Definition 1]).

2. Projective Wellorders with Large Continuum

Throughout the paper we work over the constructible universe L, thus unless otherwise specified $V = L$. Let $\langle G_\xi : \xi \in \omega_2 \cap \mathrm{cof}(\omega_1) \rangle$ be a $\diamondsuit_{\omega_2}(\mathrm{cof}(\omega_1))$ sequence which is Σ_1 definable over L_{ω_2}. For every $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding the ordinal α. Let $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3 \rangle$ be the sequence of stationary subsets of ω_2 defined as follows: $S_\alpha = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}$. In particular, the sets S_α are stationary subsets of $\mathrm{cof}(\omega_1) \cap \omega_2$ which are mutually almost disjoint (that is, for all $1 < \alpha, \beta < \omega_3$, $\alpha \neq \beta$, we have that $S_\alpha \cap S_\beta$ is bounded). Let $S_{-1} = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = \emptyset\}$. Note that S_{-1} is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ disjoint from all S_α's.

Say that a transitive ZF$^-$ model $\mathcal{M}$ is *suitable* if $\omega_3^\mathcal{M}$ exists and $\omega_3^\mathcal{M} = \omega_3^{L^\mathcal{M}}$. From this it follows, of course, that $\omega_1^\mathcal{M} = \omega_1^{L^\mathcal{M}}$ and $\omega_2^\mathcal{M} = \omega_2^{L^\mathcal{M}}$.

Step 0. For every $\alpha : \omega_2 \leq \alpha < \omega_3$ shoot a closed unbounded set C_α disjoint from S_α via a poset $\mathbb{P}^0_\alpha$. The poset $\mathbb{P}^0_\alpha$ consists of all bounded, closed subsets of ω_2, which are disjoint from S_α. The extension relation is end-extension. Note that $\mathbb{P}^0_\alpha$ is countably closed and $\aleph_2$-distributive (see [3]). For every $\alpha \in \omega_2$ let $\mathbb{P}^0_\alpha$ be the trivial poset.

Let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1. Then $\mathbb{P}^0$ is countably closed and by the Δ-system Lemma, also ω_3-c.c. Its ω_2-distributivity is easily established using the stationary set $S_{-1} \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$.

Step 1. We begin by fixing some notation. Let X be a set of ordinals. Denote by $0(X)$, $I(X)$, and $II(X)$ the sets $\{\eta : 3\eta \in X\}$, $\{\eta : 3\eta + 1 \in X\}$ and $\{\eta : 3\eta + 2 \in X\}$, respectively. Let $Even(X)$ be the set of even ordinals in X and $Odd(X)$ be the set of odd ordinals in X.

In the following we treat 0 as a limit ordinal. For every $\alpha : \omega_2 \leq \alpha < \omega_3$ let $D_\alpha \subset \omega_2$ be a set coding the tuple $\langle C_\alpha, W_\alpha, W_\gamma \rangle$, where γ is the largest limit ordinal $\leq \alpha$. More precisely D_α is such that $0(D_\alpha)$, $I(D_\alpha)$, and $II(D_\alpha)$ equal C_α, W_α, and W_γ, respectively. Now let E_α be the club in ω_2 of intersections with ω_2 of elementary submodels of $L_{\alpha+\omega_2+1}[D_\alpha]$ which contain $\omega_1 \cup \{D_\alpha\}$ as a subset. (These elementary submodels form an ω_2-chain.) Now choose Z_α to be a subset of ω_2 such that $Even(Z_\alpha) = D_\alpha$, and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then β belongs to E_α. (This is easily done by placing in Z_α a code for a bijection $\phi : \beta_1 \to \omega_1$ on the interval $(\beta_0, \beta_0 + \omega_1)$ for each adjacent pair $\beta_0 < \beta_1$ from E_α.) Then we have:

$(*)_\alpha$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \vDash \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "$Even(X)$ codes a tuple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Indeed, given a suitable model $\mathcal{M}$ with $\omega_2^{\mathcal{M}} = \beta$ and $Z_\alpha \cap \beta \in \mathcal{M}$, note that $\beta \in E_\alpha$ by the construction of Z_α and also that $D_\alpha \cap \beta \in \mathcal{M}$. Let $\mathcal{N}$ be an elementary submodel of $L_{\alpha+\omega_2+1}[D_\alpha]$ such that $\omega_1 \cup \{D_\alpha\} \subset \mathcal{N}$ and $\mathcal{N} \cap \omega_2 = \beta$. Denote by $\bar{\mathcal{N}}$ the transitive collapse of $\mathcal{N}$. Then $\bar{\mathcal{N}} = L_\xi[D_\alpha]$ for some $\omega_2 > \xi > \beta$ and $\omega_2^{\bar{\mathcal{N}}} = \omega_2^{\mathcal{M}} = \beta$. Therefore $\bar{\mathcal{N}} \subset \mathcal{M}$. Let $Z'_\alpha \subset \omega_2$ be such that $Even(Z'_\alpha) = Odd(Z'_\alpha) = D_\alpha$. By the definition of D_α, $L_{\alpha+\omega_2+1}[D_\alpha] \vDash \psi(\omega_2, Z'_\alpha)$. By elementarity, $\bar{\mathcal{N}} \vDash \psi(\omega_2, Z'_\alpha \cap \beta)$. Since the formula ψ is Σ_1, $\omega_2^{\bar{\mathcal{N}}} = \omega_2^{\mathcal{M}}$, we conclude that $\mathcal{M} \vDash \psi(\omega_2, Z'_\alpha \cap \beta)$. Since $Z_\alpha \cap \beta \in \mathcal{M}$ and $Even(Z'_\alpha) = Even(Z_\alpha)$, we have $\mathcal{M} \vDash \psi(\omega_2, Z_\alpha \cap \beta)$, which finishes the proof of $(*)_\alpha$.

Now similarly to $\vec{S}$ we can define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of ω_1 using the "standard" $\diamondsuit$-sequence. Then in particular this sequence is nicely definable over L_{ω_1} and almost disjoint. Now we code Z_α by a subset X_α of ω_1 with the forcing $\mathbb{P}^1_\alpha$ consisting of all tuples $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. Then X_α obviously satisfies the following condition:

$(**)_\alpha$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model such that $\omega_2^{\mathcal{M}} = \beta$ and $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: " Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, whose even part $Even(\bar{Z})$ codes a tuple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha<\omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for $\alpha \in \omega_2$, be the product of the $\mathbb{P}^1_\alpha$'s with countable support. The poset $\mathbb{P}^1$ is easily seen to be countably closed. Moreover, it has the ω_2-c.c. by a standard Δ-system argument.

Step 2. Now we shall force a localization of the X_α's. Fix ϕ as in $(**)_\alpha$.

Definition 1. Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. We denote by $\mathcal{L}(X, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.

Set $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in Lim(\omega_3)\backslash\omega_2$ and $m \in \omega$. Let $\mathbb{P}^2_{\alpha+m}$ be the trivial poset for every $\alpha \in Lim(\omega_2)$ and $m \in \omega$. Let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the Δ-system Lemma in $L^{\mathbb{P}^0 * \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the ω_2-c.c.

Observe that the poset $\mathbb{P}^2_{\alpha+m}$, where $\alpha > 0$, produces a generic function from ω_1 (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$) into 2, which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

Lemma 1. The poset $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

Proof. Given a condition $p_0 \in \mathbb{P}_0$ and a collection $\{O_n\}_{n\in\omega}$ of open dense subsets of $\mathbb{P}_0$, choose the least countable elementary submodel $\mathcal{N}$ of some large L_θ (θ regular) such that $\{p_0\} \cup \{\mathbb{P}_0\} \cup \{O_n\}_{n\in\omega} \subset \mathcal{N}$. Build a subfilter g of $\mathbb{P}_0 \cap \mathcal{N}$, below p_0, which hits all dense subsets of $\mathbb{P}_0$ which belong to $\mathcal{N}$. Write g as $g(0) * g(1) * g(2)$. Now $g(0) * g(1)$ has a greatest lower bound $p(0) * p(1)$ because the forcing $\mathbb{P}^0 * \mathbb{P}^1$ is ω-closed. The condition $(p(0), p(1))$ is obviously $(\mathcal{N}, \mathbb{P}^0 * \mathbb{P}^1)$-generic.

On each component $\alpha+m \in \mathcal{N}\cap\omega_3$, where $\alpha \in Lim(\omega_3)$, $m \in \omega$, define $p(2)(\alpha+m) = \bigcup g(2)(\alpha+m)$. It suffices to verify that $p(2)(\alpha+m)$ is a condition in $\mathbb{P}^2_{\alpha+m}$, for this will give us a condition $p(2)$ so that $p(0) * p(1) * p(2)$ meets each of the O_n's.

As $(p(0)(\alpha), p(0)(\alpha+m), p(1)(\alpha), p(1)(\alpha+m))$ is a $(\mathcal{N}, \mathbb{P}^0_\alpha * \mathbb{P}^0_{\alpha+m} * \mathbb{P}^1_\alpha * \mathbb{P}^1_{\alpha+m})$-generic condition, if

$$G := G(0)(\alpha) * G(0)(\alpha+m) * G(1)(\alpha) * G(1)(\alpha+m)$$

is a $\mathbb{P}^0_\alpha * \mathbb{P}^0_{\alpha+m} * \mathbb{P}^1_\alpha * \mathbb{P}^1_{\alpha+m}$-generic filter over L containing it, then the isomorphism π of the transitive collapse $\bar{\mathcal{N}}$ of $\mathcal{N}$, onto $\mathcal{N}$ extends to an elementary embedding from

$$\bar{\mathcal{N}}_0 := \bar{\mathcal{N}}[\overline{g(0)}(\bar{\alpha}) * \overline{g(0)}(\bar{\alpha}+m) * \overline{g(1)}(\bar{\alpha}) * \overline{g(1)}(\bar{\alpha}+m)]$$

into $L_\theta[G]$. Here $\overline{g(i)} = \pi^{-1}(g(i))$, $i \in 2$, and $\bar{\xi} = \pi^{-1}(\xi)$ for all $\xi \in \mathcal{N} \cap \mathrm{Ord}$. By the genericity of G we know that, letting $X_\alpha = \bigcup G(1)(\alpha)$, $X_{\alpha+m} = \bigcup G(1)(\alpha+m)$, properties $(**)_\alpha$ and $(**)_{\alpha+m}$ hold. By elementarity, $\bar{\mathcal{N}}_0$ is a suitable model and $\bar{\mathcal{N}}_0 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$, where $x_{\bar{\alpha}} = \bigcup g(1)(\alpha) = \bigcup \overline{g(1)}(\bar{\alpha})$ and $x_{\bar{\alpha}+m} = \bigcup g(1)(\alpha+m) = \bigcup \overline{g(1)}(\bar{\alpha}+m)$. By the construction of $\mathbb{P}_0$, $\bar{\mathcal{N}}_0 = \bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ and hence $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}] \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$.

Let ξ be such that $\bar{\mathcal{N}} = L_\xi$ and let $\mathcal{M}$ be any suitable model containing $p(2)(\alpha)$, $p(2)(\alpha+m)$, and such that $\omega_1^{\mathcal{M}} = \omega_1 \cap \mathcal{N}$. We have to show that $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. Set $\eta = \mathcal{M} \cap \mathrm{Ord}$ and consider the chain $\mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ of suitable models, where $\mathcal{M}_2 = L_\eta[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ and $\mathcal{M}_1 = L_\eta[p(2)(\alpha), p(2)(\alpha+m)]$. Three cases are possible.

Case a). $\eta > \xi$. Since $\mathcal{N}$ was chosen to be the least countable elementary submodel of L_θ containing the initial condition, the poset and the sequence of dense sets, it follows that ξ (and therefore also δ) is collapsed to ω in $L_{\xi+2}$, and hence this case cannot happen.

Case b). $\eta = \xi$. In this case $\mathcal{M}_2 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. (Indeed, $\mathcal{M}_2 = L_\eta[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}] = \bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$.) Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$ and $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$.

Case c). $\eta < \xi$. In this case $\mathcal{M}_2$ is an element of $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$. Since $L_\theta[G]$ satisfies $(**)_\alpha$ and $(**)_{\alpha+m}$, by elementarity so does the model $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ with X_α replaced by $x_{\bar{\alpha}}$ and $X_{\alpha+m}$ replaced by $x_{\bar{\alpha}+m}$. In particular, $\mathcal{M}_2 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$, and $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$, which finishes our proof. $\square$

Set $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$. Let us fix $\xi \in \omega_3$ and denote by $\mathbb{P}^{0,\neq\xi}, \mathbb{P}^{1,\neq\xi}, \mathbb{P}^{2,\neq\xi}$ the following posets in L, $L^{\mathbb{P}^{0,\neq\xi}}$, and $L^{\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi}}$, respectively:

$$\prod_{\alpha\in\omega_3\setminus\{\xi\}} \mathbb{P}^0_\alpha \text{ with supports of size } \omega_1;$$

$$\prod_{\alpha\in\omega_3\setminus\{\xi\}} \mathbb{P}^1_\alpha \text{ with countable supports; and}$$

$\prod_{\alpha\in\omega_3\setminus\{\xi\}}\mathbb{P}^2_\alpha$ with countable supports.

Observe that $\tilde{\mathbb{P}}^{\neq\xi}_0 := \mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi} <_c \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 = \mathbb{P}_0$, where for posets $\mathbb{P}\subseteq\mathbb{Q}$ the notation $\mathbb{P} <_c \mathbb{Q}$ means that the identity embedding from $\mathbb{P}$ to $\mathbb{Q}$ is complete.[2] Let $\tilde{\mathbb{R}}$ be the quotient poset $\mathbb{P}_0/\tilde{\mathbb{P}}^{\neq\xi}_0$. Thus $\tilde{\mathbb{P}}^{\neq\xi}_0 * \tilde{\mathbb{R}} = \mathbb{P}_0$.

Step 3. We begin with fixing some terminology. For $\alpha : 1 < \alpha < \omega_3$ we will say that there is a stationary kill of S_α, if there is a closed unbounded set C disjoint from S_α. We will say that the stationary kill of S_α is coded by a real, if there is a closed unbounded set disjoint from S_α which is constructible from this real.

Fix a nicely definable sequence $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega\rangle$ of almost disjoint subsets of ω. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3\rangle$ such that $\mathbb{P}_0$ is as above, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset, in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals and $\mathfrak{c} = \mathfrak{b} = \aleph_3$. Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$ and we shall prove that $L[G_\alpha]\cap\omega^\omega = L[\langle \dot{u}^{G_\alpha}_\xi : \xi < \alpha\rangle]\cap\omega^\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α (see Lemma 2). This gives us a canonical wellorder of the reals in $L[G_\alpha]$, which depends only on the sequence $\langle \dot{u}^{G_\alpha}_\xi : \xi < \alpha\rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta}$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, $<^G = \bigcup\{\dot{<}^G_\alpha : \alpha < \omega_3\}$ will be the desired wellorder of the reals. Furthermore this wellorder will not depend on the generic set G (see Lemmas 4 and 5).

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along this construction we shall also define a sequence $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3)\rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha+\omega)$. For every $\omega_2 \leq \nu < \omega_3$ fix a bijection $i_\nu : \{\langle\zeta,\xi\rangle : \zeta < \xi < \nu\} \to Lim(\omega_2)$. If G_α is $\mathbb{P}_\alpha$-generic over L, $<_\alpha = \dot{<}^{G_\alpha}_\alpha$ and x, y are reals in $L[G_\alpha]$ such that $x <_\alpha y$, let $x*y = \{2n : n\in x\}\cup\{2n+1 : n \in y\}$ and $\Delta(x*y) = \{2n+2 : n \in x*y\}\cup\{2n+1 : n\notin x*y\}$.

Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Case 1. Suppose α is a limit ordinal and write it in the form $\omega_2\cdot\alpha' + \xi$, where $\xi < \omega_2$. If $\alpha' > 0$, let $i = i_{o.t.(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})}$ and $\langle \xi_0, \xi_1\rangle = i^{-1}(\xi)$. Let $A_\alpha := \dot{A}^{G_\alpha}_\alpha$ be the set $\alpha + (\omega\setminus\Delta(x_{\xi_0} * x_{\xi_1}))$, where x_ζ is the ζ-th real in $L[G_{\omega_2\cdot\alpha'}]\cap[\omega]^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'}$ (here $G_{\omega_2\cdot\alpha'} = G_\alpha\cap\mathbb{P}_{\omega_2\cdot\alpha'}$). Let also

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(x_{\xi_0}*x_{\xi_1})} Y_{\alpha+m}\times\{m\}]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if $s_1 \subset t_1$, s_0 is an initial segment of t_0 and $(t_0\setminus s_0)\cap B_{\zeta,m} = \emptyset$ for all $\langle\zeta, m\rangle \in s_1$.

[2] It might seem unclear why we denote $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi}$ by $\tilde{\mathbb{P}}^{\neq\xi}_0$ and not simply by $\mathbb{P}^{\neq\xi}_0$. It is to reserve the notation $\mathbb{P}^{\neq\xi}_0$ for a certain restriction of $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi}$ appearing naturally in the proof of Lemma 3.

Case 2. If α is not of the form above, i.e. α is a successor or $\alpha < \omega_2$, then $\dot{A}_\alpha$ is a name for the empty set and $\dot{\mathbb{Q}}_\alpha$ is a name for the following poset adding a dominating real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in \omega^{<\omega}, s_1 \in [o.t.(\dot{<}_\alpha^{G_\alpha})]^{<\omega}\},$$

where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subset t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \mathrm{dom}(t_0) \setminus \mathrm{dom}(s_0)$ and $\xi \in s_1$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}_\alpha^{G_\alpha}$.

In both cases $\mathbb{Q}_\alpha$ adds the generic real[3] $u_\alpha = \bigcup\{s_0 : \exists s_1\ \langle s_0, s_1\rangle \in g_\alpha\}$, where g_α is $\mathbb{Q}_\alpha$-generic over $V[G_\alpha]$ and $L[G_\alpha][u_\alpha] = L[G_\alpha][g_\alpha]$.

With this the definitions of $\mathbb{P} = \mathbb{P}_{\omega_3}$ and $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3)\rangle$ are complete.

Remark 1. Note that if the first case in the definition of $\dot{\mathbb{Q}}_\alpha$ above takes place, then in $L^{\mathbb{P}_\alpha}$ the poset $\dot{\mathbb{Q}}_\alpha$ produces a real r_α, which for certain reals x, y codes $Y_{\alpha+m}$ for all $m \in \Delta(x * y)$.

Let $\mathbb{H}$ be a poset. An $\mathbb{H}$-name $\dot{f}$ is called a *nice name for a real* if $\dot{f} = \bigcup_{i\in\omega}\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i(\dot{f})\}$ where for all $i \in \omega$, $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $\mathbb{H}$, $j^i_p \in \omega$ and for all $p \in \mathcal{A}_i(\dot{f})$, $p \Vdash \dot{f}(i) = j^i_p$. From now on we will assume that all names for reals are nice.

Using the fact that for every $p \in \mathbb{P}$ and $\alpha > 0$ the coordinate $p(\alpha)$ is a $\mathbb{P}_\alpha$-name for a finite set of ordinals, one can show that the set $\mathcal{D}$ of conditions p fulfilling the following properties is dense in $\mathbb{P}$:

- For every $\alpha > 0$ in the support of p, $p(\alpha) = \langle s_0, \check{s}_1\rangle$ for some $s_1 \in [\mathrm{Ord}]^{<\omega}$ and $s_0 \in [\omega]^{<\omega}$ or $s_0 \in \omega^{<\omega}$ depending on $\dot{\mathbb{Q}}_\alpha$.

Lemma 2. Let $\gamma \leq \omega_3$ and let G_γ be a $\mathbb{P}_\gamma$-generic filter over L. Then $L[G_\gamma] \cap \omega^\omega = L[\langle \dot{u}_\delta^{G_\gamma} : \delta < \gamma\rangle] \cap \omega^\omega$.

Proof. Let $\dot{f} = \bigcup_{i\in\omega}\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i(\dot{f})\}$ be a nice $\mathbb{P}_\gamma$-name for a real such that $\bigcup_{i\in\omega} \mathcal{A}_i(\dot{f}) \subset \mathcal{D}$, $f = \dot{f}^{G_\gamma}$ and let p_i be the unique element of $\mathcal{A}_i(\dot{f}) \cap G_\gamma$. Set $u_\xi = \dot{u}_\xi^{G_\gamma}$ for all $\xi < \gamma$. Since $\mathbb{P}_0$ is countably distributive, there exists $q \in \mathbb{P}_0 \cap G_\gamma$ such that $q \leq p_i(0)$ for all $i \in \omega$.

Observe that $\langle i, j\rangle \in f$ if and only if there exists $p \in \mathcal{A}_i(\dot{f})$ such that $p(0) \geq q$ and for every α in the support of p the following holds:

If $p \restriction \alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be an almost disjoint coding, i.e. $\alpha = \omega_2 \cdot \alpha' + i(\beta_0, \beta_1)$ for some $\alpha' > 0$ and $\beta_0 < \beta_1 < o.t.(\dot{<}^{G_\gamma}_{\omega_2\cdot\alpha'})$ and $\mathbb{Q}_\alpha$ produces a real coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x_{\beta_0} * x_{\beta_1})$, where x_δ is the δ-th real in $L[\langle u_\xi : \xi < \omega_2 \cdot \alpha'\rangle]$, ***then*** $p(\alpha)_0$ is an initial segment of u_α and $u_\alpha \setminus p(\alpha)_0$ is disjoint from $B_{\zeta,m}$ for all $\langle \zeta, m\rangle \in p(\alpha)_1$; and

[3] $u_\alpha \in [\omega]^\omega$ in the first case and $u_\alpha \in \omega^\omega$ in the second case.

If $p \restriction \alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be a poset adding a dominating function, i.e. $\mathbb{Q}_\alpha$ produces a real u_α dominating all reals in $L[\langle u_\xi : \xi < \alpha\rangle]$, ***then*** $p(\alpha)_0$ is an initial segment of u_α and $u_\alpha(n) > x_\xi(n)$ for all $\xi \in p(\alpha)_1$ and $n \geq \mathrm{dom}(p(\alpha)_0)$, where x_ξ is the ξ-th real in $L[\langle u_\zeta : \zeta < \alpha\rangle]$ according to the wellorder $\dot{<}_\alpha^{G_\gamma}$.

Since $\dot{<}_\beta^{G_\gamma}$ depends only on the sequence $\langle u_\zeta : \zeta < \beta\rangle$ for all $\beta < \gamma$, the definition of f above implies that $f \in L[\langle u_\zeta : \zeta < \gamma\rangle]$, which finishes our proof. □

Lemma 3. Let G be a $\mathbb{P}$-generic filter over L. Then for $\xi \in \bigcup_{\alpha\in Lim(\omega_3)} \dot{A}_\alpha^G$ there is no real coding a stationary kill of S_ξ.

Proof. Let $p \in G$ be a condition forcing

$$\xi \in \bigcup_{\alpha\in Lim(\omega_3)} \dot{A}_\alpha^G.$$

Suppose that $\xi = \beta + 2n - 1$ for some limit β and $n \in \omega$. Without loss of generality, $p \in \mathbb{P}_\beta \cap \mathcal{D}$.

We define a finite support iteration of a countably distributive poset followed by c.c.c. posets $\langle \bar{\mathbb{P}}_\alpha, \dot{\bar{\mathbb{Q}}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3\rangle$, where $\bar{\mathbb{P}}_0 = \mathbb{P}_0 \restriction p(0)$ and in $L^{\bar{\mathbb{P}}_\alpha}$ we have $\bar{\mathbb{Q}}_\alpha = \mathbb{Q}_\alpha \restriction p(\alpha)$. Such an iteration is just another way of thinking of the poset $\mathbb{P} \restriction p$ which will appear useful for further considerations.

Let $p_0^{\neq\xi}, p_0^\xi$ be such that $p_0^{\neq\xi} \in \tilde{\mathbb{P}}_0^{\neq\xi}$, $p_0^{\neq\xi} \Vdash p_0^\xi \in \tilde{\mathbb{R}}$ and $\langle p_0^{\neq\xi}, p_0^\xi\rangle = p(0)$, where $\tilde{\mathbb{R}}$ is the quotient poset $\mathbb{P}_0/\tilde{\mathbb{P}}_0^{\neq\xi}$. Denote by $\mathbb{P}_0^{\neq\xi}$ the restriction $\tilde{\mathbb{P}}_0^{\neq\xi} \restriction p_0^{\neq\xi}$ and let $\mathbb{R}$ be the $\mathbb{P}_0^{\neq\xi}$-name for $\tilde{\mathbb{R}} \restriction p_0^\xi$. Note that $\mathbb{P}_0^{\neq\xi} * \mathbb{R} = \bar{\mathbb{P}}_0$[4].

Now we define a finite support iteration $\langle \mathbb{P}_\alpha^{\neq\xi}, \dot{\mathbb{Q}}_\gamma^{\neq\xi} : \alpha \leq \omega_3, \gamma < \omega_3\rangle$, where $\mathbb{P}_0^{\neq\xi}$ is as above and $\dot{\mathbb{Q}}_\gamma^{\neq\xi}$ is a name for a σ-centered poset. Also we define a sequence $\langle \dot{A}_\alpha^{\neq\xi} : \alpha \in Lim(\omega_3)\rangle$, where $\dot{A}_\alpha^{\neq\xi}$ is a $\mathbb{P}_\alpha^{\neq\xi}$-name for a subset of $[\alpha, \alpha + \omega)$. The intention is to show that in $\bar{\mathbb{P}} = \bar{\mathbb{P}}_{\omega_3}$ the components $\mathbb{P}_\xi^0$, $\mathbb{P}_\xi^1$, $\mathbb{P}_\xi^2$ of $\mathbb{P}^0$, $\mathbb{P}^1$, $\mathbb{P}^2$, respectively, can be left out in a certain sense. Thus the iteration $\langle \mathbb{P}_\alpha^{\neq\xi}, \dot{\mathbb{Q}}_\gamma^{\neq\xi} : \alpha \leq \omega_3, \gamma < \omega_3\rangle$ will be introduced along the lines of the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3\rangle$. In particular, every $\mathbb{Q}_\alpha^{\neq\xi}$ will add a generic real with $\mathbb{P}_\alpha^{\neq\xi} * \mathbb{Q}_\alpha^{\neq\xi}$-name $\dot{u}_\alpha^{\neq\xi}$. Given a $\mathbb{P}_\alpha^{\neq\xi}$-generic filter $G = G_\alpha^{\neq\xi}$, this gives us a canonical wellorder of the reals in $L[\langle \dot{u}_\zeta^{\neq\xi^G} : \zeta < \alpha\rangle]$ which depends only on the sequence $\langle \dot{u}_\zeta^{\neq\xi^G} : \zeta < \alpha\rangle$, whose $\mathbb{P}_\alpha^{\neq\xi}$-name will be denoted by $\dot{<}_\alpha^{\neq\xi}$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta^{\neq\xi}}$ forces $\dot{<}_\alpha^{\neq\xi}$ to be an initial segment of $\dot{<}_\beta^{\neq\xi}$. Along the recursive construction for every $\gamma < \omega_3$ we will establish the following properties:

1. $\mathbb{P}_\gamma^{\neq\xi} <_c \bar{\mathbb{P}}_\gamma$;

---

[4] In fact, one can prove that $\Vdash_{\tilde{\mathbb{P}}_0^{\neq\xi}} \tilde{\mathbb{R}} = \mathbb{P}_0^\xi * \mathbb{P}_1^\xi * \mathbb{P}_2^\xi$, but this does not simplify the proof.

2. $\dot{u}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}} = \dot{u}_\gamma^{H_\gamma}$, $\dot{<}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}} = \dot{<}_\gamma^{H_\gamma}$ and $\dot{A}_\gamma^{H_\gamma} = \dot{A}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}}$ for limit γ, where $H_\gamma^{\neq\xi} \subseteq \mathbb{P}_\gamma^{\neq\xi}$ is the preimage of the $\bar{\mathbb{P}}_\gamma$-generic filter H_γ under the complete embedding from (1);
3. Let $\mathbb{P}^{\neq\xi}_{[1,\gamma)}$, $\bar{\mathbb{P}}_{[1,\gamma)}$ be the quotient posets $\mathbb{P}^{\neq\xi}_\gamma/\mathbb{P}^{\neq\xi}_0$ and $\bar{\mathbb{P}}_\gamma/\bar{\mathbb{P}}_0$ respectively. Then $\Vdash_{\bar{\mathbb{P}}_0} \mathbb{P}^{\neq\xi}_{[1,\gamma)} = \bar{\mathbb{P}}_{[1,\gamma)}$; and
4. $L[H_\gamma] \cap [\mathrm{Ord}]^\omega = L[H^{\neq\xi}_\gamma] \cap [\mathrm{Ord}]^\omega$ where H_γ, $H^{\neq\xi}_\gamma$ are as in (2).

For $\gamma = 0$ the properties above follow from the corresponding definitions. Suppose that (1)-(4) are established for all $\eta < \gamma$.

Case 1. If γ is a limit, there is nothing to prove except for (4) (To see that $\mathbb{P}^{\neq\xi}_\gamma$ is completely embedded in $\bar{\mathbb{P}}_\gamma$ refer to the inductive hypothesis and [2, Lemma 10]). Let $H^{\neq\xi}_0 = H^{\neq\xi}_\gamma \cap \mathbb{P}^{\neq\xi}_0$, $H_0 = H_\gamma \cap \mathbb{P}_0$ and let K be an $\mathbb{R}$-generic filter over $L[H^{\neq\xi}_0]$ such that $L[H_0] = L[H^{\neq\xi}_0][K]$. Let $\mathbb{E}$ be the poset $(\mathbb{P}^{\neq\xi}_{[1,\gamma)})^{H^{\neq\xi}_0} = \bar{\mathbb{P}}^{H_0}_{[1,\gamma)} \in L[H^{\neq\xi}_0]$ (the latter equality follows from (3)). Then $H_{[1,\gamma)}(= H_\gamma/H_0)$ is $\mathbb{E}$-generic over $L[H^{\neq\xi}_0][K]$. Therefore $L[H^{\neq\xi}_0][K][H_{[1,\gamma)}] = L[H^{\neq\xi}_0][H_{[1,\gamma)}][K]$.

The following standard fact may be compared to [9, Lemma 15.19].

Claim. Suppose that $\mathbb{P}, \mathbb{Q}$ are in V, $\mathbb{P}$ is ω-distributive and $\mathbb{Q}$ is c.c.c. in $V^{\mathbb{P}}$. Then $\mathbb{P}$ is ω-distributive in $V^{\mathbb{Q}}$. In particular, if $\mathbb{P}$ is ω-distributive and $\mathbb{Q}$ is a finite support iteration of σ-centered posets, then $\mathbb{P}$ is ω-distributive in $V^{\mathbb{Q}}$.

Proof. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic. Let $f : \omega \to \mathrm{Ord}$ be in $V[H][G] = V[G][H]$ and σ be a $\mathbb{Q}$-name for f in $V[G]$. Without loss of generality, σ is a nice name which can be written as $\bigcup_{i\in\omega}\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i\}$, where j^i_p is an ordinal and $\mathcal{A}_i \in V[G]$ is a maximal antichain in $\mathbb{Q}$. As $\mathbb{Q}$ is c.c.c. in $V[G]$, each $\mathcal{A}_i$ is countable in $V[G]$, and hence σ is countable in $V[G]$. Therefore $\sigma \in V$ by the countable distributivity of $\mathbb{P}$. It follows that f belongs to $V[H]$. □

By the above Claim, $\mathbb{R}$ is countably distributive in $L[H^{\neq\xi}_0][H_{[1,\gamma)}] = L[H^{\neq\xi}_\gamma]$ and hence $L[H_\gamma] \cap [\mathrm{Ord}]^\omega = L[H^{\neq\xi}_\gamma] \cap [\mathrm{Ord}]^\omega$.

Case 2). $\gamma = \eta + 1$.

Let $H^{\neq\xi}_\eta$ be a $\mathbb{P}^{\neq\xi}_\eta$-generic filter over L and let K be a $\mathbb{R}$-generic filter over $L[H^{\neq\xi}_0]$, where $H^{\neq\xi}_0 = H^{\neq\xi}_\eta \cap \mathbb{P}^{\neq\xi}_0$. In $L[H^{\neq\xi}_0]$, the quotient poset $\mathbb{P}_{[1,\eta)} = \mathbb{P}_\eta/\mathbb{P}_0$ is a finite support iteration of σ-centered posets. Since $\mathbb{P}^{\neq\xi}_{[1,\eta)}$ has c.c.c. in $L[H^{\neq\xi}_0][K]$ and $\mathbb{R}$ is ω-distributive, $H^{\neq\xi}_{[1,\eta)}$ is $\mathbb{P}^{\neq\xi}_{[1,\eta)}$-generic over $L[H^{\neq\xi}_0][K]$. By (3), the equality $\mathbb{P}^{\neq\xi}_{[1,\eta)} = \bar{\mathbb{P}}_{[1,\eta)}$ holds in $L[H^{\neq\xi}_0][K]$. Therefore $H_\eta := H^{\neq\xi}_0 * K * H^{\neq\xi}_{[1,\eta)}$ is $\bar{\mathbb{P}}_\eta$-generic over L.

Since $p \in \mathcal{D}$, one of the following alternatives holds.

Case a). $\dot{\mathbb{Q}}_\eta$ is a name for an almost disjoint coding below the condition $p(\eta) = \langle s_0^\eta, \check{s}_1^\eta\rangle$. Set $\bar{\mathbb{Q}}_\eta = \dot{\mathbb{Q}}_\eta^{H_\eta}$, $u_\delta = \dot{u}_\delta^{H_\eta}$, $A_\delta = \dot{A}_\delta^{H_\eta}$, and $<_\delta = \dot{<}_\delta^{H_\eta}$ for all $\delta \le \eta$.

It follows that:

• η is a limit ordinal that can be written in the form $\eta = \omega_2 \cdot \nu + \zeta$, where $\zeta = i(\zeta_0, \zeta_1)$ for some $\zeta_0, \zeta_1 < o.t.(<_{\omega_2\cdot\nu}^{H_\eta})$ and $i = i_{o.t.(<_{\omega_2\cdot\nu}^{H_\eta})}$;

• $A_\eta = \eta + (\omega \setminus \Delta(x_{\zeta_0} * x_{\zeta_1}))$, where x_ϵ is the ϵ-th real in $L[\langle u_\delta : \delta < \omega_2 \cdot \nu] \cap \omega^\omega$ according to the natural wellorder $<_{\omega_2\cdot\nu}^{H_\eta}$ of this set;

• $\bar{\mathbb{Q}}_\eta = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega},\ s_1 \in \big[\bigcup_{m\in\Delta(x_{\zeta_0} * x_{\zeta_1})} Y_{\eta+m} \times \{m\}\big]^{<\omega},\ s_0$ end-extends s_0^η, $s_1 \supseteq s_1^\eta$ and $s_0 \setminus s_0^\eta \cap B_{\epsilon,m} = \emptyset$ for all $\langle \epsilon, m\rangle \in s_1^\eta\}$ ordered as before.

Our choice of p and the fact that the upwards closure of H_η in $\mathbb{P}_\eta$ is a $\mathbb{P}_\eta$-generic filter containing p imply that Y_ξ is not among the $Y_{\eta+m}$'s involved into the definition of $\bar{\mathbb{Q}}_\eta$. Thus $\bar{\mathbb{Q}}_\eta \in L[H_\eta^{\neq\xi}]$. Moreover, $\bar{\mathbb{Q}}_\eta$ is fully determined by the relevant $Y_{\eta+m}$'s and the sequence $\langle u_\delta : \delta < \eta\rangle$ which belongs to $L[H_\eta^{\neq\xi}]$ and does not depend on K by (2). Therefore $\bar{\mathbb{Q}}_\eta$ does not depend on K and hence we may set $\mathbb{Q}_\eta^{\neq\xi} := \bar{\mathbb{Q}}_\eta$, $A_\eta^{\neq\xi} := A_\eta$. Let $\dot{\mathbb{Q}}_\eta^{\neq\xi}$, $\dot{A}_\eta^{\neq\xi}$ be $\mathbb{P}_\eta^{\neq\xi}$-names for $\mathbb{Q}_\eta^{\neq\xi}$ and $A_\eta^{\neq\xi}$ respectively. By the definition, (3) and the third part of (2) hold true.

The equality $L[H_\eta] \cap [\mathrm{Ord}]^\omega = L[H_\eta^{\neq\xi}] \cap [\mathrm{Ord}]^\omega$ and the σ-centeredness of $\bar{\mathbb{Q}}_\eta$ imply that any $\mathbb{Q}_\eta^{\neq\xi}$-generic over $L[H_\eta^{\neq\xi}]$ is $\bar{\mathbb{Q}}_\eta$-generic over $L[H_\eta]$ and vice versa. Therefore $\mathbb{P}_{\eta+1}^{\neq\xi} <_c \bar{\mathbb{P}}_{\eta+1}$ (note that H_η may be thought of as being an arbitrary $\bar{\mathbb{P}}_\eta$-generic filter over L). This establishes (1).

Let h_η be a $\mathbb{Q}_\eta^{\neq\xi}$-generic over $L[H_\eta^{\neq\xi}]$ (or, equivalently, $\bar{\mathbb{Q}}_\eta$-generic filter over $L[H_\eta]$). Since a (nice) $\bar{\mathbb{Q}}_\eta$-name for a countable set of ordinals in $L[H_\eta]$ can be naturally identified with a countable set of ordinals, every $\bar{\mathbb{Q}}_\eta$-name $\sigma \in L[H_\eta]$ for a countable set of ordinals is in fact in $L[H_\eta^{\neq\xi}]$. Therefore $L[H_{\eta+1}] \cap [\mathrm{Ord}]^\omega = L[H_{\eta+1}^{\neq\xi}] \cap [\mathrm{Ord}]^\omega$, where $H_{\eta+1} = H_\eta * h_\eta$. This proves (4).

Let us denote by $u_\eta^{\neq\xi} \in [\omega]^\omega \cap L[H_{\eta+1}^{\neq\xi}]$ the union of the first coordinates of elements of h_η. By the maximality principle, this gives us a $\mathbb{P}_{\eta+1}^{\neq\xi}$-name $\dot{u}_\eta^{\neq\xi}$. By the definitions of $\dot{u}_\eta$ and $\dot{u}_\eta^{\neq\xi}$, $\dot{u}_\eta^{H_\eta * h_\eta} = \dot{u}_\eta^{\neq\xi H_\eta^{\neq\xi} * h_\eta}$, which proves the first part of (2). By (4) and Lemma 2,

$$\begin{aligned} L[H_\eta^{\neq\xi} * h_\eta] \cap [\omega]^\omega &= (L[H_\eta^{\neq\xi} * h_\eta] \cap [\mathrm{Ord}]^\omega) \cap [\omega]^\omega = \\ &= (L[H_\eta * h_\eta] \cap [\mathrm{Ord}]^\omega) \cap [\omega]^\omega = L[H_\eta * h_\eta] \cap [\omega]^\omega = \\ = L[\langle \dot{u}_\delta^{H_\eta * h_\eta} : \delta \le \eta\rangle] \cap [\omega]^\omega &= L[\langle \dot{u}_\delta^{\neq\xi H_\eta^{\neq\xi} * h_\eta} : \delta \le \eta\rangle] \cap [\omega]^\omega, \end{aligned}$$

which implies the second equality in (2) and thus concludes *Case a).*

Case b). $\dot{\mathbb{Q}}_\eta$ is a name for a poset adjoining a dominating function restricted to the condition $p(\eta) = \langle s_0^\eta, \check{s}_1^\eta\rangle$. This case is analogous to, but easier than the *Case a)* (here we do not have to worry about Y_ξ) and we leave it to the reader.

This finishes our construction of $\langle \mathbb{P}^{\neq\xi}_\alpha, \dot{\mathbb{Q}}^{\neq\xi}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$. Observe that conditions (1)-(4) hold for $\gamma = \omega_3$. In particular, $L[G] \cap \omega^\omega = L[G^{\neq\xi}] \cap \omega^\omega$, where $G^{\neq\xi} \subset \mathbb{P}^{\neq\xi}_{\omega_3}$ is the preimage of the $\bar{\mathbb{P}}_{\omega_3}$-generic filter G under the complete embedding from (1). So it is sufficient to show that in $L[G^{\neq\xi}]$ there is no real coding a closed unbounded subset disjoint from S_ξ. Since $\mathbb{P}^{\neq\xi}_{[1,\omega_3)}$ is a $\mathbb{P}^{\neq\xi}_0$-name for a c.c.c poset and $\mathbb{P}^{2,\neq\xi}, \mathbb{P}^{1,\neq\xi}$ are $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi}, \mathbb{P}^{0,\neq\xi}$-names for ω_2-c.c. posets, respectively, every closed unbounded subset of ω_2 in $L[G^{\neq\xi}]$ contains a closed unbounded subset of ω_2 in $L[G^{0,\neq\xi}]$, see [9, Lemma 22.25]. (Here $G^{0,\neq\xi} = G^{\neq\xi} \cap \mathbb{P}^{0,\neq\xi}$ is the $\mathbb{P}^{0,\neq\xi}$-generic filter over L induced by $G^{\neq\xi}$). Thus it suffices to verify that S_ξ is stationary in $L^{\mathbb{P}^{0,\neq\xi}}$. We shall use here an idea from [6].

Fix $p \in \mathbb{P}^{0,\neq\xi}$ and let $\dot{C}$ be a name for a club in ω_2. We would like to find $q \in \mathbb{P}^{0,\neq\xi}$ such that $q \leq p$ and $q \Vdash_{\mathbb{P}^{0,\neq\xi}} \dot{C} \cap S_\xi \neq \emptyset$. Let $\langle \mathcal{M}_i : i < \omega_2 \rangle$ be a continuous chain of elementary submodels of some large L_θ such that $\mathcal{M}_0$ contains $p, \alpha, \dot{C}$, $\omega_1 + 1 \subset \mathcal{M}_0$, $\gamma_i := \mathcal{M}_i \cap \omega_2 \in \omega_2$, $\mathrm{cof}(\gamma_i) = \omega_1$, and $\mathcal{M}_i^{<\omega_1} \subset \mathcal{M}_i$ for all $i \in \omega_2$. Set $S^0_\xi = \{i \in S_\xi : \gamma_i = i\}$ and note that S^0_ξ is stationary.

Claim. There exists $i \in S^0_\xi$ such that $i \notin S_\alpha$ for all $\alpha \in \mathcal{M}_i \setminus \{\xi\}$.

Proof. Note that $\alpha \in \mathcal{M}_i$ is equivalent to $\alpha < \gamma_i$, and hence to $\alpha < i$ since $i \in S^0_\xi$. Suppose that for every $i \in S^0_\xi$ there exists $f(i) < i$ such that $i \in S_{f(i)}$ and $f(i) \neq \xi$. By Fodor's Lemma there exists $j \in \omega_2$ and a stationary $T \subset S^0_\xi$ such that $f(i) \equiv j$ for all $i \in T$. It follows that $T \subset S_j$, and hence $T \subset S_j \cap S_\xi$, a contradiction. □

Choose i as in the Claim above. We shall build an ω_1-sequence $p = p_0 \geq p_1 \geq \cdots$ with a lower bound forcing $i \in \dot{C}$. Let $\langle i_\alpha : \alpha < \omega_1 \rangle$ be an increasing continuous sequence of ordinals such that $\sup_{\alpha\in\omega_1} i_\alpha = i$. Given p_α, let $p_{\alpha+1} \leq p_\alpha$ be such a condition in $\mathbb{P}^{0,\neq\xi} \cap \mathcal{M}_i$ such that $p_{\alpha+1}$ forces some ordinal $j_{\alpha+1} \in [i_{\alpha+1}, i)$ to belong to $\dot{C}$. For limit α and $\zeta \in i \setminus \{\xi\}$ set

$$p_\alpha(\zeta) = \bigcup_{\beta<\alpha} p_\beta(\zeta) \cup \{\sup \bigcup_{\beta<\alpha} p_\beta(\zeta),\ i_\alpha\}.$$

Since S_ζ's consist of ordinals of cofinality ω_1 and $\mathcal{M}_i$ is closed under countable sequences of its elements, $p_\alpha \in \mathbb{P}^{0,\neq\xi} \cap \mathcal{M}_i$. This finishes our construction of the sequences $\langle p_\alpha : \alpha < \omega_1 \rangle \in \mathcal{M}_i^{\omega_1}$ and $\langle j_\alpha : \alpha < \omega_1 \rangle \in i^{\omega_1}$. Set $q(\zeta) = \bigcup_{\alpha\in\omega_1} p_\alpha(\zeta) \cup \{i\}$ for all $\zeta \in i \setminus \xi$. Since $i \notin S_\zeta$ for all $\zeta \in i \setminus \{\xi\}$, we conclude that $q(\zeta) \cap S_\zeta = \emptyset$ for all $\zeta \in i \setminus \{\xi\}$. From the above it follows that $q \in \mathbb{P}^{0,\neq\xi}$ and $q \Vdash_{\mathbb{P}^{0,\neq\xi}} i \in \dot{C}$, which finishes our proof. □

Corollary 1. Let G be a $\mathbb{P}$-generic filter over L and let x, y be reals in $L[G]$. Then $x <^G y$ if and only if there is $\alpha < \omega_3$ such that for all m, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$.

Proof. Suppose that $x <^G y$. Let $\alpha' > 0$ be minimal such that $x, y \in L[G_{\omega_2\cdot\alpha'}]$ and let $i = i_{o.t.(\dot{<}^G_{\omega_2\cdot\alpha'})}$. Find $\xi \in Lim(\omega_2)$ such that $i(\xi) = (\xi_x, \xi_y)$ where x and y are the ξ_x-th and ξ_y-th real respectively in $L[G_{\omega_2\cdot\alpha'}]$ according to the wellorder $\dot{<}^G_{\omega_2\cdot\alpha'}$. (By Lemma 2 such a ξ exists). Let $\alpha = \omega_2 \cdot \alpha' + \xi$. Then $\mathbb{Q}_\alpha$ adds a real coding a stationary kill for $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. On the other hand if $m \notin \Delta(x * y)$, then $\alpha + m \in \dot{A}^G_\alpha = \alpha + (\omega \backslash \Delta(x * y))$ and so by Lemma 3, there is no real in $L[G]$ coding the stationary kill of $S_{\alpha+m}$.

Now suppose that there exists α such that the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$. Since the stationary kill of some $\alpha + m$'s is coded by a real in $L[G]$, Lemma 3 implies that $\dot{\mathbb{Q}}^G_\alpha$ introduced a real coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}^G_\alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a*b)$. Therefore $\Delta(a * b) = \Delta(x * y)$ and hence $a = x$ and $b = y$, and consequently $x \dot{<}^G_\alpha y$. □

Lemma 4. Let G be $\mathbb{P}$-generic over L and let x, y be reals in $L[G]$. If $x <^G y$, then there is a real r such that for every countable suitable model $\mathcal{M}$ such that $r \in \mathcal{M}$, there is $\bar{\bar{\alpha}} < \omega_3^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$,

$$(L[r])^{\mathcal{M}} \models S_{\bar{\bar{\alpha}}+m} \text{ is not stationary.}$$

Proof. By Corollary 1, there exists $\alpha < \omega_3$ such that $\dot{\mathbb{Q}}^G_\alpha$ adds a real r coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. Let $\mathcal{M}$ be a countable suitable model containing r. It follows that $Y_{\alpha+m} \cap \omega_1^{\mathcal{M}} \in \mathcal{M}$ and hence $X_\alpha \cap \omega_1^{\mathcal{M}}$, $X_{\alpha+m} \cap \omega_1^{\mathcal{M}}$ also belong to $\mathcal{M}$. Observe that these sets are actually in $\mathcal{N} := (L[r])^{\mathcal{M}}$. Note also that $\mathcal{N}$ is a countable suitable model and consequently by the definition of $\mathcal{L}(X_{\alpha+m}, X_\alpha)$ we have that for every $m \in \Delta(x * y)$, $\mathcal{N} \models$

> " Using the sequence $\vec{A}$, $X_{\alpha+m} \cap \omega_1$ (resp. $X_\alpha \cap \omega_1$) almost disjointly codes a subset $\bar{Z}_m$ (resp. $\tilde{Z}_0$) of ω_2, whose even part $Even(\bar{Z}_m)$ (resp. $Even(\tilde{Z}_0)$) codes a tuple $\langle \bar{C}, \bar{W}_m, \bar{\bar{W}}_m \rangle$ (resp. $\langle \tilde{C}, \tilde{W}_0, \tilde{\tilde{W}}_0 \rangle$), where $\bar{W}_m$ and $\bar{\bar{W}}_m$ are the L-least codes of ordinals $\bar{\alpha}_m, \bar{\bar{\alpha}}_m < \omega_3$ (resp. $\tilde{W}_0 = \tilde{\tilde{W}}_0$ is the L-least code for a limit ordinal $\tilde{\tilde{\alpha}}_0$) such that $\bar{\bar{\alpha}}_m = \tilde{\tilde{\alpha}}_0$ is the largest limit ordinal not exceeding $\bar{\alpha}_m$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}_m}$.[5]"

Note that in particular for every $m \neq m'$ in $\Delta(x * y)$, $\bar{\bar{\alpha}}_m = \bar{\bar{\alpha}}_{m'}$. □

Lemma 5. Let G be $\mathbb{P}$-generic over L and let x, y be reals in $L[G]$. If there is a real r such that for every countable suitable model $\mathcal{M}$ containing r as an element, there is $\bar{\bar{\alpha}} < \omega_3^{\mathcal{M}}$ such that for every $m \in \Delta(x * y)$,

$$(L[r])^{\mathcal{M}} \models S_{\bar{\bar{\alpha}}+m} \text{ is not stationary,}$$

[5] In the above, $\vec{A}$, $S_{\bar{\alpha}_m}$, $S_{\bar{\bar{\alpha}}_m}$, ω_1, ω_2, ω_3 refer of course to their interpretations in the model $\mathcal{N}$.

then $x <^G y$.

Proof. Suppose that there is such a real r. By the Löwenheim-Skolem theorem, it has the property described in the formulation with respect to *all* suitable models $\mathcal{M}$, in particular for $\mathbb{H}_\Theta^{\mathbb{P}}$, where Θ is sufficiently large (here $\mathbb{H}_\Theta$ denotes the set of all sets hereditarily of cardinality $< \Theta$). That is there is $\alpha < \omega_3$ such that for every $m \in \Delta(x * y)$

$$L_\Theta[r] \models S_{\alpha+m} \text{ is not stationary.}$$

Thus in particular the stationary kill of at least some $S_{\alpha+m}$ was coded by a real. Lemma 3 implies that $\dot{\mathbb{Q}}_\alpha^G$ introduced a real u_α (perhaps different from r) coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}_\alpha^G b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a*b)$. Therefore $\Delta(a*b) \supset \Delta(x*y)$, which yields $\Delta(a*b) = \Delta(x*y)$. From the above, it follows that $a = x$, $b = y$ and hence $x \dot{<}_\alpha^G y$, which finishes our proof. □

Combining Lemmata 4,5 and the fact that we have added dominating reals cofinally often, we get the following result.

Theorem 1. It is consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_3$, that there is a projective (indeed Δ_3^1-definable) wellorder of the reals.

3. Projective mad families

The main result of this section and of the whole paper is the following theorem which answers [7, Question 19] in the positive.

Theorem 2. It is consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_3$, that there is a Δ_3^1-definable wellorder of the reals and a Π_2^1-definable ω-mad subfamily of $[\omega]^\omega$ (resp. ω^ω).

The proof is completely analogous to that of Theorem 2. Moreover, we believe that adding the argument responsible for ω-mad families would just make the proof in the previous section messier without introducing any new ideas besides those used in the proof of Theorem 1 and in [7]. Therefore the proof of Theorem 2 is just sketched here. More precisely, we shall define the corresponding poset $\mathbb{P}_{\omega_3}$ and leave it to the reader to verify that the proof of Theorem 1 can be carried over.

Let $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega \rangle$ be as in the proof of Theorem 1. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset and in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ_3^1-definable wellorder of the reals, a Π_2^1-definable ω-mad subfamily of $[\omega]^\omega$ (the case of subfamilies of ω^ω is completely analogous, see [7]), and $\mathfrak{c} = \mathfrak{b} = \aleph_3$.

$\mathbb{P}_0$ is a three step iteration $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, where $\mathbb{P}^0$ and $\mathbb{P}^1$ are exactly the same as in the proof of Theorem 1. The poset $\mathbb{P}^2$ uses the following modification of Definition 1, where ϕ is as in $(**)_\alpha$ from the previous section.

Definition 2. Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. Let also η be a countable limit ordinal. We denote by $\mathcal{L}_\eta(X, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. $|r| \geq \eta$
2. if $\gamma < \eta$ then $r(\gamma) = 0$
3. if $\gamma < |r|$ then $\gamma \in X$ iff $r(\eta + 3\gamma) = 1$
4. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(\eta + 3\gamma + 1) = 1$
5. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$ holds in $\mathcal{M}$.

The extension relation is end-extension.

For $\alpha \in Lim(\omega_3) \backslash \omega_2$ and $m \in \omega$ set $\mathbb{P}^2_{\alpha+m} = \prod_{\eta \in Lim(\omega_1)} \mathcal{L}_\eta(X_{\alpha+m}, X_\alpha)$. If $\alpha \in Lim(\omega_2)$ and $m \in \omega$, let $\mathbb{P}^2_{\alpha+m}$ be the trivial poset. Then let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the Δ-system Lemma in $L^{\mathbb{P}^0 * \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the ω_2-c.c. Analogously to Lemma 1 we conclude that $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

If α is limit and $m \in \omega$, we shall refer to the localizing set for $X_{\alpha+m}$ produced by $\mathcal{L}_\eta(X_{\alpha+m}, X_\alpha)$ as $Y_{\alpha+m,\eta}$. That is $Y_{\alpha+m,\eta} \subseteq \omega_1 \setminus \eta$ and $Y_{\alpha+m,\eta}$ codes both $X_{\alpha+m}$ and X_α.

Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$ and similarly to the proof of Lemma 2 one can prove that $L[G_\alpha] \cap \omega^\omega = L[\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha \rangle] \cap \omega^\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α. This gives us a canonical wellorder of the reals in $L[G_\alpha]$ which depends only on the sequence $\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha \rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta}$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, $<^G = \bigcup \{ \dot{<}_\alpha^G : \alpha < \omega_3 \}$ will be the desired wellorder of the reals.

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along this construction we shall also define a sequence $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3) \rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha + \omega)$. Let $i : \omega \times \omega \to \omega$ and

$$j_\nu : \nu \cup \{ \langle \zeta, \xi \rangle : \zeta < \xi < \nu \} \to Lim(\omega_2)$$

be some bijections, where $\nu \in [\omega_2, \omega_3)$. Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Case 1. α is a limit ordinal that can be written in the form $\omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi < \omega_2$, and the preimage $j^{-1}(\xi)$ is a tuple $\langle \xi_0, \xi_1 \rangle$ for some $\xi_0 \dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'} \xi_1$, where $j = j_{o.t.(\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'})}$. In this case the definition of $\dot{\mathbb{Q}}_\alpha$ is the same as in the proof of Theorem 1.

Case 2. α is a limit ordinal that can be written in the form $\omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$ and the preimage $j^{-1}(\xi)$ is an ordinal $\zeta \in o.t.(\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'})$, where $j = j_{o.t.(\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'})}$. In this case we use a simplified version of the poset from [7, Theorem 1]. More precisely, ordinals fulfilling the condition above will be used for the construction of a Π^1_2 definable ω-mad family $\mathcal{A}$.

For a subset s of ω and $l \in |s|$ ($= \mathrm{card}(s) \leq \omega$) we denote by $s(l)$ the l-th element of s. In what follows we shall denote by $E(s)$ and $O(s)$ the sets $\{s(2i) : 2i \in |s|\}$ and $\{s(2i+1) : 2i+1 \in |s|\}$, respectively. Let $\mathcal{A}_\alpha$ be the approximation to $\mathcal{A}$ constructed thus far. Suppose also that

$$(*) \qquad \forall \mathcal{D} \in [\mathcal{A}_\alpha]^{<\omega}\ \forall B \in \vec{B}\ (|E(B) \setminus \cup\mathcal{D}| = |O(B) \setminus \cup\mathcal{D}| = \omega).$$

Observe that equation $(*)$ yields $|E(B) \setminus \cup\mathcal{D}| = |O(B) \setminus \cup\mathcal{D}| = \omega$ for every $\mathcal{D} \in [\vec{B} \cup \mathcal{A}_\alpha]^{<\omega}$ and $B \in \vec{B} \setminus \mathcal{D}$. Let x_ζ be the ζ-th real in $L[G_{\omega_2 \cdot \alpha'}] \cap [\omega]^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$. Set $C_n = \{x_\zeta(i(n,m)) : m \in \omega\} \in [\omega]^\omega$ and $C = \{C_n : n \in \omega\}$. Unless the following holds, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset: none of the C_n's is covered by a finite subfamily of $\mathcal{A}_\alpha$. In the latter case $\mathbb{Q}_\alpha := \dot{\mathbb{Q}}^{G_\alpha}_\alpha$ is defined as follows.

Let us fix a limit ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets $J, \mathcal{E}$ of $(\omega_1 \setminus \eta_\alpha) \times \omega$, $\mathcal{A}_\alpha$, respectively and $n \in \omega$, such that $C_n \subset \bigcup_{\langle \eta, m \rangle \in J} B_{\eta,m} \cup \bigcup \mathcal{E}$. (The almost disjointness of the $B_{\eta,m}$'s imply that if $C_n \subset \bigcup \mathcal{B}' \cup \bigcup \mathcal{A}'$ for some $\mathcal{B}' \in [\vec{B}]^{<\omega}$ and $\mathcal{A}' \in [\mathcal{A}_\alpha]^{<\omega}$, then $C_n \setminus \bigcup \mathcal{A}'$ has finite intersection with all elements of $\vec{B} \setminus \mathcal{B}'$. This easily yields the existence of such an η_α.) Let I_α be an infinite subset of ω coding a surjection from ω onto η_α. For a subset s of ω we denote by Δs the set $\{2k+1 : k \in (\sup s \setminus s)\} \cup \{2k+2 : k \in s\}$.

In $V[G_\alpha]$, $\mathbb{Q}_\alpha$ consists of pairs $\langle s, s^* \rangle$ such that $s \in [\omega]^{<\omega}$, $s^* \in \big[\{B_{\beta,m} : m \in \Delta(s), \beta \in Y_{\alpha+m,\eta_\alpha}\} \cup \mathcal{A}_\alpha\big]^{<\omega}$, and for every $2n \in |s \cap B_{0,0}|$, $n \in I_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap B_{0,0})(2n) = B_{0,0}(2m)$. For conditions $p = \langle s, s^* \rangle$ and $q = \langle t, t^* \rangle$ in $\mathbb{Q}_\alpha$, we let $q \leq p$ if and only if t is an end-extension of s and $t \setminus s$ has empty intersection with all elements of s^*.

Let h_α be a $\mathbb{Q}_\alpha$-generic filter over $L[G_\alpha]$. Set $u_\alpha = \bigcup_{\langle s, s^* \rangle \in h_\alpha} s$, $A_\alpha = \alpha + (\omega \setminus \Delta(u_\alpha))$, and $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{u_\alpha\}$. As a consequence of the definition of $\mathbb{Q}_\alpha$ and the genericity of

h_α we get[6]

(1) $u_\alpha \in [\omega]^\omega$, u_α is almost disjoint from all elements of $\mathcal{A}_\alpha$, and has infinite intersection with C_n for all $n \in \omega$;

(2) If $m \in \Delta(u_\alpha)$, then $|u_\alpha \cap B_{\beta,m}| < \omega$ if and only if $\beta \in Y_{\alpha+m,\eta_\alpha}$;

(3) For every $n \in \omega$, $n \in I_\alpha$ if and only if there exists $m \in \omega$ such that $(u_\alpha \cap B_{0,0})(2n) = B_{0,0}(2m)$; and

(4) Equation $(*)$ holds for $\alpha + 1$, i.e. for every $B \in \vec{B}$ and a finite subfamily $\mathcal{A}'$ of $\mathcal{A}_{\alpha+1}$, $\mathcal{A}'$ covers neither a cofinite part of $E(B)$ nor of $O(B)$.

By (2) u_α codes $Y_{\alpha+m,\eta_\alpha}$ for all $m \in \Delta(u_\alpha)$.

Case 3. If α is not of the form above, i.e. α is a successor or $\alpha < \omega_2$, then $\dot{A}_\alpha$ is a name for the empty set and $\dot{\mathbb{Q}}_\alpha$ is a name for the poset adding a dominating real defined in *Case 2* of the proof of Theorem 1.

With this the definitions of $\mathbb{P} = \mathbb{P}_{\omega_3}$ and $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3) \rangle$ are complete. Let G be a $\mathbb{P}$-generic over L.

Just as in the proof of Theorem 1 one can verify that Lemmata 2 and 3 hold true. These were of crucial importance for the proof of Corollary 1, which in turn was used in the proofs of Lemmata 4 and 5. Again, a direct verification shows that all of these statements still hold and hence $<^G$ is a Δ^1_3-wellorder of the reals in $L[G]$.

Lemma 2 implies that the family $\mathcal{A}$ we construct in the instances of *Case 2* is an ω-mad subfamily of $[\omega]^\omega$. Condition (3) above yields $\eta_\alpha < \omega_1^{\mathcal{M}}$ for all countable suitable models $\mathcal{M}$ containing $\dot{u}_\alpha^G$ provided that at stage α, *Case 2* took place (i.e., there is a condition in G which forces this). Combining this with the ideas of the proofs of Lemmata 4 and 5 we get that $a \in \mathcal{A}$ *iff for every countable suitable model* $\mathcal{M}$ *containing* a *as an element there exists* $\bar{\alpha} < \omega_3^{\mathcal{M}}$ *such that* $S^{\mathcal{M}}_{\bar{\alpha}+k}$ *is nonstationary in* $(L[a])^{\mathcal{M}}$ *for all* $k \in \Delta(a)$. This provides a Π^1_2 definition of $\mathcal{A}$, which finishes our proof of Theorem 2.

4. Questions

The consistency of the existence of a Δ^1_3-definable wellorder of the reals in the presence of $\mathfrak{c} \geq \aleph_3$ and MA, is still open. A second question naturally emerging from the developed techniques is the existence of a model in which a desired inequality betwen the cardinal characteristics of the real line holds, there is a Δ^1_3-definable wellorder of the

[6]See [7, Claim 11] for an analogous argument.

reals and $\mathfrak{c} \geq \aleph_3$. Note that the bookkeeping argument which we have used in Theorems 1 and 2 allows only for handling of countable objects, which presents an additional difficulty in obtaining such models.

References

[1] A. Blass *Combinatorial Cardinal Characteristics of the Continuum*, in: M. Foreman, A. Kanamori (Eds.), Handbook of Set Theory, Springer (2010), 395-489.

[2] J. Brendle, V. Fischer *Mad families, splitting families and large continuum*, Journal of Symbolic Logic, 76, 1 (2011), 198-208.

[3] J. Cummings *Iterated forcing and elementary embeddings*, in: M. Foreman, A. Kanamori (Eds.), Handbook of Set Theory, Springer (2010), 775-883.

[4] R. David *A very absolute* Π^1_1 *real singleton*, Ann. Math. Logic 23 (1982) 101-120.

[5] V. Fischer, S. D. Friedman *Cardinal characteristics and projective wellorders*, Annals of Pure and Applied Logic 161 (2010) 916-922.

[6] S. D. Friedman, *Lecture notes on definable wellorders*, http://www.logic.univie.ac.at/~sdf/

[7] S. D. Friedman, L. Zdomskyy *Projective Mad Families*, Annals of Pure and Applied Logic 161(2010), 1581-1587.

[8] L. Harrington *Long Projective Wellorderings,* Ann. Math. Logic 12 (1977), 1-24.

[9] T. Jech *Set Theory* Springer, 2002.