A Concise Analysis of Popper's Qualitative Theory of Verisimilitude

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1 Popper's Qualitative Definition of Verisimilitude

Popper [3] offers a qualitative definition of the relation "$p \prec q$" = "p is (strictly) closer to the truth than (*i.e.*, strictly more verisimilar than) q", using the notions of truth (in the actual world) and classical logical consequence ($\vDash$), as follows:

Definition. *$p \prec q$ if and only if the following three conditions are satisfied:*

① *For all r, if $q \vDash r$ and r is true, then $p \vDash r$. That is, p has all the true consequences that q has.*

② *For all r, if $p \vDash r$ and r is false, then $q \vDash r$. That is, q has all the false consequences that p has.*

③ *Either:*

③.1 *For some r, $p \vDash r$, r is true, and $q \nvDash r$. That is, p has some true consequences that q lacks.*
or

③.2 *For some r, $q \vDash r$, r is false, and $p \nvDash r$. That is, q has some false consequences that p lacks.*

In section 2, I will prove a Theorem which shows that Popper's somewhat cumbersome definition of $\prec$ reduces to a relatively simple disjunction. In section three, I will use this simplifying Theorem to provide straightforward proofs of some important and well-known Corollaries about $\prec$. Many useful facts about $\prec$ can be established quite easily with the help of our Theorem.

2 An Exhaustive Account of the Consequences of Popper's Definition of $\prec$

Theorem. *$p \prec q$* ***if and only if*** *either:*

(i) *$p \vDash q$, $q \nvDash p$, and p, q are both true.*
or

(ii) *p is true, q is false, and ① obtains (i.e., p has all the true consequences that q has).*

Proof. There are 16 cases (of which, only 12 are logically possible) to consider. Below, I will work through all 12 possible cases exhaustively. Before I do this, however, it will help to visualize the 16 cases, in a sort of "truth-table" structure. Table 1 provides both a handy summary of the results, and a perspicuous outline of the subsequent proofs.

	$p \vDash q$	$q \vDash p$	p	q	$p \prec q$
Case 1.	**T**	**T**	**T**	**T**	**F**
Case 2.†	**T**	**T**	**T**	**F**	**F**
Case 3.†	**T**	**T**	**F**	**T**	**F**
Case 4.*	**T**	**T**	**F**	**F**	**F**
Case 5 [(i)].	**T**	**F**	**T**	**T**	**T**
Case 6.†	**T**	**F**	**T**	**F**	**F**
Case 7.	**T**	**F**	**F**	**T**	**F**
Case 8.*	**T**	**F**	**F**	**F**	**F**
Case 9.	**F**	**T**	**T**	**T**	**F**
Case 10 [(ii)].	**F**	**T**	**T**	**F**	**T** *iff* ①
Case 11.†	**F**	**T**	**F**	**T**	**F**
Case 12.*	**F**	**T**	**F**	**F**	**F**
Case 13.	**F**	**F**	**T**	**T**	**F**
Case 14 [(ii)].	**F**	**F**	**T**	**F**	**T** *iff* ①
Case 15.	**F**	**F**	**F**	**T**	**F**
Case 16.*	**F**	**F**	**F**	**F**	**F**

Table 1: Visualizing the 16† cases.

*Both Miller [1] and Tichý [4] (famously) published proofs that $p \not\prec q$ in cases 4, 8, 12, and 16 (in which p and q are both false). The only non-trivial case of this kind is case 12 (as I show below, even this case is not very difficult to see). See Corollary 1, below, for a stronger result.

†Cases 2, 3, 6, and 11 are logically impossible, and so are ruled out *a priori*. This leaves 12 logically possible cases to analyze.

Now, we're ready for the analyses of the 12 logically possible cases.

Cases 1 and 4 ($p \vDash q$ and $q \vDash p$). In these two cases, p and q are *logically equivalent*. Hence, p and q have *exactly the same set of logical consequences*. Therefore, condition ③ will inevitably be violated in these cases. So, $p \nprec q$.

Case 5 ($p \vDash q$, $q \nvDash p$, p, q). In this case, we have: ① *All* consequences of q (including the true ones) are consequences of p, since $p \vDash q$, and $\vDash$ is transitive. Moreover, ② all false consequences of p are consequences of q. This is true *vacuously*, since p (being *true*) can have *no* false consequences. Finally, we have ③.1, since p has a true consequence (namely, p itself) that q does not have. Therefore, $p \prec q$. This is the *only* case [(i)] in which $p \prec q$ is *guaranteed* by Popper's definition alone.

Case 7 ($p \vDash q$, $q \nvDash p$, $\bar{p}$, q). In this case, p has a false consequence (namely, p itself) that q does not have. Therefore, $p \nprec q$.

Case 8 ($p \vDash q$, $q \nvDash p$, $\bar{p}$, $\bar{q}$). In this case, p has a false consequence (namely, p itself) that q does not have. Therefore, $p \nprec q$.

Case 9 ($p \nvDash q$, $q \vDash p$, p, q). In this case, q has a true consequence (namely, q itself) that p does not have. Therefore, $p \nprec q$.

Case 10 ($p \nvDash q$, $q \vDash p$, p, $\bar{q}$). In this case, we have ③.2, since q has a false consequence (namely, q itself) that p does not have. Moreover, ② all false consequences of p are consequences of q [*vacuously*, since p (being *true*) can have *no* false consequences]. However, we do *not* automatically have ① here. Here's a recipe for generating possible counterexamples to ①. Since q is false and strictly logically stronger than p, q will be equivalent to $p \,\&\, s$, for some s that is false and logically independent of p. Now, let $r = s \vee t$, where t is true, and $p \nvDash s \vee t$. This gives us a true consequence r of q that p doesn't have. Therefore, we will have $p \prec q$ in this case [(ii)] *if and only if* p happens to have *all* the true consequences that q has (*i.e.*, *iff* ① obtains).

Case 12 ($p \nvDash q$, $q \vDash p$, $\bar{p}$, $\bar{q}$). In this case, q has a true consequence (namely, $p \to q$, *i.e.*, $q \vee \bar{p}$) that p does not have. It is obvious that $q \vDash q \vee \bar{p}$. And, since p is false in this case, it follows that $p \to q$ is a *true* consequence of q in this case. To see that $p \nvDash p \to q$ in this case, remember that $p \nvDash q$. Thus, there is a world w in which p is true and q is false — *i.e.*, there is a world w in which p is true, and $p \to q$ is false. Therefore, $p \nprec q$ in this case.

Case 13 ($p \nvDash q$, $q \nvDash p$, p, q). In this case, q has a true consequence (namely, q itself) that p does not have. Therefore, $p \nprec q$.

Case 14 ($p \nvDash q$, $q \nvDash p$, p, $\bar{q}$). This case is similar to case 10, above. But, in this case, we also have ③.1, since p has a true consequence (p itself) that q does not have. To obtain a possible counterexample to ① here, let $r = q \vee t$ (where t is true, and $p \nvDash q \vee t$, as in case 10). Again, this would give us a true consequence r of q that p lacks. Therefore, we will have $p \prec q$ in this case [(ii)] *if and only if* p happens to have *all* the true consequences that q has (*i.e.*, *if and only if* ① obtains).

Case 15 ($p \nvDash q$, $q \nvDash p$, $\bar{p}$, q). In this case, q has a true consequence (namely, q itself) that p does not have. Therefore, $p \nprec q$.

Case 16 ($p \nvDash q$, $q \nvDash p$, $\bar{p}$, $\bar{q}$). In this case, p has a false consequence (namely, p itself) that q does not have. $\therefore p \nprec q$. ❑

3 Two Easy But Important Corollaries of Our Theorem

Corollary 1 (Miller–Tichý, *strengthened*). *If p is false, then $p \nprec q$, for* ***any*** *q.*

Proof. See cases 4, 7, 8, 12, 15, and 16, above. ❑

Corollary 2. *$\prec$ is a strict partial order. That is, $\prec$ is* (1) *asymmetric and* (2) *transitive.*

Proof. (1) Assume $p \prec q$ for some p, q. Then, by our Theorem, we must have either (i) or (ii). If (i), then $q \nprec p$, since p has a true consequence (p itself) that q does not have. If (ii), then $q \nprec p$, since q is false (Corollary 1). $\therefore p \prec q \Rightarrow q \nprec p$.

(2) Assume $p \prec q$ and $q \prec r$, for some p, q, r. Since $q \prec r$, q must be true (Corollary 1). Since q is true and $p \prec q$, it follows from our Theorem that p is true, $p \vDash q$, and $q \nvDash p$ [case (i)]. Now, either r is true or r is false. If r is true, then, since $q \prec r$, our Theorem implies that $q \vDash r$ and $r \nvDash q$ [case (i)]. In this case, $p \vDash r$ (by the transitivity of $\vDash$), and $r \nvDash p$ (since $q \vDash r$, but $q \nvDash p$). So, we have $p \prec r$, by case (i) of our our Theorem. Otherwise, if r is false, then, since $q \prec r$, our Theorem implies that q must have all the true consequences that r has [case (ii)]. In this case, p must also have all the true consequences that r has (since q does, and $p \vDash q$). So, we have $p \prec r$, by case (ii) of our Theorem. $\therefore p \prec q$ and $q \prec r \Rightarrow p \prec r$.[‡] ❑

References

[1] Miller, D. (1974) "Popper's Qualitative Theory of Verisimilitude", *British Journal for the Philosophy of Science*, *25*:166–167.

[2] Miller, D. (1994) *Critical Rationalism: A Restatement and Defence*, Open Court.

[3] Popper, K. (1963) *Conjectures and Refutations*, New York: Harper and Row; 2d ed., 1965.

[4] Tichý, P. (1974) "On Popper's Definitions of Verisimilitude", *British Journal for the Philosophy of Science*, *25*:155–160.

[‡] Miller [2, page 203] characterizes $\prec$ by saying that (i), (ii), and (2) are "all the [and the only] ways in which one theory can be more truthlike than another." We see here that Miller's characterization of $\prec$ is redundant, since (2) follows logically from (i) and (ii).