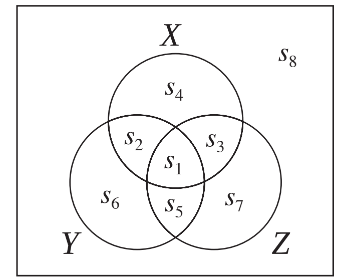
Overview of Finite Propositional Boolean Algebras I

- Consider a logical language $\mathcal{L}$ containing n atomic sentences. These may be sentence letters (X, Y, Z, *etc.*), or they may be atomic sentences of monadic or relational predicate calculus (Fa, Gb, Rab, Hcd, *etc.*).
- The Boolean Algebra $\mathcal{B}_{\mathcal{L}}$ set-up by such a language will be such that:
  - $\mathcal{B}_{\mathcal{L}}$ will have 2^n *states* (corresponding to the *state descriptions* of $\mathcal{L}$)
  - $\mathcal{B}_{\mathcal{L}}$ will contain 2^{2^n} *propositions*, in total.
    - ∗ This is because each proposition p in $\mathcal{B}_{\mathcal{L}}$ is equivalent to a disjunction of state descriptions. Thus, each subset of the set of state descriptions of $\mathcal{L}$ corresponds to a proposition of $\mathcal{B}_{\mathcal{L}}$.
    - ∗ Note: there are 2^{2^n} subsets of a set of size 2^n.
      - · The empty set $\varnothing$ of state descriptions corresponds to "the empty disjunction", which corresponds to *the logical falsehood*: $\bot$.
      - · Singelton sets of state descriptions correspond to "disjunctions with one member". [All other subsets are "normal" disjunctions.]

Overview of Finite Propositional Boolean Algebras II

- Example. Let $\mathcal{L}$ have three atomic sentences: X, Y, and Z. Then, $\mathcal{B}_{\mathcal{L}}$ is:

X	Y	Z	States
T	T	T	s_1
T	T	F	s_2
T	F	T	s_3
T	F	F	s_4
F	T	T	s_5
F	T	F	s_6
F	F	T	s_7
F	F	F	s_8



- Examples of reduction to disjunctions of state descriptions of $\mathcal{L}$:
  - '$X \,\&\, {\sim}X$' is equivalent to the *empty* disjunction: $\bot$.
  - '$X \,\&\, ({\sim}Y \,\&\, Z)$' is equivalent to the *singleton* disjunction: s_3.
  - '$X \equiv (Y \vee Z)$' is equivalent to: $s_1 \vee s_2 \vee s_3 \vee s_8$.
- In general: $p \dashv\vDash \bigvee\{s_i \mid s_i \vDash p\}$. And, if $\{s_i \mid s_i \vDash p\} = \varnothing$, then $p \dashv\vDash \bot$.

The Probability Calculus: An Algebraic Approach I

- Once we grasp the concept of a finite Boolean algebra of propositions, understanding the probability calculus *algebraically* is very easy.
- The central concept is a *finite probability model*. A finite probability model $\mathcal{M}$ is a finite Boolean algebra of propositions $\mathcal{B}$, together with a function $\Pr(\cdot)$ which maps elements of $\mathcal{B}$ to the unit interval $[0,1] \in \mathbb{R}$.
- This function $\Pr(\cdot)$ must be a *probability function*. It turns out that a probability function $\Pr(\cdot)$ on $\mathcal{B}$ is just a function that assigns a real number on $[0,1]$ to each state s_i of $\mathcal{B}$, such that $\sum_i \Pr(s_i) = 1$.
- Once we have $\Pr(\cdot)$'s *basic assignments* to the states of $\mathcal{B}$ (s.d.'s of $\mathcal{L}$), we define $\Pr(p)$ for *any* statement $\mathcal{L}$ of the language of $\mathcal{B}$, as follows:

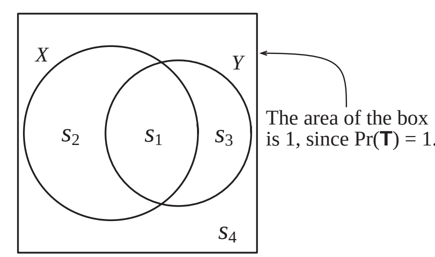
$$\Pr(p) = \sum_{s_i \vDash p} \Pr(s_i) \qquad \text{[note: if } p \dashv\vDash \bot\text{, then } \Pr(p) = 0]$$

- In other words, $\Pr(p)$ is the sum of the probabilities of the state descriptions in p's (equivalent) disjunction of state descriptions.

The Probability Calculus: An Algebraic Approach II

- Here's an example of a finite probability model $\mathcal{M}$, whose algebra $\mathcal{B}$ is characterized by a language $\mathcal{L}$ with two atomic letters "X" and "Y":

X	Y	States	$\Pr(s_i)$
T	T	s_1	$\frac{1}{6}$
T	F	s_2	$\frac{1}{4}$
F	T	s_3	$\frac{1}{8}$
F	F	s_4	$\frac{11}{24}$



- On the left, a *stochastic truth-table* (STT) representation of $\mathcal{M}$; on the right, a *stochastic Venn Diagram* (SVD) representation, in which *area is proportional to probability*. This is a *regular* model: $\Pr(s_i) > 0$, for all i.
- $\mathcal{M}$ determines a *numerical* probability for *each* p in $\mathcal{L}$. Examples?
- We can also use STTs to furnish an algebraic method for *proving **general** facts* about *all* probability models — *the algebraic method*.

The Probability Calculus: An Algebraic Approach III

- Let $a_i = \Pr(s_i)$ be the probability [under the probability assignment $\Pr(\cdot)$] of state s_i in $\mathcal{B}$ — *i.e.*, the area of region s_i in our SVD.
- Once we have real variables (a_i) for each of the basic probabilities, we can not only calculate probabilities relative to *specific* numerical models — *we can say* ***general*** *things, using only simple high-school algebra.*
- That is, we can *translate* any expression ⌜$\Pr(p)$⌝ into a *sum* of some of the a_i, and thus we can *reduce probabilistic* claims about the p's in $\mathcal{B}/\mathcal{L}$ into simple, high-school-*algebraic* claims about the real variables a_i.
- This allows us to be able to prove general claims about *probability functions*, by proving their corresponding *algebraic theorems*.
- Method: translate the probability claim into a claim involving sums of the a_i, and determine whether the corresponding claim is a theorem of algebra (assuming only that the a_i are on $[0, 1]$ and that they sum to 1).

The Probability Calculus: An Algebraic Approach IV

- Here are two simple/obvious examples involving two atomic sentences:

  Theorem. $\Pr(X \vee Y) = \Pr(X) + \Pr(Y) - \Pr(X \,\&\, Y)$.

  Proof. $\Pr(X \vee Y) = a_1 + a_2 + a_3 = (a_1 + a_2) + (a_1 + a_3) - a_1$.

  Theorem. $\Pr(X) = \Pr(X \,\&\, Y) + \Pr(X \,\&\, {\sim}Y)$.

  Proof. $a_1 + a_2 = a_1 + a_2$.

- Here are two general facts that are also obvious from the set-up:

  Theorem. If $p \dashv\vDash q$, then $\Pr(p) = \Pr(q)$.

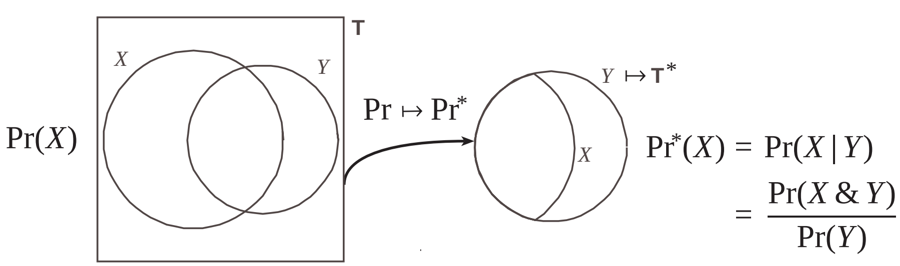
  Proof. Obvious, since the same regions always have the same areas, and the algebraic translation is *the same* for logically equivalent p/q.

  Theorem. If $p \vDash q$, then $\Pr(p) \leq \Pr(q)$.

  Proof. Since $p \vDash q$, the set of state descriptions entailing p is a subset of the set of state descriptions entailing q. Thus, the set of a_i in the summation for $\Pr(p)$ will be a subset of the a_i in the summation for $\Pr(q)$. Thus, since all the $a_i \geq 0$, $\Pr(p) \leq \Pr(q)$.

The Probability Calculus: An Algebraic Approach V

- **Conditional Probability**. $\Pr(p \mid q) \overset{\text{def}}{=} \dfrac{\Pr(p \,\&\, q)}{\Pr(q)}$, *provided that* $\Pr(q) > 0$.
- Intuitively, $\Pr(p \mid q)$ is supposed to be the probability of p ***given that*** q *is true*. So, *conditionalizing* on q is like "supposing q to be true".
- Using Venn diagrams, we can explain: "Supposing Y to be true" is like "treating the Y-circle as if it is the bounding box of the Venn Diagram".
- This is like "moving to a new $\Pr^*(\cdot)$ such that $\Pr^*(Y) = 1$." Picture:



The Probability Calculus: An Algebraic Approach VI

- There may be other ways of defining conditional probability, which may also seem to capture the "supposing q to be true" intuition.
- But, any such definition must make $\Pr(\cdot \mid q)$ itself a *probability function, for all propositions* q. This proves to be quite a strong constraint.
- Algebraically, we can see just how strong this constraint is. Recall:

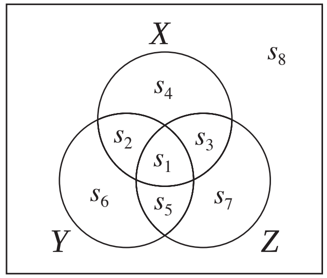
$$\Pr(X \vee Y) = \Pr(X) + \Pr(Y) - \Pr(X \,\&\, Y).$$

- Therefore, if $\Pr(\cdot \mid q)$ is to be a *probability* function *for all* q, then we must also have the following equality (in general), *for all* Z:

$$\Pr(X \vee Y \mid Z) = \Pr(X \mid Z) + \Pr(Y \mid Z) - \Pr(X \,\&\, Y \mid Z).$$

- Using our algebraic method, we can *prove* this. We just need to remind ourselves of what the 3-atomic sentence algebra looks like, and how the algebraic translation of this equation would go. Let's do that …

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	a_1
T	T	F	s_2	a_2
T	F	T	s_3	a_3
T	F	F	s_4	a_4
F	T	T	s_5	a_5
F	T	F	s_6	a_6
F	F	T	s_7	a_7
F	F	F	s_8	a_8



- By our definition of conditional probability, we have:

$$\Pr(X \vee Y \mid Z) = \frac{\Pr((X \vee Y) \,\&\, Z)}{\Pr(Z)} = \frac{\Pr((X \,\&\, Z) \vee (Y \,\&\, Z))}{\Pr(Z)} = \frac{a_1 + a_3 + a_5}{a_1 + a_3 + a_5 + a_7}$$

and

$$\Pr(X \mid Z) + \Pr(Y \mid Z) - \Pr(X\&Y \mid Z) = \frac{\Pr(X \,\&\, Z)}{\Pr(Z)} + \frac{\Pr(Y \,\&\, Z)}{\Pr(Z)} - \frac{\Pr(X \,\&\, Y \,\&\, Z)}{\Pr(Z)}$$

$$= \frac{\Pr(X \,\&\, Z) + \Pr(Y \,\&\, Z) - \Pr(X \,\&\, Y \,\&\, Z)}{\Pr(Z)}$$

$$= \frac{(a_1 + a_3) + (a_1 + a_5) - a_1}{a_1 + a_3 + a_5 + a_7} = \frac{a_1 + a_3 + a_5}{a_1 + a_3 + a_5 + a_7}$$

The Probability Calculus: An Algebraic Approach VII

- We can use our algebraic method to demonstrate that our definition of $\Pr(\cdot \mid q)$ yields a probability function, *for all* q, in the following way.
- Intuitively, think about what an "unconditional" and a "conditional" stochastic truth-table must look like, for any pair of sentences p and q.

p	q	$\Pr(s_i)$
T	T	a_1
T	F	a_2
F	T	a_3
F	F	a_4

$\xrightarrow{\cdot \mid q}$

p	q	$\Pr(s_i \mid q)$
T	T	$\Pr(s_1 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_1 \& q)}{\Pr(q)} = \frac{a_1}{a_1+a_3}$
T	F	$\Pr(s_2 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_2 \& q)}{\Pr(q)} = 0$
F	T	$\Pr(s_3 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_3 \& q)}{\Pr(q)} = \frac{a_3}{a_1+a_3}$
F	F	$\Pr(s_4 \mid q) \overset{\text{def}}{=} \frac{\Pr(s_4 \& q)}{\Pr(q)} = 0$

- Note: the new basic probabilities assigned to the state descriptions, under our "conditionalized" $\Pr(\cdot \mid q)$ satisfy the requirements for being a *probability* function, since $\frac{a_1}{a_1+a_3} + \frac{a_3}{a_1+a_3} = 1$, and $\frac{a_1}{a_1+a_3}, \frac{a_3}{a_1+a_3} \in [0,1]$.

The Probability Calculus: An Algebraic Approach VIII

- Here's a neat theorem of the probability calculus, proved algebraically.

Theorem. $\Pr(X \supset Y) \geq \Pr(Y \mid X)$. [Provided that $\Pr(X) > 0$, of course.]

Proof. $\Pr(X \supset Y) = \Pr(\sim X \vee Y) = \Pr(s_1 \vee s_3 \vee s_4) = a_1 + a_3 + a_4$.

$$\Pr(Y \mid X) = \frac{\Pr(Y \,\&\, X)}{\Pr(X)} = \frac{\Pr(s_1)}{\Pr(s_1 \vee s_2)} = \frac{a_1}{a_1 + a_2}.$$

So, we need to prove that $\boxed{a_1 + a_3 + a_4 \geq \frac{a_1}{a_1 + a_2}}$.

- First, note that $a_4 = 1 - (a_1 + a_2 + a_3)$, since the a_i's must sum to 1.
- Thus, we need to show that $a_1 + a_3 + 1 - a_1 - a_2 - a_3 \geq \frac{a_1}{a_1 + a_2}$.
- By simple algebra, this reduces to showing that $\boxed{1 - a_2 \geq \frac{a_1}{a_1 + a_2}}$.
- If $a_1 + a_2 > 0$ and $a_i \in [0,1]$, this must hold, since then we must have: $\boxed{a_2 \geq a_2 \cdot (a_1 + a_2)}$, and then the boxed formulas are equivalent. □

The Probability Calculus: An Algebraic Approach IX

- Here are some further fundamental theorems of probability calculus, involving 2 or 3 atomic sentences and CP. Easy, given defn. of CP.
  - **The Law of Total Probability** (LTP):

$$\Pr(X \mid Y) = \Pr(X \mid Y \,\&\, Z) \cdot \Pr(Z \mid Y) + \Pr(X \mid Y \,\&\, {\sim}Z) \cdot \Pr({\sim}Z \mid Y)$$

  - Note: $\Pr(X \mid \top) = \Pr(X)$. Why? So, the LTP has a *special case*:

$$\Pr(X \mid \top) = \Pr(X) = \Pr(X \mid \top \,\&\, Z) \cdot \Pr(Z \mid \top) + \Pr(X \mid \top \,\&\, {\sim}Z) \cdot \Pr({\sim}Z \mid \top)$$

$$= \Pr(X \mid Z) \cdot \Pr(Z) + \Pr(X \mid {\sim}Z) \cdot \Pr({\sim}Z)$$

  - Two forms of **Bayes's Theorem**. The second one *follows*, using (LTP):

$$\Pr(X \mid Y) = \frac{\Pr(Y \mid X) \cdot \Pr(X)}{\Pr(Y)}$$

$$= \frac{\Pr(Y \mid X) \cdot \Pr(X)}{\Pr(Y \mid Z) \cdot \Pr(Z) + \Pr(Y \mid {\sim}Z) \cdot \Pr({\sim}Z)}$$

The Probability Calculus: An Algebraic Approach X

- One more interesting theorem (due to Popper & Miller), algebraically.
- Let $d(X,Y) \overset{\text{def}}{=} \Pr(X \mid Y) - \Pr(X)$. Then, we have the following theorem:

  Theorem (PM). $d(X,Y) = d(X \vee Y, Y) + d(X \vee \sim Y, Y)$.

  Proof (algebraic, using STT from X/Y language, above).

$$d(X,Y) \overset{\text{def}}{=} \Pr(X \mid Y) - \Pr(X) = \frac{a_1}{a_1 + a_3} - (a_1 + a_2)$$

$$d(X \vee Y, Y) \overset{\text{def}}{=} \Pr(X \vee Y \mid Y) - \Pr(X \vee Y) = 1 - a_1 - a_2 - a_3$$

$$d(X \vee \sim Y, Y) \overset{\text{def}}{=} \Pr(X \vee \sim Y \mid Y) - \Pr(X \vee \sim Y) = \frac{a_1}{a_1 + a_3} - (a_1 + a_2 + a_4)$$

$$\therefore d(X \vee Y, Y) + d(X \vee \sim Y, Y) = 1 - a_1 - a_2 - a_3 + \frac{a_1}{a_1 + a_3} - a_1 - a_2 - a_4$$

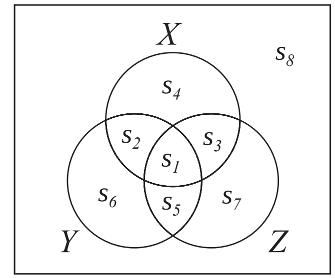
$$= \frac{a_1}{a_1 + a_3} + 1 - a_1 - a_2 - a_3 - a_1 - a_2 - (1 - (a_1 + a_2 + a_3))$$

$$= \frac{a_1}{a_1 + a_3} - (a_1 + a_2). \quad \square$$

The Probability Calculus: An Algebraic Approach XI

- The algebraic approach for *refuting* general claims involves two steps:
  1. Translate the claim from probability notation into algebraic terms.
  2. Find a (numerical) probability model on which the translation is *false*.
- Show that $\Pr(X \mid Y \,\&\, Z) = \Pr(X \mid Y \vee Z)$ can be *false*. Here's a model $\mathcal{M}$:

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	$a_1 = 1/6$
T	T	F	s_2	$a_2 = 1/6$
T	F	T	s_3	$a_3 = 1/4$
T	F	F	s_4	$a_4 = 1/16$
F	T	T	s_5	$a_5 = 1/6$
F	T	F	s_6	$a_6 = 1/12$
F	F	T	s_7	$a_7 = 1/24$
F	F	F	s_8	$a_8 = 1/16$



(1) Algebraic Translation: $\dfrac{a_1}{a_1 + a_5} = \dfrac{a_1 + a_2 + a_3}{a_1 + a_2 + a_3 + a_5 + a_6 + a_7}$.

(2) This claim is *false* on $\mathcal{M}$, since $1/2 \neq 2/3$. I used PrSAT to find $\mathcal{M}$.

The Probability Calculus: An Algebraic Approach XII

- There are *decision procedures* for Boolean propositional logic, based on truth-tables. These methods are *exponential* in the number of atomic sentences (n), because truth-tables grow exponentially in n (2^n).
- It would be nice if there were a decision procedure for probability calculus, too. In algebraic terms, this would require a decision procedure for the salient fragment of high-school (real) algebra.
- As it turns out, high-school (real) algebra (HSA) *is* a decidable theory. This was shown by Tarski in the 1920's. But, it's only been very recently that computationally feasible procedures have been developed.
- In my "A Decision Procedure for Probability Calculus with Applications", I describe a user-friendly decision procedure (called `PrSAT`) for probability calculus, based on recent HSA procedures.
- My implementation is written in *Mathematica* (a general-purpose mathematics computer programming framework). It is freely downloadable from my website, at: `http://fitelson.org/PrSAT/`.

The Probability Calculus: An Algebraic Approach XIII

- I encourage the use of `PrSAT` as a tool for finding counter-models and for establishing theorems of probability calculus. It is not a requirement of the course, but it is a useful tool that is worth learning.
- `PrSAT` doesn't give readable proofs of theorems. But, it will find concrete numerical counter-models for claims that are not theorems.
- `PrSAT` will also allow you to calculate probabilities that are determined by a *given* probability assignment. And, it will allow you to do algebraic and numerical "scratch work" without making errors.
- I have created a *Mathematica* notebook which contains some examples from algebraic probability calculus that we see in this lecture.
- Let's have a look at this first notebook (`examples_lmu.nb`). I will now go through the examples in this notebook, and demonstrate some of the features of `PrSAT`. I encourage you to play around with it.

Probabilistic Independence

Definition. p and q are *probabilistically independent, given* r ($p \perp\!\!\!\perp q \mid r$) iff $\Pr(p \,\&\, q \mid r) = \Pr(p \mid r) \cdot \Pr(q \mid r)$. [Note: We will use ⌜$p \perp\!\!\!\perp q$⌝ as an abbreviation for ⌜$p \perp\!\!\!\perp q \mid \top$⌝.]

- If $\Pr(p) > 0$ and $\Pr(q) > 0$, then $p \perp\!\!\!\perp q$ is equivalent to all of the following:
  - ∗ $\Pr(p \mid q) = \Pr(p)$ [Why? Because this is just: $\frac{\Pr(p\&q)}{\Pr(q)} = \Pr(p)$]
  - ∗ $\Pr(q \mid p) = \Pr(q)$ [ditto.]
  - ∗ $\Pr(p \mid q) = \Pr(p \mid {\sim}q)$ [Not as obvious. See next slide.]
  - ∗ $\Pr(q \mid p) = \Pr(q \mid {\sim}p)$ [ditto.]
- Closely related fact about independence. If $p \perp\!\!\!\perp q$, then we also must have: $p \perp\!\!\!\perp {\sim}q$, $q \perp\!\!\!\perp {\sim}p$, and ${\sim}p \perp\!\!\!\perp {\sim}q$. See next slide for algebraic set-up.
- A set of propositions $\mathbf{P} = \{p_1, \ldots p_n\}$ is *mutually independent* if all subsets $\{p_i, \ldots, p_j\} \subseteq \mathbf{P}$ are s.t. $\Pr(p_i \,\&\cdots\&\, p_j) = \Pr(p_i) \cdot \cdots \cdot \Pr(p_j)$. For sets with 2 propositions, pairwise independence is equivalent to mutual independence. But, not for 3 or more propositions. Example given below.

p	q	States	$\Pr(s_i)$
T	T	s_1	a_1
T	F	s_2	a_2
F	T	s_3	a_3
F	F	s_4	$a_4 = 1 - (a_1 + a_2 + a_3)$

$$\therefore \Pr(p \mid q) = \Pr(p \mid {\sim}q) \Leftrightarrow \frac{a_1}{a_1 + a_3} = \frac{a_2}{a_2 + a_4} = \frac{a_2}{1 - (a_1 + a_3)}$$
$$\Leftrightarrow a_1 \cdot (1 - (a_1 + a_3)) = a_2 \cdot (a_1 + a_3)$$
$$\Leftrightarrow a_1 = a_2 \cdot (a_1 + a_3) + a_1 \cdot (a_1 + a_3) = (a_2 + a_1) \cdot (a_1 + a_3)$$
$$\Leftrightarrow \Pr(p \,\&\, q) = \Pr(p) \cdot \Pr(q) \quad \square$$

- If p and q are independent, then so are p and ${\sim}q$. Prove this algebraically.
- More generally, if $\{p, q, r\}$ are mutually independent, then p is independent of *any* Boolean function β of q and r, *e.g.*, $p \perp\!\!\!\perp q \vee r$.
- How might one prove this more general theorem? And, is there an even more general theorem to be proved here?

- To wit: is it the case that if $\mathbf{P} = \{p_1, \ldots, p_n\}$ is a mutually independent set, then *any β-functions of any two disjoint subsets of* $\mathbf{P}$ are independent?
- So far, we've seen a some *proofs* of *true* general claims about independence, correlation, *etc.* Now, for some *counterexamples*!
- As always, these are numerical probability models in which some claim *fails*. We have seen two false claims about $\perp\!\!\!\perp$ already. Let's prove them.
- **Theorem**. Pairwise independence of a collection of three propositions $\{X, Y, Z\}$ does not entail mutual independence of the collection. That is to say, there exist probability models in which (1) $\Pr(X \,\&\, Y) = \Pr(X) \cdot \Pr(Y)$, (2) $\Pr(X \,\&\, Z) = \Pr(X) \cdot \Pr(Z)$, (3) $\Pr(Y \,\&\, Z) = \Pr(Y) \cdot \Pr(Z)$, but (4) $\Pr(X \,\&\, Y \,\&\, Z) \neq \Pr(X) \cdot \Pr(Y) \cdot \Pr(Z)$. *Proof*. Here's a counterexample.
- Suppose a box contains 4 tickets labelled with the following numbers:

  112, 121, 211, 222

  Let us choose one ticket at random (*i.e.*, each ticket has an *equal* probability of being chosen), and consider the following propositions:

X = "1" occurs at the first place of the chosen ticket.
Y = "1" occurs at the second place of the chosen ticket.
Z = "1" occurs at the third place of the chosen ticket.

Since the ticket #'s are 112, 121, 211, 222, we have these probabilities:

$$\Pr(X) = \tfrac{1}{2}, \Pr(Y) = \tfrac{1}{2}, \Pr(Z) = \tfrac{1}{2}$$

Moreover, each of the three conjunctions determines a unique ticket #:

$X \,\&\, Y$ = the ticket is labeled #112
$X \,\&\, Z$ = the ticket is labeled #121
$Y \,\&\, Z$ = the ticket is labeled #211

Therefore, since each ticket is equally probable to be chosen, we have:

$$\Pr(X \,\&\, Y) = \Pr(X \,\&\, Z) = \Pr(Y \,\&\, Z) = \tfrac{1}{4}$$

So, the three events X, Y, Z are pairwise independent (*why?*). But,

$X \,\&\, Y \,\&\, Z \dashv\vDash \bot$, since X, Y, and Z are jointly inconsistent.

Hence,

$\Pr(X \,\&\, Y \,\&\, Z) = \Pr(\mathsf{F}) = 1 - \Pr(\mathsf{T}) = 0 \neq \Pr(X) \cdot \Pr(Y) \cdot \Pr(Z) = (\tfrac{1}{2})^3 = \tfrac{1}{8}$

- This information determines a *unique* probability function. Can you specify it? Algebra (7 equations, 7 unknowns — see STT below).

$$\Pr(X) = a_4 + a_2 + a_3 + a_1 = \tfrac{1}{2}, \Pr(Y) = a_2 + a_6 + a_1 + a_5 = \tfrac{1}{2}$$
$$\Pr(Z) = a_3 + a_1 + a_5 + a_7 = \tfrac{1}{2}, \Pr(X \,\&\, Y \,\&\, Z) = a_1 = 0$$
$$\Pr(X \,\&\, Y) = a_2 + a_1 = \tfrac{1}{4}, \Pr(X \,\&\, Z) = a_3 + a_1 = \tfrac{1}{4}, \Pr(Y \,\&\, Z) = a_1 + a_5 = \tfrac{1}{4}$$

- Here's the STT. [This (and other models) can be found with PrSAT.]

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	$\Pr(s_1) = a_1 = 0$
T	T	F	s_2	$\Pr(s_2) = a_2 = 1/4$
T	F	T	s_3	$\Pr(s_3) = a_3 = 1/4$
T	F	F	s_4	$\Pr(s_4) = a_4 = 0$
F	T	T	s_5	$\Pr(s_5) = a_5 = 1/4$
F	T	F	s_6	$\Pr(s_6) = a_6 = 0$
F	F	T	s_7	$\Pr(s_7) = a_7 = 0$
F	F	F	s_8	$\Pr(s_8) = a_8 = 1/4$

- **Theorem**. $\perp\!\!\!\perp$ is *not* transitive. Example in which $\Pr(X \,\&\, Y) = \Pr(X) \cdot \Pr(Y)$, $\Pr(Y \,\&\, Z) = \Pr(Y) \cdot \Pr(Z)$, but $\Pr(X \,\&\, Z) \neq \Pr(X) \cdot \Pr(Z)$ $[X \neq Y \neq Z]$:

X	Y	Z	States	$\Pr(s_i)$
T	T	T	s_1	$\Pr(s_1) = a_1 = 3/32$
T	T	F	s_2	$\Pr(s_2) = a_2 = 9/32$
T	F	T	s_3	$\Pr(s_3) = a_3 = 3/32$
T	F	F	s_4	$\Pr(s_4) = a_4 = 9/32$
F	T	T	s_5	$\Pr(s_5) = a_5 = 2/32$
F	T	F	s_6	$\Pr(s_6) = a_6 = 2/32$
F	F	T	s_7	$\Pr(s_7) = a_7 = 2/32$
F	F	F	s_8	$\Pr(s_8) = a_8 = 2/32$

$$\begin{aligned}\Pr(X \,\&\, Y) &= a_2 + a_1 = \frac{3}{8} = \frac{3}{4} \cdot \frac{1}{2}\\ &= (a_4 + a_2 + a_3 + a_1) \cdot (a_2 + a_1 + a_6 + a_5) = \Pr(X) \cdot \Pr(Y)\\ \Pr(Y \,\&\, Z) &= a_1 + a_5 = \frac{5}{32} = \frac{1}{2} \cdot \frac{5}{16}\\ &= (a_2 + a_1 + a_6 + a_5) \cdot (a_3 + a_1 + a_5 + a_7) = \Pr(Y) \cdot \Pr(Z)\\ \Pr(X \,\&\, Z) &= a_3 + a_1 = \frac{3}{16} \neq \frac{3}{4} \cdot \frac{5}{16}\\ &= (a_4 + a_2 + a_3 + a_1) \cdot (a_3 + a_1 + a_5 + a_7) = \Pr(X) \cdot \Pr(Z)\end{aligned}$$

Measures of Confirmation I

- As I mentioned in my previous lectures this week, we can define a notion of "support" or "confirmation" as *probabilistic relevance*.

  Definition. E *confirms* H iff $\Pr(H \mid E) > \Pr(H)$. E *dis*confirms H iff $\Pr(H \mid E) < \Pr(H)$. E is *neutral/irrelevant* to H iff $\Pr(H \mid E) = \Pr(H)$.

- Given this qualitative definition of "confirms", it is natural to think about *quantitative measures* of *degree of confirmation*.
- This involves adopting some *function* $\mathfrak{c}$ of $\Pr(H \mid E)$ and $\Pr(H)$.
- We will use the notation $\mathfrak{c}(H, E)$ to denote the degree to which E confirms H, according to some function $\mathfrak{c}$ of $\Pr(H \mid E)$ and $\Pr(H)$.
- We'll adopt the following convention about the *range* of $\mathfrak{c}(H, E)$:

$$(\mathcal{R}) \qquad \mathfrak{c}(H,E) \in \begin{cases} (0,1] & \text{if } \Pr(H \mid E) > \Pr(H), \\ \{0\} & \text{if } \Pr(H \mid E) = \Pr(H), \\ [-1,0) & \text{if } \Pr(H \mid E) < \Pr(H). \end{cases}$$

Measures of Confirmation II

- A large number of measures of confirmation have been proposed in the literature (in statistics, cognitive science, philosophy, *etc.*). Here are the four most popular measures (*up to ordinal equivalence* — see below):
  - $d(H,E) \overset{\text{def}}{=} \Pr(H \mid E) - \Pr(H)$
  - $r(H,E) \overset{\text{def}}{=} \dfrac{\Pr(H \mid E) - \Pr(H)}{\Pr(H \mid E) + \Pr(H)} \doteq \dfrac{\Pr(H \mid E)}{\Pr(H)}$
  - $l(H,E) \overset{\text{def}}{=} \dfrac{\Pr(E \mid H) - \Pr(E \mid {\sim}H)}{\Pr(E \mid H) + \Pr(E \mid {\sim}H)} \doteq \dfrac{\Pr(H \mid E) \cdot (1 - \Pr(H))}{(1 - \Pr(H \mid E)) \cdot \Pr(H)}$
  - $s(H,E) \overset{\text{def}}{=} \Pr(H \mid E) - \Pr(H \mid {\sim}E)$
- If two measures $\mathfrak{c}_1$ and $\mathfrak{c}_2$ agree on *all comparisons*, then we say that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are *ordinally equivalent* ($\mathfrak{c}_1 \doteq \mathfrak{c}_2$). More precisely, we define:

  $\mathfrak{c}_1 \doteq \mathfrak{c}_2 \overset{\text{def}}{=} \mathfrak{c}_1(H_1,E_1) \geq \mathfrak{c}_1(H_2,E_2)$ *iff and only if* $\mathfrak{c}_2(H_1,E_1) \geq \mathfrak{c}_2(H_2,E_2)$

- **Exercises**: (i) prove that $\{d, r, l, s\}$ all satisfy $(\mathcal{R})$, and (ii) prove the two "$\doteq$" claims about r and l, above. *Hint*. Use $\frac{x-y}{x+y} = \tanh\left[\frac{1}{2}\log\left(\frac{x}{y}\right)\right]$.

Measures of Confirmation III

- **Fact**. *No two of* $\{d, r, l, s\}$ *are ordinally equivalent.* [Use PrSAT!]
- This ordinal disagreement between the most popular measures is what I have called "the plurality of Bayesian measures of confirmation".
- This was the topic of my dissertation [link on my mathcamp webpage].
- Here are eight important properties of measures of confirmation:

(1) If $E \vDash H_1$ and $E \nvDash H_2$, then $\mathfrak{c}(H_1, E) \geq \mathfrak{c}(H_2, E)$.

(2) If $\Pr(E \mid H_1) > \Pr(E \mid H_2)$, then $\mathfrak{c}(H_1, E) > \mathfrak{c}(H_2, E)$.

(3) If $\Pr(H \mid E_1) > \Pr(H \mid E_2)$, then $\mathfrak{c}(H, E_1) > \mathfrak{c}(H, E_2)$.

(4) $\mathfrak{c}(H, E) = \mathfrak{c}(E, H)$.

(5) $\mathfrak{c}(H, E) = -\mathfrak{c}(H, \sim E)$.

(6) $\mathfrak{c}(H, E) = -\mathfrak{c}(\sim H, E)$.

(7) If $H \vDash E$, then $\mathfrak{c}(H, E) > \mathfrak{c}(H \& X, E)$, for any X.

(8) If $\Pr(E \mid H_1) > \Pr(E \mid H_2)$ **&** $\Pr(E \mid \sim H_1) \leq \Pr(E \mid \sim H_2)$, then $\mathfrak{c}(H_1, E) > \mathfrak{c}(H_2, E)$.

Measures of Confirmation IV

	Does $\mathfrak{c}$-Measure have property?							
$\mathfrak{c}$-Measures	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$d(H,E)$	No	No	Yes	No	No	Yes	Yes	Yes
$r(H,E)$	No	Yes	Yes	Yes	No	No	No	Yes
$l(H,E)$	Yes	No	Yes	No	No	Yes	Yes	Yes
$s(H,E)$	No	No	No	No	Yes	Yes	Yes	Yes

- One can settle these (and many other) questions using PrSAT.
- Property (8) is the property that underlies the *robust* theorem about the conjunction fallacy that I discussed at the end of yesterday's lecture. [It's one of very few robust properties one finds in the literature.]
- **Exercise**. (iii) Define a relatively simple ($\mathcal{R}$)-measure that *violates* (8).
  - *Hint*. Try $\Pr(H \mid E)^n - \Pr(H)^n$ for (any) $n > 1$. See:

http://fitelson.org/crupi.pdf