Partial Belief, Full Belief, and Accuracy–Dominance

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- Arguments for probabilism aim to undergird/motivate a *synchronic probabilistic coherence* norm for partial beliefs.
- Standard arguments for probabilism are all of the form:
  - An agent S has a non-probabilistic partial belief function b iff ($\Longleftrightarrow$) S has some "bad" property B (*in virtue of* the fact that their p.b.f. b has a certain kind of *formal* property F).
- These *arguments* rest on *Theorems* ($\Rightarrow$) and *Converse Theorems* ($\Leftarrow$): b is non-Pr $\Longleftrightarrow$ b has formal property F.
  - **Dutch Book Arguments**. B is *susceptibility to sure monetary loss* (in a certain betting set-up), and F is the formal role played by non-Pr b's in the DBT and the Converse DBT.
  - **Representation Theorem Arguments**. B is *having preferences that violate some of Savage's axioms* (and/or *being unrepresentable as an expected utility maximizer*), and F is the formal role played by non-Pr b's in the RT.
- To the extent that we have reasons to avoid these B's, these arguments provide reasons (not) to have a(n) (in)coherent b.
- Today, we're talking about *accuracy-dominance* arguments.

- In "Accuracy-Dominance Arguments" (ADAs) for probabilism:
  - B is *having an "accuracy-dominated" partial belief function* b. This means one's b is *less accurate* than some alternative candidate credence function $b^\star$ — *come what may*.
  - F is *inadmissibility*: the existence of an alternative candidate partial belief function $b^\star$ such that $b^\star$ s*-dominates* b, where s is some "good" scoring rule that is adopted by S. (we will adopt the *Brier score* — over *an entire algebra* $\mathbb{A}$).
  - We'll give examples, below, to illustrate these concepts.
- We'll focus on the relationship between the *inconsistency* of S's *full* beliefs and the *incoherence* of S's *partial* beliefs.

☞ The upshot will be that ADAs for probabilistic coherence *break down* (for *certain sorts* of agents), when we think carefully about this connection. First, some set-up.

- We'll consider *logically omniscient* agents S, with languages $\mathcal{L}$ & *total* credence functions b such that: (i) $b : \mathcal{L} \mapsto [0, 1]$, (ii) $b(p) = b(q)$ *if* $p \dashv\vDash_{\mathcal{L}} q$, (iii) $b(\top) = 1$ and $b(\bot) = 0$.

- The locution "b is incoherent on the algebra $\mathbb{A}$ of propositions expressible in $\mathcal{L}$" (or "b is incoherent on $\mathbb{A}$", for short) means that b is *not a probability function* on $\mathbb{A}$.
- A credence function b is said to be *extremal* just in case it assigns either 1 or 0 to each (and every) proposition in $\mathbb{A}$.
- Thus, the *truth-value assignments* on $\mathcal{L}$ correspond (exactly) to the *coherent* extremal credence functions on $\mathbb{A}$.
- Now, we will be contrasting two kinds of agents:
  - **Extremal agents** are agents whose credence functions are extremal, and for whom non-extremal credence functions aren't even so much as *candidate* alternative credence functions (*i.e.*, extremal S's "*necessarily*" have extremal b's).
  - **Non-extremal agents** are agents that "*can*" (and *typically do*) have non-extremal credence functions.
- The important contrast here will be between *incoherent* extremal agents and *incoherent* non-extremal agents.
- OK, now we're ready for a concrete example...

- We begin with a (concrete) *incoherent non*-extremal agent S, to vividly illustrate the concepts we've been talking about.
- Consider an agent S with a 2-atomic-sentence (X,Y) $\mathcal{L}$, and a d.o.b. function b on $\mathcal{L}$, which satisfies these six constraints:

$b(X \& Y) = \frac{1}{10}$	$b(X \& {\sim}Y) = \frac{2}{5}$	$b({\sim}X \& Y) = \frac{1}{5}$
$b({\sim}X \& {\sim}Y) = \frac{3}{10}$	$b(X) = \frac{1}{2}$	$b({\sim}X) = \frac{2}{5}$

- Note: b is coherent *on the partition of state descriptions* of $\mathcal{L}$, but b will have to be incoherent *on the full algebra* $\mathbb{A}$.
- On the next slide, we will fill-in the values of b on the rest of $\mathbb{A}$, so as to make S (intuitively) "close" to being coherent.
- Accuracy-dominance theorems (going back to de Finetti) will entail the existence of *alternative, non-extremal* credence functions $b^\star$ that will be more accurate than b (in Brier score) — *in all possible worlds* (call this *Brier–dominance*).
- On the next slide, we look at b and two "close" (in Euclidean distance) alternative, coherent, non-extremal functions on $\mathbb{A}$.

p	$b_{\mathbb{A}}(p)$	$b'_{\mathbb{A}}(p)$	$b^{\dagger}_{\mathbb{A}}(p)$
${\sim}X \& {\sim}Y$	3/10	3/10	23/80
$X \& {\sim}Y$	2/5	2/5	33/80
$X \& Y$	1/10	1/10	9/80
${\sim}X \& Y$	1/5	1/5	3/16
${\sim}Y$	7/10	7/10	7/10
$X \equiv Y$	2/5	2/5	2/5
${\sim}X$	**2/5**	**1/2**	**19/40**
X	1/2	1/2	21/40
${\sim}(X \equiv Y)$	3/5	3/5	3/5
Y	3/10	3/10	3/10
$X \vee {\sim}Y$	4/5	4/5	13/16
${\sim}X \vee {\sim}Y$	9/10	9/10	71/80
${\sim}X \vee Y$	3/5	3/5	47/80
$X \vee Y$	7/10	7/10	57/80
$X \vee {\sim}X$	1	1	1
$X \& {\sim}X$	0	0	0

- $b_{\mathbb{A}}$ is a completion of b that is (intuitively) "close" to coherent.
- $b'_{\mathbb{A}}$ is a Pr-f that's (intuitively) "close" to $b_{\mathbb{A}}$, but does ***not*** *Brier-dominate* $b_{\mathbb{A}}$.
- $b^{\dagger}_{\mathbb{A}}$ is the *Euclidean-closest* Pr-f to $b_{\mathbb{A}}$, and ∴ it *Brier-dominates* $b_{\mathbb{A}}$.

- Next, consider an *extremal* agent S who assigns credence 1 to propositions he believes and credence 0 to propositions he disbelieves (*i.e.*, S is *dogmatic/opinionated and extremal*).
- Place S in a "*preface context*" where S believes each member of a set of propositions, but disbelieves their conjunction.
- Because S is dogmatic, extremal, and has inconsistent full beliefs, it follows that S's credence function b is *incoherent*.

☞ This sort of agent allows us to forge an interesting (and theoretically clean and revealing) connection between *inconsistency* of full belief and *incoherence* of partial belief.

- We will now focus on agents S of this sort, with an eye toward investigating the following questions.
  - What do such agents look like, from an ADA point of view?
  - Specifically, can ADAs furnish such agents with reasons to have probabilistically coherent partial beliefs (and, hence, reasons to have logically consistent full beliefs)?
  - I'll hand it off to Kenny now, to deliver the punch-line…

p	$\beta(p)$	$\beta'(p)$	$\beta^{\dagger}(p)$
${\sim}X \& {\sim}Y$	0	0	1/8
$X \& {\sim}Y$	0	0	1/8
$X \& Y$	**0**	**1**	**5/8**
${\sim}X \& Y$	0	0	1/8
${\sim}Y$	0	0	1/4
$X \equiv Y$	1	1	3/4
${\sim}X$	0	0	1/4
X	1	1	3/4
${\sim}(X \equiv Y)$	0	0	1/4
Y	1	1	3/4
$X \vee {\sim}Y$	1	1	7/8
${\sim}X \vee {\sim}Y$	**1**	**0**	**3/8**
${\sim}X \vee Y$	1	1	7/8
$X \vee Y$	1	1	7/8
$X \vee {\sim}X$	1	1	1
$X \& {\sim}X$	0	0	0

- β represents a (toy!) dogmatic, extremal agent in a "preface case".
- β' is the coherent, extremal function that is closest to β.

☞ ***No*** extremal credence function $\beta^\star$ Brier-dominates β.

- $\beta^{\dagger}$ is the *Euclidean-closest* Pr-f to β, and ∴ it *Brier-dominates* β.

- This (toy) "preface case" can be generalized to larger $\mathbb{A}$'s.
  - The algebra $\mathbb{A}$ above had four state descriptions. And, the agent assigned credence 1 to all propositions entailed by a majority of state descriptions, and 0 to all propositions incompatible with a majority of state descriptions.
- **Theorem.** β's assigning 1 to propositions entailed by most state descriptions, and 0 to those incompatible with most, is a *sufficient* condition for β's being *non*-Brier-dominated.
  - Calculate the *average* Brier score across states; if β' dominates β, then it must have a lower average score.
  - But the average Brier score of β is just the sum of the components for each proposition.
  - The component for a given p is the proportion of states in which p's truth-value is the opposite of that assigned by β.
  - So to minimize this average, it is sufficient to believe every proposition true in a majority of states and disbelieve every proposition false in a majority of states, *QED*.

    [What β does on p's true in *exactly* half the states is *irrelevant*!]

- Such β's are "preface-like", since they commit themselves to many weak propositions, but not their conjunctions.
- ☞ For *extremal* agents, *both* "preface-like" belief functions *and* coherent belief functions are *admissible* (*non*-dominated).
- In this sense, the ADA does not generate a reason for *extremal* incoherent agents to be coherent (*per se*).
  - In particular, it even allows for an extremal agent to believe both a proposition and its negation, or to disbelieve both!
- But, ADAs *do* motivate *some* wide-scope, "on pain of Brier-domination" norms — *even for extremal agents.*
- **Norm 1.** If A and B are incompatible, then (*even an extremal*) S ought (*either* disbelieve A, disbelieve B *or* believe $A \vee B$).
  - Let β be an extremal belief function with $\beta(A) = \beta(B) = 1$ and $\beta(A \vee B) = 0$ [*i.e.*, S believes A, B; but disbelieves $A \vee B$].
  - Let β' have identical values to β on all other propositions, but assign $\beta'(A) = \beta'(B) = 0$ and $\beta'(A \vee B) = 1$.
  - Then in every case, β' gets two of these beliefs right while β only gets one right, so β' dominates β.

- The vast majority (14796/16384 in $\mathbb{A}$) of extremal functions are dominated by some extremal function. Breakdown:
  - **Non-Brier-dominated extremal pbf's (1588/16384 in $\mathbb{A}$):**
    - Coherent functions (4/16384 in $\mathbb{A}$).
    - "Preface-like" (Theorem) incoherent pbf's (64/16384 in $\mathbb{A}$).
    - Others??? (1520/16384 in $\mathbb{A}$).
  - **Brier-Dominated extremal pbf's (14796/16384 in $\mathbb{A}$):**
    - Dominated by a *single* coherent extremal β. [284/16384 in $\mathbb{A}$]
    - Dominated by *every* coherent extremal β. [8/16384 in $\mathbb{A}$]
    - Dominated by *no* coherent extremal β. [14504/16384 in $\mathbb{A}$]
- In every state, one coherent function gets every proposition right while all the others get exactly half of them wrong.
- Thus, if a belief set is dominated by two distinct coherent sets, then it must get more than half wrong in every state, and thus be dominated by *all* coherent sets.
- On the next slide, we examine examples of Brier-dominated extremal belief functions of the last two types…

p	$\gamma(p)$	$\delta(p)$	$\delta'(p)$
$\sim X \,\&\, \sim Y$	1	1	0
$X \,\&\, \sim Y$	1	1	0
$X \,\&\, Y$	1	1	0
$\sim X \,\&\, Y$	1	0	0
$\sim Y$	0	1	1
$X \equiv Y$	0	1	1
$\sim X$	0	0	0
X	0	1	1
$\sim(X \equiv Y)$	0	0	0
Y	0	0	0
$X \vee \sim Y$	0	1	1
$\sim X \vee \sim Y$	0	1	1
$\sim X \vee Y$	0	1	1
$X \vee Y$	0	1	1
$X \vee \sim X$	1	1	1
$X \,\&\, \sim X$	0	0	0

- In every state, γ gets 7 p's right, while every coherent extremal β gets exactly 8 p's right in any state other than its own (where it gets 16).
- $\therefore$ γ is dominated by *every* coherent extremal β.
  - γ also *violates* Norm 1.
- δ is a belief function that is dominated by *no* ***coherent*** extremal β, but δ is dominated by δ'.
  - In fact, δ' *uniquely* dominates δ.
- Interestingly, δ *satisfies* Norm 1. Therefore, Norm 1 is *insufficient* for being *non*-dominated.
- Indeed, δ satisfies this *even stronger norm*:
  - **Norm 2.** If A and B are incompatible, and $A \vee B \vDash p$, then S ought (*either* disbelieve A *or* disbelieve B *or* believe p).
- But, δ *violates* the following additional norm:
  - **Norm 3.** S ought not believe any three pairwise incompatible propositions.

- All 3 of our Norms (so far) are *special cases* of the following:
  - **Norm**. If there exists a set of propositions **P** such that β is *incorrect on a majority* of $p \in \mathbf{P}$ — *in all possible worlds*, then (an extremal) S should not conform their beliefs to β.
- ☞ We know **Norm** is *necessary and sufficient* for (an extremal β) being non-dominated in $\mathbb{A}$ (by another extremal β').
- So, while ADAs don't provide reason for extremal S's to have *coherent* β's, they *do* provide reason to obey **Norm**.
  - Disanalogy: Dutch Book Arguments give extremal *and* non-extremal agents *alike* reason to have *coherent* b/β's.
- We anticipate the following objection. *Extremal agents are crazy*, so why should an advocate of ADAs care about them?
- Note: *we're not advocating extremality*. There may be some *epistemic* norms that extremal agents are bound to violate.
- That's not the issue. ADAs are supposed to be (aim to be) arguments against *incoherence* ***per se***. And, while extremality may be "bad", it's *not incoherent* ***per se***.

- To sum up: ADAs for probabilism (dating back to de Finetti) all share an important (hitherto unappreciated) *asymmetry*.
- Every *non*-extremal incoherent partial belief function b is Brier-dominated by a *non*-extreme partial belief function $b^\star$. [Indeed, there will always be a *coherent*, dominating $b^\star$.]
- However, some (albeit a minority of) *extremal* incoherent partial belief functions β *fail* to be Brier-dominated by *any extremal* partial belief function $\beta^\star$ — *coherent or otherwise*.
- ☞ So, while ADAs provide reason for non-extremal agents to be coherent (to the extent that they disvalue Brier-domination), they provide extremal agents with no such reason.
- Nonetheless, some (wide-scope) coherence norms for the belief functions β of extremal agents *are* appropriately motivated by ADAs. The strongest of these norms is:
  - **Norm**. If there exists a set of propositions **P** such that β is *incorrect on a majority* of $p \in \mathbf{P}$ — *in all possible worlds*, then (an extremal) S should not conform their beliefs to β.