Opinion

Are Perceptual Fields Quantum Fields?

"Out of the multitude of our sense experiences we take, mentally and arbitrarily, certain repeatedly occurring complexes of sense impression... we attribute to them a meaning — the meaning of the bodily object. Considered logically this concept is not identical with the totality of sense impressions referred to; but it is an arbitrary creation of the human (or animal) mind. On the other hand, the concept owes its meaning and its justification exclusively to the totality of the sense impressions which we associate with it"

Albert Einstein

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Abstract
We discuss Chalmers' "hard problem" in respect of a mind/brain identity theory. We examine a handful of mathematical parallels between the visual field and its concomitant quantum field, and argue for an identity between the two fields.

Key Words: Perceptual Fields, Quantum Fields, Vision, Space, Mind-Brain Identity, Hard Problem

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I.Introduction

In a widely quoted *Scientific American* article of recent vintage, David Chalmers suggests we look to information states as a possible means of bridging mind and matter:

> "The abstract notion of information, as put forward by Claude E. Shannon of MIT, is that a of a set of separate states with a basic structure of similarities and differences between them. We can think of a 10-bit binary code as an information state, for example. Such information can be embodied in the physical world. This happens whenever they correspond to physical states (voltages, say); the differences between them can be transmitted along some pathway, such as a

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telephone line".

We can also find information embodied in conscious experience. The pattern of color patches in a visual field, for example, can be seen as analogous to that of pixels covering a display screen. Intriguingly, it turns out that we find the same information states embodied in conscious experience and in underlying physical processes in the brain. The three-dimensional encoding of color spaces, for example, suggests that the information state in a color experience corresponds directly to an information state in the brain. We might even regard the two states as distinct aspects of a single information state, which is simultaneously embodied in both physical processing and conscious experience (Chalmers, 1995).

Chalmers is best known for his formulation of the "hard problem", i.e., how to explain the brain's phenomenal states? Let us compare Chalmers' views with a proposed solution to the mind/body problem from Herbert Feigl, one of the most eloquent proponents of mind/brain identity theory:

> The solution that appears most plausible to me, and that is consistent with a thoroughgoing naturalism, is an identity theory of the mental and the physical, as follows: Certain neuro-physiological terms denote (refer to) the very same events that are also denoted (referred to) by certain phenomenal terms. The identification of the objects of this twofold reference is of course logically contingent, although it constitutes a very fundamental feature of our world as we have come to conceive it in the modern scientific outlook. Using Frege's distinction between Sinn ('meaning', 'sense', 'intension'), and Bedeutung ('referent', 'denotatum', 'extension'), we may say that neurophysiological terms and the corresponding phenomenal terms, though widely differing in sense ... do have identical referents. I take these referents to be the immediately experienced qualities, or their configurations in the various phenomenal fields (Feigl, 1970).

On the present view, the key to the reconciliation of mind and matter is contained in embryo in Chalmers' remarks, specifically: "the three-dimensional encoding of color spaces, for example, suggests that the information state in a color experience corresponds directly to an information state in the brain. We might even regard the two states as distinct aspects of a single information state, which is simultaneously embodied in both physical processing and conscious experience". And now Feigl again: "we may say that neurophysiological terms and the corresponding phenomenal terms, though widely differing in sense ... do have identical referents. I take these referents to be the immediately experienced qualities, or their configurations in the various phenomenal fields".

It is worth noting that the views expressed above are consonant with those found in Bohm and Hiley: "one may then ask what is the relationship between the physical and the mental processes? The answer that we propose here is that there are not two processes. Rather, it is being suggested that both are essentially the same".

We consider that one might well consider Chalmers' "two" information states as complementary aspects of a single state. More particularly, we ask whether one might regard the states of perceptual fields and their concomitant quantum fields as being somehow the same. Feigl's "immediately experienced qualities, or their configurations in the various phenomenal fields" might then be identified with 'observables, or their configurations, in various quantum fields' - what amounts to a mind/brain identity theory wherein the matter of the brain is viewed at the quantum level.

Intriguingly, the mathematics of quantum mechanics and quantum field theory offers a number of striking parallels with the mathematics of color and the visual field. We shall have a look at a handful of such parallels, in respect of a few elementary aspects of vectors, manifolds, fields, symmetry, logic, and group theory. Given the broad interest in such matters today, we will light our way with the assistance of a number of introductory texts drawn from a dazzling array of luminaries past and present.

II.Apology

How might quantum theory help us with neuroscience? At the present historical juncture there are not a few who would readily reply 'not at all'. And indeed the polemical storms concerning the issues at hand have been something to behold these last few years. By way of reply, we refer first to Bohr, the father of quantum mechanics (QM) who, as Bohm tells us, "suggests that thought involves such small amounts of energy that quantum-theoretical limitations play an essential role in determining its character" (Bohm, 1951).

Next, we bring in Freeman Dyson, who states quite clearly that, from the perspective of quantum field theory (QFT) everything in the physical universe just is a quantum field: "There is nothing else except these fields: the whole of the material universe is built of them" (Dyson, 1953). Thus, the brain just is a collection of fields, which is interesting, given that the objects of our immediate awareness are also fields, viz., perceptual fields.

We consider that the most relevant physical fields must be electromagnetic (EM) fields, and in this wise we quote Abdus Salam, who reminds us that "all chemical binding is electromagnetic in origin, and so are all phenomena of nerve impulses" (Salam, 1990). We reason that, if perceptual fields are "phenomena of nerve impulses", as would seem altogether plausible, then it would seem to follow that perceptual fields are "electromagnetic in origin".

Further support for our thesis comes from Karl Pribram, who, in a classic modern work, writes: "The text of this volume claims that the mathematical formulations that have been developed for quantum mechanics and quantum field theory can go a long way toward describing neural processes due to the functional organization of the cerebral cortex" (Pribram, 1991).

We want to go a little further than Pribram and regard the brain and sense organs as thoroughly quantum, through and through. So saying, we produce a passage from Umezawa's highly accessible work on *Advanced Field Theory:*

Among the many biological objects a particularly interesting one is the brain. For any theory to be able to claim itself as a brain theory, it should be able to explain the origin of such fascinating properties as the mechanism for creation and recollection of memories and consciousness. For many years it was believed that brain function is controlled solely by the classical neuron system which provides the pathway for neural impulses. This is frequently called the neuron doctrine. The most essential one among many facts is the nonlocality of memory function discovered by Pribram (...)

There have been many models based on quantum theories, but many of them are rather philosophically oriented. The article by Burns...provides a detailed list of papers on the subject of consciousness, including quantum models. The incorrect perception that the quantum system has only microscopic manifestations considerably confused this subject. As we have seen in preceding sections, manifestation of ordered states is of quantum origin. *When we recall that almost all of the macroscopic ordered states are the result of quantum field theory, it seems natural to assume that macroscopic ordered states in biological systems are also created by a similar mechanism (Umezawa, 1993).*

We argue herein that a scientific account of the secondary properties of observation entails an overhaul of the traditional ontology of physics. In this connection we spring Eugene Wigner, whom Feynman called the most gifted physicist he ever met: "let us now turn to the assumption opposite to the "first alternative" considered so far: that the laws of physics will have to be modified drastically if they are to account for the phenomena of life. Actually, I believe that this second assumption is the correct one".

Can arguments be adduced to show the need for modification? There seem to be two such arguments. The first is that, if one entity is influenced by another entity, in all known cases the latter one is also influenced by the former. The most striking and originally the least expected example for this is the influence of light on matter, most obviously in the form of light pressure. That matter influences light is an obvious fact — if it were not so, we could not see objects. *The influence of light on matter is, however, a more subtle effect and is virtually unobservable under the conditions which surround us ... Since matter clearly influences the content of our consciousness, it is natural to assume that the opposite influence also exists, thus demanding the modification of the presently accepted laws of nature which disregard this influence (Wigner, 1970).*

Why should we change the ontology of physics, though? Consider the following passage from Michael Lockwood, whose thought runs in tracks remarkably parallel to our own:

Consciousness, in other words, provides us with a kind of 'window' on to our brains, making possible a transparent grasp of a tiny corner of a material reality that is in general opaque to us, knowable only at one remove. The qualities of which we are immediately aware, in consciousness, precisely are some at least of the intrinsic qualities of the states and processes that go to make up the material world — more specifically, states and processes within our own brains.

The psychologist Pribram . . . has made an interesting attempt to revive an idea originally put forwar terms of the present proposal, is that we have a 'special' or 'privileged' access, via some of our own brain activity, to the intrinsic character of, say, electromagnetism. Put like that, the idea sounds pretty fanciful. But make no mistake about it: whether about electromagnetism or about other such phenomena, that is just what the Russellian view ostensibly commits one to saying.

There are, however, two things that must now be emphasized. In the first place, it is a clear implicatic doubt whether the stock of fundamental attributes countenanced within contemporary physical science is, in principle, adequate to the task of accounting for the qualitative diversity that introspection reveals. The current trend, within physics, is towards ever greater unification of the fundamental forces (Lockwood, 1989).

Lockwood cited Pribram just now, apropos "certain fields, in the physicist's sense, within the cerebra *scale structure of space-time:* "the view of physics that is most generally accepted at the moment is that one can divide the discussion of the universe into two parts. *First, there is the question of the local laws satisfied by the various physical fields.* These are usually expressed in the form of differential equations (Hawking, 1973).

A most explicit and philosophically sound expression of our working scientific framework is found in a wonderfully erudite essay by Simon Saunders on "The Algebraic Approach to Quantum Field Theory": "Our basic ontology is that all systems, macroscopic structures included, are quantum fields . . ." (Brown, 1988).

Finally, and somewhat ironically, we note our agreement with Paul Churchland in respect of: (a) the neural implementation of matrix operations on input vectors; and (b) the observation, apropos the work of Pellionisz and Llinas (1979) that the cerebellum's job is "the systematic transformation of vectors in one neural hyperspace into vectors in another neural hyperspace"; and the notion that (c) "the tensor calculus emerges as the natural framework with which to address such matters" and (d) the characterization of phenomenal properties as vectors (Churchland, 1989).

If Paul Churchland is correct about the neural implementation of matrix-valued operators, then that is rather interesting, since that is precisely the sort of mathematics we find at work at the quantum level of neural function. Which would seem to make a kind of sense, if, as we suggest, the form of neural networks follows the underlying function of those quantum processes which mediate neural activity. Given that that the dendritic forms of neurons are aptly captured by the mathematics of fractals, we might expect this kind of self-similarity across scales. And so we might borrow from Paul to appease Patricia as to the relevance of QM to consciousness.

I suspect we will continue to encounter a measure of not unintelligent resistance from those who, perhaps not wishing to trouble themselves with the admitted difficulties of quantum theory, may reply that, even if we do have a remarkable correspondence between the mathematics of neural networks and QFT (as would seem evident), we have no need to pursue matters at the quantum level, because we can say all that needs to be said at the neural level. We are not much moved by such replies, which would seem to have more to do with intellectual inertia than with logic or reason, let alone the spirit of scientific inquiry. For after all, mind would seem to be connected to the brain somehow, and if the matter of the brain resides at the quantum level, then the quantum level would seem to be a logical place to look for mind. And then, who can say before the fact what discoveries might await us in pursuing this line of research?

Consider, then, that QFT tells us that all is field phenomena. Dyson said so, just now. Again, this would seem quite suggestive, when one reflects upon the observation that the objects of our immediate awareness just are perceptual fields. Then, too, we know our perceptual fields are clearly related to those photonic fields which “excite” our perceptions (as is most easily seen in the case of vision). We have two fields, then, two sets of states, always varying together — mechanically, predictably, quantifiably. Is this because the two are actually one? Perhaps, but let us see whether we can make our proposal more particular.

III.Vectors

Feynman tells us that colors behave like vectors, which is interesting, because photons happen to behave like vectors, too. Here is what Feynman wrote:

The second principle of color mixing of lights is this: any color at all can be made from three different colors, in our case, red, green, and blue lights. By suitably mixing the three together we can make anything at all, as we demonstrated...

Further, these laws are very interesting mathematically. For those who are interested in the mathematics of the thing, it turns out as follows. Suppose that we take our three colors, which were red, green, and blue, but label them A, B, and C, and call them our primary colors. Then any color could be made by certain amounts of these three: say an amount a of color A, an amount b of color B, and an amount c of color C makes X:

$$X = aA + bB + cC.$$

Now suppose another color Y is made from the same three colors:

$$Y = a'A + b'B + c'C.$$

Then it turns out that the mixture of the two lights (it is one of the consequences of the laws that we have already mentioned) is obtained by taking the sum of the components of X and Y:

$$Z = X + Y = (a + a')A + (b + b')B + (c + c')C.$$

It is just like the mathematics of the addition of vectors, where (a, b, c) are the components of one vector, and (a', b', c') are those of another vector, and the new light Z is then the "sum" of the vectors. This subject has always appealed to physicists and mathematicians. In fact: Schrödinger wrote a wonderful paper on color vision in which he developed this theory of vector analysis as applied to the mixing of colors. [15]

Now let us consider that mixing lights of different colors is accomplished by superposing photonic state vectors. Here is the luminous Weyl to help us along with the physics:

> Monochromatic light is completely determined as to its quality by the wave-length, because its oscillation law with regard to time and its wave structure have a definite simple mathematical form which is given by the function sine or cosine. Every physical effect of such light is completely determined by the wave-length together with the intensity. To monochromatic light corresponds in the acoustic domain the simple tone. Out of different kinds of monochromatic light composite light may be mixed, just as tones combine to a composite sound. This takes place by superposing simple oscillations of different frequency with definite intensities. The simple color qualities form a one-dimensional manifold, since within it the single individual can be fixed by one continuously variable measuring number, the wave-length. The composite color qualities, however,

> form a manifold of infinitely many dimensions from the physical point of view (Weyl, 1934).

Weyl writes that "monochromatic light is completely determined as to its quality by the wave-length." But red and green light, e.g., may be combined to give yellow. Weyl offers a way out of this dilemma:

> This discrepancy between the abundance of physical "color chords" and the dearth of the visually perceived colors must be explained by the fact that very many physically distinct colors release the same process in the retina and consequently produce the same color sensation. By parallel projection of space on to a plane, all space points lying on a projecting ray are made to coincide in the same point on a plane; similarly this process performs a kind of projection of the domain of physical colors with its infinite number of dimensions on to the two-dimensional domain of perceived colors whereby it causes many physically distinct colors to coincide. . . .
>
> It seems useful to me to develop a little more precisely the "geometry" valid in the two-dimensional manifold of perceived colors (Weyl, 1934, p10).

Since we make so much of the vectorial aspects of color, let's refresh our memories as to the central role of vectors in the axioms of QM, with a peek at a splendid modern text on the mathematics of classical and quantum physics:

Axiom I. Any physical system is completely described by a normalized vector (the *state vector* or *wave function)* in Hilbert space. All possible information about the system can be derived from this state vector by rules (...) (Byron, 1970).

As to the superposition of state vectors, we read in another source:

When a state is formed by the superposition of two other states, it will have properties that are in some vague way intermediate between those of the original states and that approach more or less closely to those of either of them according to the greater or less 'weight' attached to this state in the superposition process. The new state is completely defined by the two original states when their relative weights in the superposition process are known, together with a certain phase difference, the exact meaning of weights and phases being provided in the general case by the mathematical theory (Dirac, 1958).

Is the paragraph immediately above referring to the state vectors of QM? Or to color vectors? The answer is the former, but notice that Dirac's remarks apply equally well to our common experience with mixing lights of different colors.

Years ago, when I first broached this similarity of color vectors and QM state vectors, a persistent comeback went something like, 'So what? Colors behave like vectors - lots of things behave like vectors, that doesn't prove anything.' My considered reply was and is that, yes, many things do behave like vectors. The point is, again, that color vectors and photonic state vectors demonstrably co-vary in a manner altogether predictable, reliable, and really quite quantifiable.

A second and related mathematical correspondence presents itself, one which is both wonderfully simple and which goes to the heart of our use of vectors in physical theory generally. The example of classical mechanics shows us that there are possible representations of physical theories which do not involve Hilbert spaces. Of course, this doesn't mean that classical mechanics could not be reformulated in this way. In fact, our strategy for providing a partial answer to the question, "Why Hilbert spaces?" will be to show that the theory of vectors has very general application. We will take as an example a particular physical situation and model it mathematically. The situation will be paradigmatically of the kind with which physical theory deals, but our description will be general enough to leave open the question of what sorts of processes, deterministic or indeterministic, are involved. Similarly, its representation, in terms of vector space, will be general enough to be employed for a variety of physical theories; the particular features of quantum mechanics on the one hand, or classical mechanics on the other, will then appear as additional constraints on these mathematical structures, as proposed by Feynman:

The key to the representation is the fact that Pythagoras' theorem, or its analogue, holds in any vector space equipped with an inner product. Consider the space R^3. For any vector v in R^3,

$$v = v_x + v_y + v_z$$

Here v_x, v_y, and v_z are the projections of v onto an orthogonal triple of rays spanning R^3 — or, as we can call them, the *axes* of our coordinate system.

Pythagoras' theorem tells us that

$$|v_x|^2 + |v_y|^2 + |v_z|^2 = |v|^2,$$

and so, if v is normalized,

$$|v_x|^2 + |v_y|^2 + |v_z|^2 = 1$$

Let us now assume that we wish to represent three mutually exclusive events that together exhaust all possibilities, and that each event has a certain probability. If we use the axes of R^3 to represent the events x, y, and z, we can construct a normalized vector v to represent any probability assignment to these events. We simply take vectors v_x, v_y, and v_z along these axes such that :

$$|v_x|^2 = p(x),\ |v_y|^2 = p(y) \text{ and } |v_z|^2 = p(z),$$

and then add them (vectorially) to yield v.

Since the events x, y, and z are mutually exclusive and jointly exhaustive, we know that p(x) + p(y) + p(z) = 1 and it follows...that v is normalized. *This almost trivial construction lies at the heart of the use of vector spaces in physical theory (Hughes, 1989).*

Then considering the space R^3, for any vector v in R^3:

$v = v_x + v_y + v_z$"

that takes us full circle back to Feynman above, where any color vector v can be produced by adding suitable amounts of red, green, and blue vectors. So we have an obvious parallel between quantum states and visual states. We might pursue this analogy and construct a normalized color sphere with red, green, and blue serving duty as the projections of v onto an "orthogonal triple of rays" (RGB, e.g.) spanning R^3, as in Hughes' just quoted remarks.

We might also reflect on the fact of observation that no "point" in the visual field can be more than one color simultaneously — though it must have some color, as Wittgenstein remarks: "a speck in the visual field, though it need not be red must have some color; it is, so to speak, surrounded by color-space. Notes must have some pitch, objects of the sense of touch some degree of hardness, and so on" (Wittgenstein, 1974).

So: a speck in the visual field must have (with probability 1!) some color. It need not be red, but it must be some color, and cannot be more than one color at the same time. So we might say each x, y, z, t of the visual field is "coordinated" with one color and no other. Thus, the colors at each spacetime point are "mutually exclusive and jointly exhaustive" — another essential parallel between the states of the visual field and those of a quantum field.

Now, since any point in the visual field can be a different color from its neighbor, it is rather as though each point in the visual field has an "internal" color space attached to it — perhaps a color sphere such as we have just now mentioned, with red, green and blue for its principle axes. Curiously, this picture is strikingly similar to the mathematics of gauge theory, with its algebra of "internal" spaces, an algebra which determines the form of all particle interactions.

Our color sphere would also seem to resemble the compactified (very small) orbifolds of string/M-theory, which are thought to sit over each point in spacetime. So that's kind of suggestive. But is the spacetime of physics the same thing as the spacetime of perception? As Lindsay and Margenau argue in their scholarly and companionable work on the *foundations of physics,* there are important respects in which perceptual space and physical space would seem not identical (though they also state, together with Einstein, Mach, and others, that "public or physical space is an abstraction by the mind of the aggregate of the various modes of sense perception").

We believe that the answer to their reasonable concerns was foreshadowed just now by Weyl, and is furthered in Poincaré, where he writes:

> It is often said that we "project" into geometric space the objects of our external perception; that we "localize" them.
>
> Has this a meaning, and if so what?
>
> Does it mean that we represent to ourselves external objects in geometrical space?

> Our representations are only the reproduction of our sensations; they can therefore be ranged only in the same frame as these, that is to say, in perceptual space.
>
> It is as impossible for us to represent to ourselves external bodies in geometric space, as it is for a painter to paint on a plane canvass objects with their three dimensions.
>
> Perceptual space is only an image of geometric space, an image altered in shape by a sort of perspective (Poincaré 1913).

We would argue that our perceptual spaces are projections of a *complete* spacetime, and in a wholly physical, quantum mechanical sense of "projection". Unfortunately, a discussion of the mathematics of the "projection postulate" of QM would lead us too far afield here, though we have touched upon it just now, in our discussion of Hilbert spaces and the projection of v onto a triplet of rays. We would simply note that we are quite accustomed to using photons to project images, in movies and television, e.g., and then of course the visual world is projected onto our retinas by photons, which retinal image is then projected to other areas of our brains by processes "electromagnetic in origin". And then, of course, in projecting an image on a screen, we routinely project the colors of that image.

Let's look in on Weyl again and see what he has to say about colors and projective geometry:

> Mathematics has introduced the name 'isomorphic representation' for the relation which according t points of a convex section of the projective plane — which are bound up with one another by certain fundamental relations R, R', . . .; here, besides the continuous connection of the points, it is only the one fundamental relation: 'The point *C* lies on the segment *AB*'. In projective geometry no notions occur except such as are defined on this basis. On the other side, there is given a second system 2 of objects — the manifold of colors — within which certain relations *R, R',* . . . prevail which shall be associated with those of the first domain by equal names, although of course they have entirely different intuitive content. Besides the continuous connection, it is here the fundamental relation: '*C* arises by a mixture from *A* and *B*'; let us therefore express it somewhat strangely by the same words we used in projective geometry: 'The color *C* lies on the segment joining the colors *A* and *B*'. If now the elements of the second system 2 are made to correspond to the elements of the first system 1 in such a way, that to elements in 1 for which the relation *R,* or *R'*, or . . . holds, there always correspond elements in 2 for which the homonymous relation is satisfied, then the two domains of objects are isomorphically represented on one another. *In this sense the projective plane and the color continuum are isomorphic with one another. Every theorem which is correct in the one system 1 is transferred unchanged to the other (*our emphasis) (Weyl, 1934).

Weyl says that the "projective plane and the color continuum are isomorphic with one another." It may be worthwhile to reflect on the central role of such dualities in string/M-theory and to consider that the compactified orbifolds of that theory of are known to be projective spaces. Then, too, the Riemannian geometry of relativity is known to be a special case of projective geometry, as Kline relates:

It became possible to affirm that projective geometry is indeed logically prior to Euclidean geometry and that the latter can be built up as a special case. Both Klein and Arthur Cayley showed that the basic non-Euclidean geometries developed by Lobachevsky and Bolyai and the elliptic non-Euclidean geometry created by Riemann can also be derived as special cases of projective geometry. No wonder that Cayley exclaimed, "Projective geometry is all geometry" (Kline, 1955).

Are perceptual fields quantum fields? Well, maybe, but there are other reasons to call upon quantum theory in discussions of consciousness, for as Lockwood related just now, "it is a clear implication of the Russellian view that the material world, or more specifically, that part of it that lies within the skull, cannot possess less diversity than is exhibited amongst the phenomenal qualities that we encounter within consciousness". Later, he writes: "If mental states are brain states, then introspection is already, it seems to me, telling us that there is more to the matter of the brain than there is currently room for in the physicist's philosophy."

Just so. And here is the crux of our problem: remember that we are routinely told that the wavefunction provides the most complete description of the system that is, in principle, possible. As is well known, Einstein found all this quite implausible: "one arrives at very implausible theoretical conceptions, if one attempts to maintain the thesis that the statistical quantum theory is in principle capable of producing a complete description of an individual physical system" (Holland, 1993).

But so did the author of the Schrödinger wave function:

If you ask a physicist what is his idea of yellow light, he will tell you that it is transversal electromagnetic waves of wavelength in the neighborhood of 590 millimicrons. If you ask him: But where does yellow come in? he will say: In my picture not at all, but these kinds of vibrations, when they hit the retina of a healthy eye, give the person whose eye it is the sensation of yellow (Schrödinger, 1959).

IV.A little knowledge

There are those who may object at this point, saying something like 'but everybody knows that color is just the frequency (or wavelength) of light.' To which we reply that Schrödinger seems to have known a thing or two about physics and was, moreover, the leading authority on color science in his day, so we might ask how this fact escaped him? Then again, if color has been identified as somehow being equal to (say) wavelength, how was this identity established? And by whom? And when? And why is color sometimes identified with the frequency of light, whereas other authorities casually identify color with wavelength or energy? And why are such well-established facts nowhere supported in the literature of physics? And how is it possible to write entire texts on optics and never once employ the word, "color"? More importantly for science, since a wavelength is a length, and therefore a scalar quantity, why does color behave like a vector quantity?

Here is Weyl to help us out again:

> The idea of the merely subjective, immanent nature of sense qualities, as we have seen, always occurred in history woven together with the scientific doctrine

> about the real generation of visual and other sense perceptions…Locke's standpoint in distinguishing primary and secondary qualities corresponds to the physics of Galileo, Newton, and Huyghens; for here all occurrences in the world are constructed as intuitively conceived motions of particles in intuitive space. Hence an absolute Euclidean space is needed as a standing medium into which the orbits of motion are traced. One can hardly go amiss by maintaining that the philosophical doctrine was abstracted from or developed in close connection with the rise of this physics (Weyl, 1934).

Let us pause to sweep away a few centuries of dust. Rather than indulge the herd, we prefer to follow the lead of Maxwell, Schrödinger, Russell, Weyl, Einstein et al., all of whom devoted serious thought to these matters, and generally dismiss what "everyone knows" as a kind of shorthand for "so much sloppy thinking". We prefer to point out the historical facts, and foremost among these must be the fact that the "hard problem" flows directly from the old division of the elements of perception into two camps, viz., the primary and the secondary. According to this doctrine, bequeathed to us by the fathers of science, such "primary" qualities as extension in space and duration in time are thought to inhere in material objects themselves, to be "physical quantities" as we now say. Whereas such "secondary" qualities as color and sound have long been thought to exist only "in the mind", perhaps as a result of the movements of the brain's material constituents — even though those constituents have, *by definition*, no such secondary properties.

This old division, of primary and secondary, has its roots in Euclid and the Greek atomists, but it was Kepler, Galileo, Newton and Boyle, together with Descartes, Locke, Hobbes, and that crew who enshrined this duality at the foundations of physical theory. So the doctrine has a fine pedigree, and persists in our own time, in a largely unexamined light, as part of what "everyone knows" - though as scientists, our first question must be: is the division of primary and secondary true to nature? A modern statement of the primary/secondary doctrine, and a telling response to it, are found in Einstein's essay "On Russell's Theory of Knowledge". Einstein quotes Russell:

> We all start from 'naive realism,' i.e., the doctrine that things are what they seem. We think that grass is green, that stones are hard, and that snow is cold. But physics assures us that the greenness of grass, the hardness of stones, and the coldness of snow, are not the greenness, hardness, and coldness that we know in our own experience, but something very different. The observer, when he seems to himself to be observing a stone, is really, if physics is to be believed, observing the effects of the stone upon himself. Thus science seems to be at war with itself: when it means to be most objective, it finds itself plunged into subjectivity against its will.

> Apart from their masterful formulation these lines say something which had never previously occurred to me. For, superficially considered, the mode of thought of Berkeley and Hume seems to stand in contrast to the mode of thought in the natural sciences. However, Russell's just cited remark uncovers a connection: If Berkeley relies upon the fact that we do not directly grasp the "things" of the external world through our senses, but that only events causally connected with the presence of "things" reach our sense-organs, then this is a consideration which gets its persuasive character from

our confidence in the physical mode of thought.

What aspect(s) of the "physical mode of thought" might be at fault? We would urge that the ontology of QM is *prima facie* incomplete, and in the sense of EPR, for not every "element of reality" is represented within the theory — namely, greenness, hardness, and coldness, together with the other secondary properties.

Such statements as the foregoing are apt to result in hands waving, tempers flaring, and pedants thundering in an impressive display of cognitive dissonance. All very entertaining, but typically not terribly enlightening. Alas, the issue will not go away, for the textbooks tell us that all we can know about a state is contained "in principle" in Schrödinger's equation, and yet Schrödinger himself says otherwise. Then, too, many of our most stellar scientists past and present have pondered these issues, and with divergent results. The moral would seem to be that while we are certainly in good respectable scientific company here, these foundational issues are by no means settled, and so a measure of controversy is to be expected, if not deliberately incited.

The fact that so many intelligent, scientifically minded people will try to dodge the mind/body issues altogether, however, might suggest that we are, perhaps, not all that confident in the "physical mode of thought" when it comes to greenness and coldness and hardness and so forth. Then again, there would seem to be those who are altogether too confident in that mode of thought — or, at any rate, its standard ontology. How else to explain those learned persons who reason that, if colors have no place in science, then colors must be nonexistent or illusionary? At a glance, and put like that, such a stance must strike us as absurd. But celebrated scholarly works have been written in recent years which make just such arguments, albeit in more sophisticated terms, and though the absurdity of the conclusion remains the same. But we needn't be too harsh on our esteemed colleagues. For as an eminent contemporary authority argues, this old doctrine of Newton and company, this divorce between the primary and secondary qualities, made practical good sense at the time, when our fledgling science was just getting on its feet, though it obviously raises a number of serious issues once we attempt to discover a science of consciousness:

> The world as described by natural science has no obvious place for colors, tastes, or smells. Problems with sensory qualities have been philosophically and scientifically troublesome since ancient times, and in modern form at least since Galileo in 1623 identified some sensory qualities as characterizing nothing real in the objects themselves...
>
> The qualities of size, figure (or shape), number, and motion are for Galileo the only real properties of objects. All other qualities revealed in sense perception — colors, tastes, odors, sounds, and so on — exist only in the sensitive body, and do not qualify anything in the objects themselves. They are the effects of the primary qualities of things on the senses. Without the living animal sensing such things, these 'secondary' qualities (to use the term introduced by Locke) would not exist.
>
> Much of modern philosophy has devolved from this fateful distinction. While it was undoubtedly helpful to the physical sciences to make the mind into a sort of dustbin into which one could sweep the troublesome sensory qualities, this stratagem created difficulties for later attempts to arrive at some scientific understanding of the mind. In

particular, the strategy cannot be reapplied when one goes on to explain sensation and perception. If physics cannot explain secondary qualities, then it seems that any science that can explain secondary qualities must appeal to explanatory principles distinct from those of physics. Thus are born various dualisms (Clark, 1993).

Can physics explain secondary qualities? It seems it must attempt to do so, for our scientific picture of the world rests on sense perception, and agreement with observation must be the ultimate basis of the empirical content of science. And sense perception regularly, mechanically discloses to us these secondary properties of color and sound and so forth, and in predictable spacetime association with physical, (quantum!) mechanical stimuli. And yet, as Schrödinger tells us, these properties have no agreed-upon place in the formalisms of traditional physics.

So it is, then, that we find ourselves still in Leibniz' mill, unable to explain such a simple thing as a perception of yellow in respect of the brain's machinery — essentially the same problem we find in Penrose' challenge to strong AI: Where in all the software and circuitry is the mind? And again in Chalmers: If the brain is a physical thing, composed of particles which, by definition, contain no yellow, then what is the yellow doing there?

And this is the crux of the mind/body duality, as Hume understood in his day, when the whole scam was going down, though we have yet to learn the lesson today:

The fundamental principle of that philosophy is the opinion concerning colors, sounds, tastes, smells, heat and cold; which it asserts to be nothing but impressions in the mind, deriv'd from the operation of external objects, and without any resemblance to the qualities of the objects.

This principle being once admitted, all other doctrines of that philosophy seem to follow by an easy c
any adequate notion…

Thus there is a direct and total opposition betwixt our reason and senses... When we reason from cau

Back to the present, we have said that the answer to the duality of mind and body is contained in its essentials in Chalmers and Feigl. Now we see that we must deal with the duality of primary and secondary, for in the case of the visual field and its concomitant photon field, we want to argue that the two information states, of color and photon, are something like distinct but complementary aspects of a single, more complete state. How to resolve the duality of primary and secondary?

V. More light

Let us attend Mach, where, in his *Analysis of Sensations*. he writes, in open contradiction to Galileo and his gang: “A color is a physical object as soon as we consider its dependence, for instance, upon its luminous source, upon temperatures, upon spaces, and so forth”.

We seem to have Mach’s OK so far as treating color as a physical, and not a mental, thing. His reasoning seems quite sound to us, though Mach was criticized in his time for confusing ‘things’ with our ‘perceptions of things’.

In what follows we cheerfully ignore Mach’s critics and, for the sake of argument, generally assume the truth of his just quoted remarks, and see where that might take us. In fact, Mach’s argument regarding the physical nature of color (and, by extension, sound, etc.,) contains the only truly fundamental change in physical theory required herein. Now, there is no escaping the fact that it is a fundamental change, for as Hume noted just now, our entire scientific world view flows from this one crucial divorce between primary and secondary. In repairing this rift, we hope to achieve a unity, harmony, and identity between mind and brain, but we cannot even mention all the many possible objections that might be raised to our course in this brief space. Rather, we take the stance that the proof is in the pudding, and argue that the inclusion of the secondary properties among the elements of physics makes for a more consistent, coherent, and complete account of nature. But since the physical nature of color is such a very important point, perhaps a few words would be in order by way of justifying our position.

Why should color not be physical? Well, the color of a thing is supposedly subjective, whereas the length of a thing, e.g., is surely an objective aspect of that thing. Yet relativity tells us that both the length and the color of a thing are dependent upon the observer's state of motion — and in a perfectly objective way. Can relativity help us out here? Indeed it can, as the redoubtable Weyl relates:

The immediately experienced is subjective but absolute; no matter how cloudy it may be, in this cloudiness it is something given thus and not otherwise. To the contrary, the objective world which we continually take into account in our practical life and which science tries to crystallize into clarity is necessarily relative; to be represented by some definite thing (numbers or other symbols) only after a system of coordinates has been arbitrarily introduced into the world. We said at an earlier place, that every difference in experience must be founded on a difference of the objective conditions; we can now add: in such a difference of the objective conditions as is invariant with regard to coordinate transformations, a difference that cannot be made to vanish by a mere change of the coordinate system used ... (Weyl, 1936).

Let's pay close attention to what Weyl said just now about "every difference in experience must be founded on a difference of the objective conditions" and "in such a difference of the objective conditions as is invariant with regard to coordinate transformations" Why is this important? Well, relativity flows from the invariance of the laws of nature with regard to coordinate transformations, and we now know that such invariances or symmetries inform the very foundations of all physical theory. So we might do well to attend to the fact that the color of a thing, e.g., remains the same under translations, rotations, and reflections in spacetime — other things being equal. Or as Helmholtz put it, with brilliant clarity: "Similar light produces under like conditions a like sensation of color."

VI.Symmetry

Now, in quoting Helmholtz on this point in the past, I have had not uneducated persons object that, if you change the lighting, or the room, the color changes. On hearing such replies, I am now inclined to think that the other party (a) is not heeding the clause concerning "under like conditions"; and (b) is not yet fully cognizant of the import of her words; and (c) constitutes part of a scientific culture which strains under a large burden of ignorance concerning the secondary properties. So what? Well, it could scarcely be more important. For yes, colors do change when the light changes — and colors always change in the same way, here or there, now or then, always and everywhere, world without end. It is very like a law of nature. It is quite a lot like an invariance or symmetry of nature, such as those now known to determine the evolution of the state vector in QM.

Furthermore, and now this is the point, this is the punch line, the symmetries determine the action. This action, this form of the dynamics, is the only one consistent with these symmetries ... This, I think, is the first time that this has happened in a dynamical theory: that the symmetries of the theory have completely determined the structure of the dynamics, i.e., have completely determined the quantity that produces the rate of change of the state vector with time.

This sounds interesting, but a natural question arises: What are these symmetries Weinberg speaks of, precisely? Earlier on, he writes:
"Increasingly, many of us have come to think that the missing element that has to be added to quantum mechanics is a principle, or several principles, of symmetry. A symmetry is a statement that there are various ways that you can change the way you look at nature, which actually change the direction the state vector is pointing, but which do not change the rules that govern how the state vector rotates with time. The set of all these changes in point of view is called the symmetry group of nature. It is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today" (Weinberg, 1987).

This sounds rather abstract, perhaps, but the symmetries Weinberg speaks of are directly akin to the usual sorts of symmetries familiar to us from daily observation, as well as from the most basic tenets of relativity:

The meaning of symmetry of a physical system is frequently influenced, if not shaped, by the guidelines of our investigation.

It is obvious that the symmetry of a physical system is closely related to the transformations of the parameters describing the system. Notice, however, that not every transformation of parameters is linked to a symmetry of the system; such symmetries have to satisfy certain conditions. The necessary condition is that the physical system remains the same object of our perception before as well as after the transformation . . .
We say that a 3-dimensional sphere has a rotational symmetry because the picture of it does not change while we rotate it through an angle around an arbitrary axis through the center of this sphere. . . Let us take as an example the relativistic field theory. This theory as a whole is symmetric with respect to the Lorentz transformations. This means that independently of the choice of frame of reference, the same field theory is the object of our investigation; changing from one frame to another the fields transform covariantly according to the rule imposed by the principle of relativity.

Let's carefully consider the following bit, in relation to perception: "The necessary condition is that the physical system remains the same *object of our perception* before as well as after the transformation" (our emphasis). We are moved to reflect that, other things being equal, the color of a laser (say) does not change as a result of moving the laser about in spacetime. Similarly, other things being equal, a tuning fork will sound the same, here or there, now or then, always and everywhere. So we might ask: Do the symmetries of color and sound and so forth also contribute to the "structure of the dynamics"? Perhaps, but we want to go a bit further and tighten up our thesis as to how such simple invariant properties as "yellow" might be incorporated into the body of quantum theory. Before doing so, we really must have a glance at Noether's theorem: "If the Schrödinger equation has a symmetry with respect to some unitary group, then the observable corresponding to the self-adjoint generator of the group is a constant of motion." On this slightly more technical note, we point out that the color vector associated with the Schrödinger state vector of a photon would seem to be symmetric — a "constant of motion" — with respect to the group of translations, rotations, and reflections.

VII.Simplify, simplify

Let us attend to the simplicity of colors. For colors are so simple, we might think of them as elemental, and so perhaps count them among the proper elements of an EPR-complete quantum theory. What does this mean? Let's remind ourselves of what Einstein & Co., said in their seminal work on the (in)completeness of QM:

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience...

Whatever the meaning assigned to the term complete, the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counterpart in the physical theory (Einsterin, 1935).

We see that our investigation naturally leads us to the question: Are the secondary properties among the "hidden variables" of QM? Well, "everybody knows" hidden variables are generally out of favor at the moment. But a glance at the history of modern physics will reveal that hidden variables theories have never been in vogue. Holland gives an account of the sober scientific reception with which discussion of hidden variables has historically been greeted.

The reaction to Bohm's work by the Copenhagen establishment was generally unfavourable, unrestrained and at times vitriolic (Rosenfeld, 1958).

We first consider the response of Heisenberg (1955; 1962, Chap. 8) to Bohm's contribution, and see in outline how the points he raises may be answered (see also Bohm (1962) for a reply). Heisenberg first questions what it means to say that a wave propagating in configuration space is 'real'. His objection to this notion is based on his assertion that only 'things' in three-dimensional space are 'real'. He offers no logical or scientific argument to show that examining the possibility of multidimensional spaces is a fruitless enterprise. It is useful to recall here the Kaluza-Klein program in general relativity where physicists contemplate spacetimes of dimension greater than four as a valuable aid to comprehending and unifying the basic physical interactions.

The 'hidden parameters', i.e., the particle orbits, are denounced by Heisenberg as a 'superfluous 'ideological superstructure'" having little to do with immediate physical reality because the causal formulation generates the same empirical results as the Copenhagen view. *But in Bohm's theory it is precisely the positions of particles that are recorded in experiments; they are the immediately sensed 'reality' (Holland, 1993).*

Well, so what of these "hidden" variables or parameters? What are they, precisely?

A "hidden-variable" theory, as the name implies, postulates that alongside (or, more graphically, beneath) the measurable quantities dealt with by the theory (position, momentum, spin, and so on) there are further quantities inaccessible to measurement, whose values determine the values yielded by individual measurements of the observables. The quantum mechanical statistics are to be obtained by "averaging" over the values of the hidden variables. The inaccessibility of these variables may be a contingent and temporary matter, to be remedied as we develop new experimental procedures, or these quantities may be in principle inaccessible (see Jammer, 1974, p. 267).

The suggestion that there may be such "hidden variables" is as old as the probabilistic interpretation of the state vector. It was made by Born (1926b, p. 825) a few months after he first proposed that interpretation: "*Anyone dissatisfied with these ideas may feel free to assume that there are additional parameters not yet introduced into the theory which determine the individual event" (Holland, 1993).* But almost as old is the denial that such hidden variables can exist.

VIII.Manifest variables

It is well worth noting that Born's just cited remark, that we "may feel free to assume that there are additional parameters not yet introduced into the theory which determine the individual event" is rarely to be found in the standard literature on the subject. Having attended for many years the ongoing discussions among those more serious thinkers who are little concerned with intellectual fashion, we are inclined to agree with Gell-Mann, who writes of Bohr and Heisenberg having "brainwashed" a generation of physicists. While we have no interest in generating a diatribe, it is at least curious to consider mainstream physicists embracing many unobserved worlds in order to skirt the observer problem, even if this move entails throwing Occam out the window. Whereas we have daily experience of (secondary) variables which, though "hidden" by dogma, are nonetheless in observed association with those quantum mechanical things known as photons. Happily, according to Wheeler and Tegmark the reign of Copenhagen appears to have come to an end, and so perhaps it is not too much to hope for a more reasoned and informed debate of foundational issues - something beyond the "shut up and calculate" approach.

Now, there are also not a few today who, upon reading these words, will object and say: Yes, but our perception of yellow depends on the neurological state of the observer, therefore color is "subjective", and therefore a subject for neurology, and not for physics. Well, this is only Galileo warmed over, but all right, we can reply that our perceptions of space and time are also dependent on the observer's neurological state. Are space and time therefore subjective? Well, no ... for scientific purposes we agree on units of space and time — the rigid rods and standard clocks of relativity, e.g.. But we also employ such standards with respect to colors and sounds. Our color televisions and stereos physically incorporate such standards, and in fact we know quite a lot about how to add waveforms in order to produce this color or that sound. With respect to a future geometry of color and sound, we are arguably today in something like the situation which obtained with Euclidean geometry before Euclid came along — we have quite a lot of practical knowledge available, but no unifying theory.

As for color perception being a problem for neurology, one could also point out that neurons are, presumably, physical things, and therefore subject matter for physics. But I have a sneaking suspicion that those who argue that color is not a problem for physics are not really troubled by issues of reason, logic, or history so much as by the (rather daunting) prospect of overhauling the standard ontology of physics — what naturally strikes many as a radical move, somehow going against the grain of what they take physics to be about. I believe that many physicists and their admirers have a problem in their gut with a physics that is full of greenness, hardness, and coldness — and they have centuries of accumulated scientific success to back them up, together with the weight of illustrious authority, which has stated from the outset that colors and sounds do not belong to the world of physics. Now, one could counter such objections by simply noting that colors and sounds are observed properties of the world, and it is the business of physics to account for such properties and their manifest regularities, and never mind the appeals to authority, or trying to sweep the problem under the rug of mind, metaphysics, or neuroscience, because we've tried all that and it simply doesn't work. And these would be reasonable replies, up to a point.

Then again, to prevail against authority we can do more than patiently argue the facts. We can bring in other, equally weighty authorities to reassure us that we are on safe, intellectually respectable premises. We can call upon the likes of Leibniz, Hume, Young, Helmholtz, Mach, Maxwell, Weyl, Schrödinger and Einstein, all of whom devoted serious thought to these matters — as did Newton himself, the father of color science (as well as a physicist of no small stature), who wrote that "the science of colors becomes a speculation as truly mathematical as any other part of physics" (Newton, 1979).

A more cogent concern from standard physics might be: QFT is our most precise science, and its many successes are legendary. How would enlarging upon the ontology of QFT help in any way? The surprising but natural reply is that the inclusion of the secondary properties of color and sound and so forth would make for a more *complete* physical theory. Where, again, "complete" is used in the sense of EPR, where every "element of reality" is represented in the theory. Again, this move amounts to an identification of the secondary properties of observation with the hypothetical and much discussed "hidden variables" of QM. (Which identification would amount to an extraordinary irony, since colors and sounds are not "hidden" from us, except in plain sight and within hearing.)

But again, as everyone knows, local hidden variables theories are thought to have been dealt a death blow by the ingenious experiments of Aspect et al., and so this loose identification of secondary properties with hidden variables would seem not very promising at the outset, and, worse, might serve only to complicate matters where no such help is needed, thank you very much.

Our reply is manifold: (1) we are in sympathy with Bell, who wrote that what "no HV theories" demonstrate is a lack of imagination; (2) we seem to be stuck with nonlocality anyway, so the door would appear to be open just there — perhaps nonlocal communication occurs via the space of secondary properties? for (3) colors are nonlocal and atemporal in the sense that two photonic vectors of identical energy will exhibit identical color vectors for the standard observer — thus, even though the photons be lightyears apart, they would nonetheless appear to occupy the same "point" in color space; and (4) proving #3 quite suggestively requires the same precise mathematics employed by relativity and gauge theory generally, i.e., parallel transport in a curved spacetime; and then (5) the compactified dimensions of string/M-theory are well accepted, and yet these additional spatial variables are, at present, "hidden"; whereas (6) the secondary properties are manifestly spatio-temporal in the sense that they predictably intersect spacetime — much like the compactified dimensions of string/M-theory are supposed to do; and (7) in an empirical science, in which sensory experience must be the final court of appeal, the existence of observed entities must carry more force than rationalistic arguments (such as Von Neumann's) which purport to deny the possibility of such entities; whereas (8) the theory of real numbers is very beautiful and powerful, but the theory's beauty and power are only enhanced by the addition of the complex numbers; and, (9) a coherent scientific account of the secondary properties and of consciousness generally would constitute nothing less than a major advance in our physical understanding of the natural world *in toto;* and, finally, (10) physicists already make routine use of colors in describing such physical things as blue dwarfs, red shifts, gold and silver, and so on.

And indeed Mendeleev, in constructing the periodic table, was guided by the secondary properties of the elements, and so, in a sense, the present proposal only amounts to giving formal recognition to variables already in daily scientific use.

Polemics aside, it would seem as though we have on one hand the secondary properties, looking for a home in the formalisms of physics, whereas on the other hand we have these empty spaces in various contemporary physical theories. So we want to see whether we can "coordinate" these empty slots with the homeless secondary properties - in the case of color and vision, we want to see whether we can match the states of the photon field with the states of the associated visual field.

IX.Manifolds

Let's pick up the mathematical line again, and remind ourselves of Riemann, who, at the beginning of his famous habilitation lecture, points out that colors and space are both manifolds:

So few and far between are the occasions for forming notions whose specialisations make up a continuous manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colors. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

Definite portions of a manifoldness, distinguished by a mark or a boundary, are called Quanta ...

The upshot would seem to be as follows: We have two states consisting of two kinds of vectors occupying two manifolds. And the two always and everywhere co-vary. Which leads us back to Weyl:

In the realm of physics it is perhaps only the theory of relativity which has made it quite clear that the two essences, space and time, entering into our intuition, have no place in the world constructed by mathematical physics. Colors are thus "really" not even aether-vibrations, but merely a series of values of mathematical functions in which occur four independent parameters corresponding to the three dimensions of space, and the one of time (Weyl, 1922).

Notice that Weyl has no trouble coordinating colors with spacetime. Notice, too, though, the tendency to diminish the role or nature of color: "Colors are "really" not even aether-vibrations" but "merely" values of functions. Our essential departure with much traditional thinking along these lines consists in regarding colors and so forth as *elemental,* though of course in saying so we are well within the tradition found in Maxwell, Wittgenstein, Weyl, and Russell & Whitehead.

So, what of these theories with their additional dimensions? What can we say for sure about them? First, a bit of background from one of the master architects of string theory:

I'm sympathetic to the view that these theories are at present very remote from being able to explain directly what is measured experimentally in accelerator laboratories. Given the fact that they are so very different from previous kinds of theories, then they ought to predict some entirely new sort of phenomenon that we haven't even thought of measuring. It was only *after* Einstein had formulated general relativity that he understood which phenomena that could be measured, would test the theory. The precession of the perihelion of the planet Mercury was already known, but it wasn't until Einstein came up with general relativity that it was realized that this peculiar anomaly was of fundamental importance. So what we need in superstring theory is the analogue of the planet Mercury. *Some distinctive piece of experimental evidence that might already be known but hasn't struck anyone as being important because no one realizes that it's of relevance to testing a fundamental theory* (our emphasis) (Davies, 1986).

On the present view, should the secondary properties turn out to reside in the additional spatial dimensions of something like M-theory, then that would seem to meet the foremost objection to string/M-theory, that it does not meet up with observation.

Green also writes, "Well, obviously the extra dimensions have to be different somehow because otherwise we would notice them." This is curiously parallel to a remark from David Bohm: "Now it may be asked why these hidden variables should have so long remained undetected."

John Schwarz, another among the principal architects of string/M-theory, provides us with a measure of context and motivation regarding the additional dimensions:

If we knew what that six-dimensional space looked like we would be in a great position for calculating all sorts of things that we want to know. This may sound surprising. After all, as I have already said, this space is completely invisible because it's too tiny to observe directly. As it turns out the details of its geometry and topology actually play a crucial role in determining the properties of observable particles at observable energies (Davies, 1986).

On the present view, the just cited remarks from Bohm, Green, and Schwarz must strike us as quite ironic, given that we do notice or detect colors and sounds all the time, as a consequence of the only directly observable particles we know — i.e., photons. So why have the secondary properties not been put forward heretofore to occupy these "hidden" variables and extra dimensions? Part of the answer must lie in the fact that colors and sounds have historically been excluded from the physical world, even though they demonstrably co-vary with other physical parameters. Another part of the answer is contained in an observation from Wittgenstein, where he writes that "the things that are most important for us are hidden from us by their simplicity and familiarity." And then, of course, the dimensions of color and sound and so forth are different from the dimensions of traditional spacetime; they are more like the "internal" dimensions of gauge theory or the compactified (very small) dimensions of string/M-theory — and like these more traditional physical dimensions, the dimensions of color and sound are tangent to the points of spacetime, suggesting that colors and sounds might be amenable to the mathematics of fiber bundles.

Another important class of field theories, having a familial relation to M-theory, are the Kaluza-Klein theories:

There are other kinds of unitary field theories, including some that today claim a great deal of interest. These utilize, in some way or other, an increase in the number of dimensions of space-time. One famous example is Kaluza's proposal. He increased the number of dimensions to five, without changing the Riemannian character of the model. He was thus able to increase the number of components of the metric so as to accommodate the electromagnetic field as well. He set one extra component equal to a constant, because he had no use for it. To account for the observed four-dimensionality of space-time, he assumed that no field depended on the fifth coordinate (Bergmann, 1943).

Note: "To account for the observed four-dimensionality of space-time, he assumed that no field depended on the fifth coordinate." But the spacetime of our perceptions is not four-dimensional, for to each point in the visual field, e.g., must correspond a color, which requires an additional parameter or dimension. The assumption that the additional spatial dimensions must be exceedingly small because we do not "see" them dates back to the seminal work of Kaluza:

Although all our previous physical experience hardly provides any suggestion of such an extra world-parameter, we are certainly free to consider our space-time to be a four-dimensional part of an R5; one then has to take into account the fact that we are only aware of the space-time variation of quantities, by making their derivatives with respect to the new parameter vanish or by considering them to be small as they are of higher order ... (Kaluza, 1987).

Cao brings out the inter-relatedness of the extra dimensions of gauge and M-theory, together with the mathematics of fiber bundles:

Just as in the original spinning string model, [the] superstring also requires ten-dimensional space-time. So, to be of relevance to physics, the extra six dimensions must compactify and be very small . . .

From the above brief review, we find there are three versions of geometrization of non-gravitational gauge interactions:

1. Fibre-bundle version, in which the gauge interactions are correlated with the geometrical structures of internal space. Since it is possible to get a non-trivial fusion of space-time with internal space, the gauge interactions also have some indirect relation with space-time geometry. But the essence of the internal space is still a vexing problem: Is it a physical reality as real as space-time, or just a mathematical structure?

2. Kaluza-Klein version, in which extra space dimensions which compactify in low-energy experiments are introduced and the gauge symmetries by which the forms of gauge interactions are fixed are just the manifestation of the geometrical symmetries of the compactified space. Here the mediator between the gauge interactions and the space-time geometry is no longer the vexing internal space but the real though compactified extra space dimensions. The assumption of the reality of the compactified space is substantial and is in principle testable, although its ad-hoc-ness makes it difficult to differentiate it from the internal space in the fibre-bundle version.

3. Superstring version, in which the introduction of extra compactified space dimensions is due to different considerations from just reproducing the gauge symmetry. Therefore, the properties and structures of the compactified dimensions are totally different from those in the Kaluza-Klein version. For example there is no symmetry in the compact dimensions from which the gauge symmetries emerge; the gauge interactions are correlated with the geometrical structure of ten-dimensional space-time as a whole but not just with the extra dimensions.

OK, but are we really justified in thinking of secondary properties as dimensions? Let us attend Minkowski in his famous essay on the unity of space and time:

We will try to visualize the state of things by the graphic method. Let x, y, z be rectangular co-ordinates for space, and let t denote time. The objects of our perception invariably include places and times in combination. Nobody has ever noticed a place except at a time, or a time except at a place. . . . The multiplicity of all thinkable x, y, z, t systems of values we will christen the world.

We might add that nobody has ever "noticed" a place except that it was associated with a secondary quality and that the "objects of our perception" invariably include such qualities. Our scientific needs are served quite simply, though: To all x, y, z, t systems of values we can simply associate appropriate vector coordinates for the corresponding secondary properties. Since the color at a point in spacetime does not depend the system of coordinates chosen, we have a straight path into the heart of both relativity and quantum theory. Moreover, "the tensor calculus emerges as the natural framework with which to address such matters ..." as P.M. Churchland says of neural networks. But is this move justified on mathematical grounds? Here is Weyl again to help us out again: "The characteristic of an n-dimensional manifold is that each of the elements composing it (in our examples, single points, conditions of a gas, colors, tones) may be specified by the giving of n quantities, the "co-ordinates," which are continuous functions within the manifold" (Nagel, 1956).

X.Logic

Let's remember Weyl in regard to the mapping of colors to the points of the projective plane: "the projective plane and the color continuum are isomorphic with one another. Every theorem which is correct in the one system is transferred unchanged to the other " With this result in mind, let's visit with Newman and Nagel in their classic essay on Gödel's proof:

How did Gödel prove his conclusions? Up to a point, the structure of his demonstration is modeled, as he himself noted, on the reasoning involved in one of the logical antinomies known as the "Richard Paradox," first propounded by the French mathematician, Jules Richard, in 1905...The reasoning in the Richard Paradox is evidently fallacious. Its construction nevertheless suggests that it might be possible to "map" (or "mirror") meta-mathematical statements *about* a sufficiently comprehensive formal system *into* the system itself. If this were possible, then metamathematical statements about a system would be *represented* by statements within the system. *Thereby one could achieve the desirable end of getting the formal system to speak about itself — a most valuable form of self-consciousness.* (Our emphasis) *The idea of such mapping is a familiar one in mathematics. It is employed in coordinate geometry, which translates geometric statements into algebraic ones, so that geometric relations are mapped onto algebraic ones. The idea is manifestly used in the construction of ordinary maps, since the construction consists in projecting configurations* on the surface of a sphere onto a plane . . . *The basic fact which underlies all these mapping procedures is that an abstract structure of relations embodied in one domain of objects is exhibited to hold between "objects" in some other domain.* In consequence, deductive relations between statements about the first domain can be established by exploring (often more conveniently and easily) the deductive relations between statements about their counterparts. For example, complicated geometrical relations between surfaces in space are usually more readily studied by way of the algebraic formulas for such surfaces (Nagel, 1956).

Just for the sake of argument, suppose we interpret the elements of a formal theory T (of a very general nature) as Feigl's "immediately experienced qualities". Say T has a sufficiently rich structure so as to allow it to frame statements about itself within itself, a la Gödel. T will be unable to define its elements, for if it could, its elements would not be elements. Are we similarly unable to define the elements of our experience, the "immediately experienced qualities" in respect of anything simpler? Is this basic fact of experience an artifact of the fundamental logic of our brains and sense organs and of matter generally?

So again we ask: Do the two vector manifolds always co-vary because they are, in fact, two aspects of one manifold? It seems an altogether reasonable question.

XI. Fields

Going a bit further, we could consider the fact that photons are routinely described in terms of fields, as in QFT, where the photon is known as the "exchange particle" of the electromagnetic (EM) field. Consider, then, that the visual field is ... well, a field - and this, according to a simple definition, such as we find in 't Hooft: "a field is simply a quantity defined at every point throughout some region of space and time" (tHooft, 1980).

Or in Dyson: "This is the characteristic mathematical property of a classical field: it is an undefined something which exists throughout a volume of space and which is described by sets of numbers, each set denoting the field strength and direction at a single point in the space" (Dyson, 1953). On the present view, Dyson's "undefined something" in the visual field is just color, because, as Maxwell, Russell, and Wittgenstein have pointed out, color is simply too simple to be defined, and is therefore mathematically "elemental".

Such questions raise many another in their wake — just what is color space, e.g.? Note that we can make a natural mapping from the spectral colors to a color sphere, where Newton's color wheel runs around the circumference, with black and white at the poles. Or such a mapping could be made with red, green and blue for the axes of a unit sphere in Hilbert space. We could then easily map those color vectors to the photonic vectors with which they are associated, remembering that these "physical" vectors recapitulate the mathematics of colors under vector addition and multiplication. Then, any operation upon the photonic vector would naturally correspond to a rotation of the color vector, in a direct analogy with the mathematics of gauge theory and quantum theory generally.

If such a color sphere were to "sit over" every point in 4D space-time, that would seem to square with the facts of our experience of the visual field and also perhaps with the mathematics of string/M-theory, where a 6-dimensional orbifold is often pictured "sitting over" every point in 4D spacetime (for a nice graphic illustration of this notion, see Brian Greene's popular work *The Elegant Universe).*

On a somewhat more technical note, we might reflect on the fact that Calabi-Yau spaces are known to be projective spaces, and to include the SU(3) group of the standard model. So what? Well, as Weyl tells us, colors are isomorphic to the points of the projective plane, SU(3) is thought to govern quantum chromodynamics. The theory is called "chromo" dynamics because quarks and gluons recapitulate the behavior of spectral colors. Now, as 't Hooft, has pointed out, it is usually thought that spectral colors and the colors of chromodynamics have nothing to do with each other ... though no one ever says why this must be so — it is just the sort of thing that everybody knows. Yet photons and gluons are both gauge particles, and photons are continually associated with spectral color. So we might defy convention and ask: Is color a kind of gauge information?

If we were to include spectral color in physical theory, we would seem to need additional dimensions to spacetime in order to account for the color perceived at a point in spacetime. As is well known, there are today various physical theories which require increasing the number of spacetime dimensions or variables, though on different grounds altogether. Thus, e.g., in the original Kaluza-Klein theory, both electromagnetism and gravity were seen to flow from a single spacetime metric, where the number of spatial dimensions is raised from three to four. A similar situation obtains today with string/M-theory, where all the known forces are represented. In both K-K and M-theory the extra dimensions are assumed to be very small because we do not "see" them. We suggest instead that we do, in fact, "see" these other dimensions all the time. We reason that these extra dimensions are perhaps where the secondary properties reside. We are clearly on speculative ground, here, but it seems at least initially plausible that what we think of as spectral color (together with those other "elements of reality" known to us as sound, heat and cold, etc.,) might be kinds of gauge information — which interpretation might then address the supposed causal inefficacy of these phenomenal properties.

Of central importance to such a research program must be the evident differences between the usual spacetime manifold of relativity and the manifolds of 'colors and tones' and so forth. Fortunately, the mathematics of differential geometry provides us with a ready guide in these matters.

Let us first remind ourselves of the proposal with which we began. The three-dimensional encoding of color spaces, for example, suggests that the information state in a color experience corresponds directly to an information state in the brain. We might even regard the two states as distinct aspects of a single information state, which is simultaneously embodied in both physical processing and conscious experience.

Let's tighten that up a bit with some help from Lockwood:

Take some range of phenomenal qualities. Assume that these qualities can be arranged according to some abstract *n*-dimensional space, in a way that is faithful to their perceived similarities and degrees of similarity — just as, according to Land, it is possible to arrange the phenomenal colors in his three-dimensional color solid. Then my Russellian proposal is that there exists, within the brain, some physical system, the states of which can be arranged in some n-dimensional state space ... And the two states are to be equated with each other: the phenomenal qualities are identical with the states of the corresponding physical system. [52]

One way of looking at our present proposal consists in considering each perceptual state space (vision, audition, etc.,) as a projection of a complete, n-dimensional state space that is identically phenomenal and quantum. Remarkably, Lockwood and I arrived at this point, and by many of the same paths, almost simultaneously, though entirely independently — a conclusion with which Lockwood concurs (private communication). All right, then: perceptual states are quantum states. We have been speaking of states and spaces. We have raised the possibility that the secondary qualities might reside in the "internal" spaces of gauge theory, wherein the symmetries of those secondary properties might then contribute to the QM action. So let us attend Atiyah, who requires no introduction among mathematicians (suffice it to say that he has made fundamental contributions to topology and has edited a collection of essays by Fields Medal laureates).

XII. Gauge Potentials and Fields

We shall now recall the data of a classical theory as understood by physicists and then reinterpret them in geometrical form. Geometrically or mechanically we can interpret this data as follows. Imagine a structured particle, that is a particle which has a location at a point x of R_4 and an internal structure, or set of states, labeled by elements g of G. We then consider the total space P of all states of such a particle. In general we conceive of the internal spaces G_x and G_y for x   y as not being identified and so we draw the picture of P as a collection of "fibers".

Here we regard the colors as observed "states" of the photon. In the absence of any external field however we consider that all G_x can be identified to each other so that in addition to the vertical lines or fibers we can also draw horizontal lines (called sections) making the usual Cartesian type of grid. Now we can imagine an external field imposed which has the effect of distorting the relative alignment of the fibers so that no coherent identification is possible between the G_x at different points. However we assume that G_x and G_y can still be identified if we choose a path in R_4 from x to y. In more physical terms we imagine the particle moving from x to y and carrying its internal space with it.

Here I imagine an interaction-free photon of constant energy carrying along its constant or invariant color in some sort of internal space — remembering that this kind of invariance leads us directly into the deep waters of contemporary physical theory, where such invariances or symmetries are understood to determine the quantum mechanical action. Back to Atiyah:

In Minkowski space such a motion would take place along the world line of the particle. This identification of fibers along paths is called "parallel transport". If we now imagine two different paths joining x to y then there is no reason for the two different parallel transports to agree and they are assumed to differ by multiplication with a group element, which could be viewed as a generalized "phase shift". This phase shift is interpreted as produced by the external field. In geometrical terms it is viewed as the total "curvature" or distortion of the fiber bundle over the region enclosed by the two paths (Atiyah, 1979).

Here I would point to the observational fact that a change in a photon's color can be brought about by an "external field" such as a gravitational field, or, by the principle of equivalence, an acceleration field. Also, it seems to make a kind of sense to regard such a change in color as a 'generalized phase shift'. When viewed in respect of the total "curvature" of the fiber bundle, such a phase shift would seem to have a natural analog in Kaluza-Klein theory, where all interactions arise from curvature in an extended metric, in direct analogy with gravity in general relativity. So we seem to have a clear path toward understanding how color might couple to the equations of relativity — i.e., via the red-shift phenomenon. Now, the prediction of the red shift was one of three predictions offered by Einstein in support of the general theory. I would extend the reach of that prediction by asserting that it ought to be impossible to tell, by judging the color of a photon, whether a shift in color to the red was due to a gravitational source, or to a receding source.

Let's get back to the internal spaces, with Cao:

In this case the question of what is the relation between internal and external geometries seems to have turned out to be quite irrelevant. But in a deeper sense the question remains profound. What is the real issue when we talk about the geometrization of gauge field theory? The geometrization thesis only makes sense if the geometrical structures in four-dimensional space-time are actually correlated to the gauge interactions other than gravity or supergravity, or equivalently, if they are mixed with the geometrical structures associated with extra dimensions.

Here I would point to the fact that photons, as "geometrical structures in four-dimensional space-time" are correlated with color vectors — which vectors we might then regard as "geometrical structures associated with extra internal: dimensions." And, as we see, "the geometrical structures in four-dimensional space-time" ... "are mixed with the geometrical structures associated with extra dimensions", in the manifest sense that, to each spacetime vector in (say) our visual manifold there corresponds a color vector.

A question arises: is color phase information? That is not an issue to be settled here, or very soon, but let's have a glance at a bit of introductory gauge theory, just to finish up and look ahead a bit:

The extension of a global symmetry to a local gauge invariance is termed 'gauging the symmetry'...The new fields that are introduced are called gauge boson fields, and the quanta of these fields are called gauge particles. Thus the photon is the gauge boson of the electromagnetic field. A natural question arises: 'when is it useful to gauge a symmetry?'...experimental observation is the arbiter: gauging the U(1) charge symmetry of QED forces the introduction of a gauge boson field with exactly the properties of the experimentally observed photon field (Guidry, 1991).

Here I would simply point to the durable fact that colors, e.g., belong among "the properties of the experimentally observed photon field". However, since colors, together with the other secondary properties, have traditionally been considered nonphysical, their evident group properties have not been taken into consideration heretofore in regard to the electromagnetic field:

Consider the field of the data of sense a field of universal interest and fundamental. We are here in the domain of sights and sounds and motions among other things...Do the colors constitute a group?...Let us pass from colors to figures or shapes to figures or shapes, I mean, of physical or material objects rocks, chairs, trees, animals and the like as known to sense perception ... And what of sounds, sensations of sound? Are sounds combinable? Is the result always a sound or is it sometimes silence? If we agree to regard silence as a species of sound — as the zero of sound — has the system of sounds the property of a group? (Keyser, 1922).

Recall that the addition of color vectors reliably depends upon the phases of their concomitant state vectors. Repairing the rift between primary and secondary, and so between mind and body, would seem to require a larger group of symmetries for the EM field and for nature as a whole. Happily, because the secondary properties are given to us in observation, in predictable conjunction with photons in characteristic *(eigen)* states, the determination of the complete symmetry group is open to our inspection - a fact of some importance given Weinberg's remark that "it is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today."

To sum up and look ahead a little bit, then, we started out with Chalmers reasoning that certain information states in the brain embody both "physical processing and conscious experience". Along with Feigl, we have considered that the relevant states are the "immediately experienced qualities, or their configurations in the various phenomenal fields" which embody the concomitant quantum fields as well. In so doing, we have looked at various points of contact between the mathematics of quantum fields and that of the visual field. However, since the secondary qualities are not mentioned in the formalisms of current quantum theory, and since these properties appear to be elemental, we have looked at the foundations of physical theory for places where additional variables or dimensions might be accommodated, viz., hidden variables theory, gauge theory, and string/M-theory. Since the first of these alternatives is weighed down with an excess of rhetorical baggage, it may be most expeditious to pursue the latter two possibilities, which offer the further advantage of being highly developed mathematically. With these points in mind, let us finish up by recalling Wittgenstein on color space: "a speck in the visual field, though it need not be red must have some color; it is, so to speak, surrounded by color-space. Notes must have some pitch, objects of the sense of touch some degree of hardness, and so on".

It is as though each speck in the visual field is tangent to color space — as though a color sphere "sits over" each spacetime coordinate of the visual field, in direct analogy with string/M-theory and Kaluza-Klein theory. Now let us reflect on the following, where TpX is the tangent space of X at p. and where v is a color vector living on a color sphere associated with a photonic state vector moving along the curve C:
First, let X be a real differentiable manifold of real dimension n, and let v TpX.
Assuming that X is equipped with a metric g and the associated Levi-Civita connection. we can imagine parallel transporting v along a curve C in X which begins and ends at p. After the journey around the curve, the vector v will generally not return to its original orientation in TpX. Rather, if X is not at, v will return to p pointing in another direction, say v' (since we are using the Levi-Civita connection for parallel transport, the length of v will not change during this process) If X is orientable, the vectors v and v' will be related by an SO(n) transformation AC, where the subscript reminds us of the curve we have moved around. That is

$$v' = AC_v : (2.44)$$

Now consider all possible closed curves in X which pass through p, and repeat the above procedure. This will yield a collection of SO(n) matrices AC1 ; AC2 ; AC3 ; :::, one for each curve. Notice that if we traverse a curve C which is the curve Ci followed by the curve Cj , the associated matrix will be ACj ACi and that if we traverse the curve Cj in reverse, the associated matrix will be $A^{-1}Cj$.

Thus, the collection of matrices generated in this manner form a group — namely, some subgroup of SO(n). Let us now take this one step further by following the same procedure at all points p on X. Similar reasoning to that just used ensures that this collection of matrices also forms a group.

Well, when Brian Greene wrote the foregoing, he was explaining Calabi-Yau spaces with respect to string theory. But if I am not mistaken, his remarks above apply equally well to color vectors, and so, echoing Weyl, it seems like it might be useful to develop this geometry a little further.

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