Preliminary version: comments welcomed.

The Dependence of Lorentz Boost Generators on the Presence and Nature of Interactions

by

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Abstract

The long established but infrequently discussed dependence of Lorentz boost generators on the presence and nature of interactions is reviewed in this tutorial note. The last third of the note presents a discussion of the covariant transformation and evolution equations for the non-conserved *partial* generators of the inhomogeneous Lorentz group for interacting subsystems.

1: Introduction: Recent encounters with literature and colleagues have led me to the view that the dependence of Lorentz boost generators on interactions is not as widely recognized within the philosophy of physics community as it deserves to be. Indeed, as it needs to be if Lorentz boosts are to be properly assessed in Lorentz covariant quantum field theory. The lapse is characterized by, and, in turn, sustains the perception of Lorentz boosts as having, in common with spatial translations and rotations, a purely kinematical significance. They are widely perceived to have none of the dynamical features of time translations which are expressed in Hamiltonian evolution and which are quantitatively manifested by the Hamiltonian, the generator of time translations, having an interaction part, i.e., being dependent on the presence and nature of interactions.

But this perception is a misperception. While accounts of the dynamical content of Lorentz boosts, i.e., <u>their</u> dependence on the presence and nature of interactions, are <u>not</u> <u>common</u> in the physics literature, authoritative accounts in standard texts are available. One of the best is provided by

Steven Weinberg (1995, vol. I, pp. 116-121, 145 and Chapter 7, especially § 7.4). I suspect that a contributing factor to the misconception of concern here is that there is no precursor in Galilean relativity. The Galilean boost generators are not dependent on interactions (Jauch, 1968, § 12-5, 13-4,5; Ballentine, 1990, § 3-3,4). But, as we shall see, the relativity of simultaneity, introduced by Einstein and expressed in the Lorentz transformations, requires the Lorentz boost generators to be interaction dependent.

A quick and easy way to see the need for interaction terms in the boost generators is to look at, in the Heisenberg picture, the commutation relations between the full set of self adjoint generators for the inhomogeneous Lorentz group (IHLG) and any local scalar field, $\hat{\varphi}(x)$. With $\hat{\mathbf{P}}, \hat{\mathbf{J}}, \hat{H}$ and $\hat{\mathbf{N}}$ representing the spatial translation, rotation, time translation and Lorentz boost generators, respectively, we have,

$$[\,\hat{\varphi}(x)\,,\hat{\mathbf{P}}\,] = -i\hbar\,(\partial/\partial\mathbf{x})\,\hat{\varphi}(x)\,, \tag{1.1a}$$

$$[\,\hat{\varphi}(x)\,,\hat{\mathbf{J}}\,] = -i\hbar\,\mathbf{x}\times(\partial/\partial\mathbf{x})\,\hat{\varphi}(x)\,, \tag{1.1b}$$

$$[\,\hat{\varphi}(x)\,,\hat{H}\,] = i\hbar\,(\partial/\partial x^0)\,\hat{\varphi}(x) \tag{1.1c}$$

and

$$[\,\hat{\varphi}(x)\,,\hat{\mathbf{N}}\,] = i\hbar\,\{\mathbf{x}\,(\partial/\partial x^0) + x^0\,(\partial/\partial\mathbf{x})\}\,\hat{\varphi}(x)\,. \tag{1.1d}$$

The first two commutators can be said to just shuffle initial 'data' while the last two, by virtue of the appearance of time derivatives on the right hand side, convert initial 'data' into evolved 'data'. Lest one think to argue that, since the field equation for a scalar field is likely to be second order in the time derivative, first order time derivatives still constitute initial 'data', note what happens upon taking the partial time derivative of both sides of the last two commutators. Those commutators with the first time derivatives of the field yield second time derivatives on the right hand side. Those second time derivatives are no longer initial 'data' but interaction dependent evolved 'data'. The boost generator, $\hat{\mathbf{N}}$, like the time translation generator (Hamiltonian), $\hat{H}$, must be interaction dependent.

In the remainder of this note I will provide a more detailed account of the interaction dependence of the Lorentz boost generators of *heuristic* local quantum field theory. It will be shown that, with careful regard for the meaning of the change in a generator when interactions are modified, a change in the form of the Lagrangian density that alters the Euler-Lagrange (E-L) equations always changes both and only the time translation and boost generators.

In section **2** I review the algebraic structure of unitary representations (up to a sign) of the IHLG, including the Lie algebra of the self adjoint generators of the group. In section **3** I review the commutation relations of arbitrary, covariant, local fields with those generators. This generalizes the argument motivated by equations (1.1). Section **4** examines the canonical construction of the generators in terms of the fields and, for the simple case of *non-derivative coupling* interactions, displays how those generators change when the Lagrangian density determining the E-L field equations change. In section **5** the general, and more subtle, case of *derivative couplings* is considered and **Appendix I** provides amplification on some of the subtleties in section **5**. Section **6** and **Appendix II** go beyond the review character of this note to discuss the covariant transformation and evolution behaviour of non-conserved *partial* generators of the IHLG (Fleming, 1966, 1968).

2: Unitary Representation of the Inhomogeneous Lorentz Transformations: I will work in the Heisenberg picture where rays of the state space represent the relationship between the physical state of affairs of an entire dynamical history of a system (hypothetically) suffering no state reductions and an inertial Minkowski coordinate system. Consequently, for a fixed physical state of affairs, the representative ray must change when the inertial coordinate system changes. For any two state vectors selected from two rays, the normalized absolute value of the inner product between the two state vectors depends only on the rays from which they were selected and represents the holistic statistical relationship between the associated physical states of affairs. Thus, the normalized absolute inner product must be invariant under a change of inertial coordinate system. This allows the change of rays to be represented by a unitary change of state vectors in which the unitary operators employed form a unitary representation, up to a phase, of the IHLG. Wigner (1939) classified the irreducible unitary representations and showed that the unitary operators could always be chosen so that the phase factors were restricted to ± 1.

Let M represent the Minkowski space-time manifold and M and M', two inertial Minkowski coordinate systems over M. With x^{μ} and x'^{μ} the coordinates of M and M', respectively, we have, for some $\Lambda^{\mu}{}_{\nu}$,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \qquad (2.1)$$

or M' = (Λ, a)M , where the coefficients, Λ, satisfy the group property,

$$(\Lambda_2 , a_2)(\Lambda_1 , a_1) = (\Lambda_2\Lambda_1 , a_2 + \Lambda_2 a_1). \qquad (2.2)$$

We can then choose equal norm state vectors, $|\Psi>$ and $|\Psi'>$, representing the same state of affairs relative to M and M', respectively, such that

$$|\Psi'> = \hat{U}(\Lambda , a)|\Psi >, \qquad (2.3)$$

where $\hat{U}(\Lambda , a)$ is unitary and satisfies,

$$\hat{U}(\Lambda_2, a_2)\hat{U}(\Lambda_1, a_1) = \pm\, \hat{U}(\Lambda_2\Lambda_1, a_2 + \Lambda_2 a_1). \qquad (2.4)$$

From this last relationship it can be inferred that $\hat{U}(\Lambda , a)$ has the form,

$$\hat{U}(\Lambda , a) = \exp[\frac{i}{\hbar}\hat{P}^{\mu} a_{\mu}]\exp[-\frac{i}{2\hbar}\hat{M}^{\mu\nu}\omega_{\mu\nu}(\Lambda)], \qquad (2.5)$$

where $\Lambda^{\mu}{}_{\nu} = (\exp\,\omega(\Lambda))^{\mu}{}_{\nu}$ and $\hat{P}^{\mu}$ and $\hat{M}^{\mu\nu}$ are the self adjoint <u>generators</u> of this unitary representation of the IHLG. One also infers the generators to satisfy,

$$\hat{U}(\Lambda , a)^{\dagger}\, \hat{P}^{\mu}\hat{U}(\Lambda , a) = \Lambda^{\mu}{}_{\lambda}\hat{P}^{\lambda}, \qquad (2.6a)$$

and

$$\hat{U}(\Lambda , a)^{\dagger}\, \hat{M}^{\mu\nu}\hat{U}(\Lambda , a) = \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\rho}\hat{M}^{\lambda\rho} + a^{\mu}\Lambda^{\nu}{}_{\rho}\hat{P}^{\rho} - a^{\nu}\Lambda^{\mu}{}_{\lambda}\hat{P}^{\lambda} \qquad (2.6b)$$

This then yields the commutation relations that comprise the Lie algebra of the generators of the IHLG (with the metric convention, $\eta^{\mu\nu} = (+, -, -, -)$),

$$[\hat{P}^{\mu}, \hat{P}^{\nu}] = 0, \tag{2.7a}$$

$$[\hat{M}^{\mu\nu}, \hat{P}^{\lambda}] = i\hbar(\eta^{\nu\lambda}\hat{P}^{\mu} - \eta^{\mu\lambda}\hat{P}^{\nu}), \tag{2.7b}$$

and

$$[\hat{M}^{\mu\nu}, \hat{M}^{\lambda\rho}] = i\hbar(\eta^{\mu\lambda}\hat{M}^{\nu\rho} - \eta^{\nu\lambda}\hat{M}^{\mu\rho} + \eta^{\nu\rho}\hat{M}^{\mu\lambda} - \eta^{\mu\rho}\hat{M}^{\nu\lambda}). \tag{2.7c}$$

3: Transformations of Field Operators and the Heisenberg Equations of Motion: I will denote a general member of the set of 'basic' fields, in terms of which <u>all</u> operators in the theory are to be expressed, by $\hat{\varphi}_A(x)$. This includes spinorial fields as well as tensorial fields and the subscript, A, carries all the indices needed to indicate the spinorial-tensorial character of the field. The first order Minkowski partial derivatives of the fields are denoted by, $\partial_\mu \hat{\varphi}_A(x)$.

The response of the field operators to the action of the unitary operators discussed in the previous section is determined by the requirement that under changes of inertial coordinate systems the expectation values of the quantum fields transform the same as the classical analogues would, i.e.,

$$< \Psi' | \hat{\varphi}_A(\Lambda x + a) | \Psi' > = C_A{}^B(\Lambda) < \Psi | \hat{\varphi}_B(x) | \Psi >, \tag{3.1}$$

where the $C_A{}^B(\Lambda)$ are the classical transformation coefficients*. From the arbitrariness of the choice of |Ψ>, and from (2.3), we obtain,

$$\hat{U}(\Lambda, a)^{\dagger}\, \hat{\varphi}_A(\Lambda x + a)\hat{U}(\Lambda, a) = C_A{}^B(\Lambda)\hat{\varphi}_B(x). \tag{3.2}$$

By considering infinitesimal transformations, $\Lambda \cong I + \varepsilon$, where,

* Notice that as a consequence of the state vectors representing the relationship between physical states of affairs and inertial coordinate systems and thus changing with the coordinate systems for fixed states of affairs, the operators representing observables and dynamical variables having a fixed defining relationship to whatever inertial coordinate system is being employed do not change with the coordinate system.

$$C_A{}^B(I+\varepsilon) :\cong \delta_A{}^B + (L^{\mu\nu})_A{}^B \varepsilon_{\mu\nu}, \text{ and (see (2.5)) } \omega_{\mu\nu}(I+\varepsilon) \cong \varepsilon_{\mu\nu}, \quad (3.3)$$

and infinitesimal translations, we obtain the generalized Heisenberg equations of motion,

$$[\,\hat{\varphi}_A(x)\,,\hat{P}^\mu\,] = i\hbar\,\partial^\mu \hat{\varphi}_A(x)\,, \tag{3.4a}$$

$$[\,\hat{\varphi}_A(x)\,,\hat{M}^{\mu\nu}\,] = i\hbar\,\{(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\varphi}_A(x) + (L^{\mu\nu})_A{}^B\hat{\varphi}_B(x)\}\,, \tag{3.4b}$$

where $\partial^\mu := \partial/\partial x_\mu$. Equations (3.4) are the generalization to fields of arbitrary tensorial/spinorial rank of equations (1.1). Again we see that for the generators $\hat{P}^0$ and $\hat{M}^{j0} = -\hat{M}^{0j}$, the right hand sides of the equations contain time derivatives. If the field satisfies a first order partial differential equation of motion, as Dirac-like spinorial fields do, then those time derivatives on the right hand sides are interaction dependent and the generators must be also. If the fields satisfy second order partial differential equations of motion, as Klein-Gordon-like tensorial fields do, then the first order time derivatives still have the status of initial 'data' but, by time differentiating (3.4), they get converted into the interaction dependent second order time derivatives under commutation with the crucial generators. So, again, $\hat{P}^0$ and $\hat{M}^{j0} = -\hat{M}^{0j}$ must be interaction dependent.

Just what form does the interaction dependence take?

4: The Form of the Interaction Dependence of the Boost Generators for non-Derivative Coupling: In the canonical formalism of *heuristic* quantum field theory (see Weinberg (1995), chapter 7, but be alert for metric and other sign convention differences between Weinberg's and the present discussion), the field equations of motion are obtained as Euler-Lagrange equations,

$$\frac{\partial \hat{L}(x)}{\partial \hat{\varphi}^n{}_A(x)} - \partial_\lambda \frac{\partial \hat{L}(x)}{\partial(\partial_\lambda \hat{\varphi}^n{}_A(x))} = 0\,, \tag{4.1}$$

from an action principle employing a Lorentz invariant Lagrangian density,

$$\hat{L}(x) = L(\hat{\varphi}^n{}_A(x), \partial_\lambda \hat{\varphi}^n{}_A(x))\,, \tag{4.2}$$

where the superscript, n, on the fields denotes distinct species of fields.

The generators of the IHLG are explicitly obtained as globally conserved functionals of the fields from an application of Noether's theorem to the IHLG transformations. They are given as volume integrals over all space of the *symmetrized* stress-energy-momentum field (SEM) or its first Minkowski moment. From Noether's theorem one first obtains the *canonical* SEM, given by,

$$\hat{T}^{\mu\nu}(x) := :\sum_{n,A} \frac{\partial \hat{L}(x)}{\partial(\partial_\nu \hat{\varphi}^n{}_A(x))} \partial^\mu \hat{\varphi}^n{}_A(x) - \eta^{\mu\nu}\, \hat{L}(x):\ , \tag{4.3}$$

where the bracketing colons on the right hand side refer to the normal ordering and/or renormalization protocols needed to make this field physically well defined. This *canonical* SEM is locally conserved,

$$\partial_\nu \hat{T}^{\mu\nu}(x) = 0 \tag{4.4}$$

but, in general, it is not symmetric in its Minkowski indices. But from the Lorentz invariance of the Lagrangian density one can show that,

$$\hat{T}^{\mu\nu}(x) - \hat{T}^{\nu\mu}(x) = \partial_\lambda G^{\lambda\mu\nu}(x), \tag{4.5}$$

where,
$$G^{\lambda\mu\nu}(x) = \sum_{n,A} \frac{\partial \hat{L}(x)}{\partial(\partial_\lambda \hat{\varphi}^n{}_A(x))} (L_n{}^{\mu\nu})_A{}^B\, \hat{\varphi}^n{}_B(x), \tag{4.6}$$

and so a *symmetric*, locally conserved SEM (Belinfante, 1939; Rosenfeld, 1940) can be defined by*,

$$\hat{\theta}^{\mu\nu}(x) := \hat{T}^{\mu\nu}(x) - \frac{1}{2}\partial_\lambda [G^{\lambda\mu\nu}(x) - G^{\nu\mu\lambda}(x) - G^{\mu\nu\lambda}(x)]. \tag{4.7}$$

*The classical analogue of this SEM field, generalized to arbitrary space-times, is the source term in Einstein's field equations for GR.

In terms of $\hat{\theta}^{\mu\nu}(x)$ the IHLG generators are given by,

$$\hat{P}^{\mu} = \int d^3x\, \hat{\theta}^{\mu 0}(x), \tag{4.8a}$$

$$\hat{M}^{\mu\nu} = \int d^3x\, \{x^{\mu}\hat{\theta}^{\nu 0}(x) - x^{\nu}\hat{\theta}^{\mu 0}(x)\}. \tag{4.8b}$$

Now, suppose the Lagrangian density is modified by the addition of a term, $-\hat{v}(x)$, a function of the original set of basic fields, $\hat{\varphi}^{n}{}_{A}(x)$. This modification would lead to altered equations of motion for the fields and represent a change in the interactions the fields engage in. If the field equations prior to the modification are for uncoupled free fields, then the modification 'contains' <u>all</u> the interactions between the fields. In general, $\hat{v}(x)$ may contain both fields <u>and</u> their Minkowski derivatives. If derivatives of the fields are absent from $\hat{v}(x)$ we say the new interactions involve only *non-derivative coupling*. In that case the modification of the Lagrangian density will make no contribution to the first term on the right hand side of (4.3) and no contribution to the G fields, introduced in (4.5,6) and used to symmetrize the SEM field.

Thus denoting the symmetric SEM field prior to the modification by $\hat{\theta}_{(0)}{}^{\mu\nu}(x)$, we find the modified symmetric SEM field given by,

$$\hat{\theta}^{\mu\nu}(x) = \hat{\theta}_{(0)}{}^{\mu\nu}(x) + \eta^{\mu\nu}\hat{v}(x). \tag{4.9}$$

Substituting (4.9) into (4.8) and denoting the generators prior to the modification by the same (0) subscript used for the SEM field, we have,

$$\hat{P}^{j} = \hat{P}_{(0)}{}^{j} \;,\quad \hat{M}^{jk} = \hat{M}_{(0)}{}^{jk} := \varepsilon^{jkl} J_l \;\; (j,k,l = 1,2,3), \tag{4.10a}$$

$$\hat{P}^{0} = \hat{P}_{(0)}{}^{0}(x^0) + \hat{V}(x^0)\;, \tag{4.10b}$$

$$\hat{M}^{j0} := \hat{N}^{j} = \hat{N}_{(0)}{}^{j}(x^0) + \hat{W}^{j}(x^0),\;(j = 1,2,3), \tag{4.10c}$$

where,

$$\hat{V}(x^0) = \int d^3x\, \hat{v}(x) \quad \text{and} \quad \hat{W}^{j}(x^0) = \int d^3x\, x^{j}\, \hat{v}(x), \tag{4.11}$$

and I have noted explicitly the time dependence of the umodified Hamiltonian, boost generators and their modifications, since, after the modification, it is only $\hat{\theta}^{\mu\nu}(x)$ and not $\hat{\theta}_{(0)}{}^{\mu\nu}(x)$ that is locally conserved.

But <u>only</u> the time translation generator (Hamiltonian) and boost generators <u>are</u> modified and <u>both</u> of them are modified, i.e., are interaction dependent!

5: The General Case with Derivative Coupling: If the modification to the Lagrangian density, $-\hat{v}(x)$, contains derivatives of the fields, then it may contribute to <u>all</u> the terms entering into the definition of the *canonical* SEM field, (4.3), and the *symmetrized* SEM field, (4.7). At first glance this seems to imply that all of the IHLG generators could be changed by the modification. Indeed, regarded as space functionals of the fields and their *time derivatives*, the generators would, in general, <u>all</u> be changed by the modification.

But there is a *second way* to regard the generators, as functionals, that turns out to be physically more important than the first way and to preserve the judgement reached in the previous section as to which generators are changed by the modification! The *second way* is to regard the generators as space functionals of the fields and their *canonically conjugate momenta*. If the modification to the Lagrangian density contains field derivatives, it will change the relationship between the canonically conjugate momenta and the time derivatives of the fields. In particular we have, as definition of the canonically conjugate momenta after the modification,

$$\hat{\pi}^{n,A}(x) := \frac{\partial(\hat{L}_{(0)}(x) - \hat{v}(x))}{\partial(\partial_0 \hat{\varphi}^n{}_A(x))}, \tag{5.1a}$$

and before the modification,

$$\hat{\pi}^{n,A}(x) := \frac{\partial \hat{L}_{(0)}(x)}{\partial(\partial_0 \hat{\varphi}^n{}_A(x))}\ . \tag{5.1b}$$

The *second way* defines the unmodified parts of the generators as being the <u>same functionals</u> of the fields and the canonically conjugate momenta as the original generators were. This is very different from defining the unmodified parts of the generators as being the <u>same functionals</u> of the fields and their time derivatives as the original generators were. **Appendix I** provides an amplification on the definition of this *second way*.

The reason the *second way* is physically more important is that what it means to be an IHLG generator, unmodified or modified, is that certain fixed commutation relations, (2.7), are satisfied among the generators. But it is the equal time commutation relations between fields, their spatial derivatives and canonically conjugate momenta *that remain unchanged through the modification*. This is a basic feature of the canonical formalism. Commutation relations involving the time derivatives of fields depend on their relationship with the canonical momenta and are modified accordingly*.

By identifying the unmodified generators, some of which are time dependent after the modification, in accordance with the *second way* we ensure that they satisfy the Lie algebra of the IHLG,

$$[\hat{P}_{(0)}{}^{\mu}(x^0), \hat{P}_{(0)}{}^{\nu}(x^0)] = 0\,, \tag{5.2a}$$

$$[\hat{M}_{(0)}{}^{\mu\nu}(x^0), \hat{P}_{(0)}{}^{\lambda}(x^0)] = i\hbar(\eta^{\nu\lambda}\hat{P}_{(0)}{}^{\mu}(x^0) - \eta^{\mu\lambda}\hat{P}_{(0)}{}^{\nu}(x^0))\,, \tag{5.2b}$$

and

$$\begin{aligned}[\hat{M}_{(0)}{}^{\mu\nu}(x^0), \hat{M}_{(0)}{}^{\lambda\rho}(x^0)]& \\ = i\hbar(\eta^{\mu\lambda}\hat{M}_{(0)}{}^{\nu\rho}(x^0) &- \eta^{\nu\lambda}\hat{M}_{(0)}{}^{\mu\rho}(x^0) \\ &+ \eta^{\nu\rho}\hat{M}_{(0)}{}^{\mu\lambda}(x^0) - \eta^{\mu\rho}\hat{M}_{(0)}{}^{\nu\lambda}(x^0))\,. \end{aligned} \tag{5.2c}$$

* An example of this sort of thing from elementary QM is provided by the preservation of the canonical commutators and the change of the commutators involving velocity upon turning on the velocity dependent interaction of a charged particle with a magnetic field.

But I also claimed that the *second way* preserved the result of the non-derivative coupling case, that only and both the time translation and boost generators suffer any change. How can we see that?

It's easy to see for the generator of spatial translations, i.e., the total 3-momentum. Looking at (4.6) and (4.7) we see that the difference between $\hat{\theta}^{\mu 0}(x)$ and $\hat{T}^{\mu 0}(x)$ is just a 3-divergence since the time derivative of the 4-divergence is zero. In (4.8a) that 3-divergence integrates to zero at spatial infinity by Gauss' theorem* and so we have, for the spatial components,

$$\hat{P}^{j} = \int d^3x\, T^{j0}(x) = \int d^3x\, \{\sum_{n,A} \hat{\pi}^{n,A}(x)\partial^j \varphi^n{}_A(x)\} = \hat{P}_{(0)}{}^{j}. \tag{5.3}$$

For the rotation generator, the total angular momentum, the intermediate steps, while of a similar nature, are more numerous. The result is,

$$\hat{M}^{jk} = \int d^3x \sum_{n,A} \{ x^j(\hat{\pi}^{n,A}(x)\partial^k \hat{\varphi}^n{}_A(x)) - x^k(\hat{\pi}^{n,A}(x)\partial^j \hat{\varphi}^n{}_A(x))$$

$$+ \hat{\pi}^{n,A}(x)(L_n{}^{jk})_A{}^B \hat{\varphi}^n{}_B(x)\} = \hat{M}_{(0)}{}^{jk}. \tag{5.4}$$

Thus <u>only</u> the time translation and boost generators can change since the space translation and rotation generators do not change when expressed in the *second way*.

To see that <u>both</u> the time translation and boost generators must change, if any do, note that if the modification, $-\hat{v}(x)$, of the Lagrangian density is not just a 4-divergence, the E-L equations will be changed. Thus, to preserve the Heisenberg equations of motion, the time translation generator, $\hat{P}^0$, will have to change. But then, since, as we just saw, the space translation generators do not change, the boost generators <u>must</u> change to preserve the Lie algebra commutation relation,

$$[\hat{M}^{j0}, \hat{P}^k] = i\hbar\, \delta^{jk}\hat{P}^0 \ , \tag{5.5}$$

from (2.7b). Alternatively, the boost generators <u>must</u> change to preserve the commutation relation,

$$[\hat{\varphi}^{n}{}_{A}(x), \hat{M}^{j0}] = i\hbar\,\{(x^{j}\partial^{0} - x^{0}\partial^{j})\hat{\varphi}^{n}{}_{A}(x) + (L_{n}{}^{j0})_{A}{}^{B}\hat{\varphi}^{n}{}_{B}(x)\}, \quad (5.6)$$

from (3.4b). In any case <u>both</u> $\hat{P}^{0}$ and $\hat{M}^{j0}$ must change.

Unlike the non-derivative coupling case, the modification in $\hat{P}^{0}$ and $\hat{M}^{j0}$ for the general case is not given by (4.11). The modification is more complex. However, it turns out that the increased complexity can be incorporated by simply changing from $\hat{v}(x)$, in the integrands of (4.11), to an alternative $\hat{u}(x)$. The modifications are then given by,

$$\hat{V}(x^{0}) = \int d^{3}x\,\hat{u}(x) \quad \text{and} \quad \hat{W}^{j}(x^{0}) = \int d^{3}x\,x^{j}\,\hat{u}(x). \quad (5.7)$$

6: Covariance of Subsystem and Unmodified Generators: From the equations (2.6) we see that the IHLG generators for a closed system transform, themselves, covariantly. The space-time translators, $\hat{P}^{\mu}$, transform as a 4-vector and the homogeneous generators, $\hat{M}^{\mu\nu}$, transform (inhomogeneously) as an antisymmetric 2nd rank tensor. This continues to hold for the *partial* generators associated with subsystems of closed systems so long as those subsystems are also closed by virtue of the absence of interactions between them. As soon, however, as interactions between subsystems are 'turned on', the *partial* generators for the subsystems, defined in the spirit of the *second way* introduced in **5** and made explicit in **Appendix I**, become time dependent and, just as for the *partial* unmodified generators of sections **4** and **5**, appear to lose all semblance of covariant transformation properties of their own. In traditional discussions of these issues, the matter is often left at that.

But it need not be. First note that the total generators, $\hat{P}^{\mu}$, and $\hat{M}^{\mu\nu}$, need not be expressed only as space functionals. As a consequence of the local conservation and symmetry of the SEM field, $\hat{\theta}^{\mu\nu}(x)$, the total generators can be expressed as space-like functionals over arbitrary space-like hyperplanes. That is, we have (Schwinger, 1948),

$$\hat{P}^{\mu} = \int d^{4}x\,\delta(\eta x - \tau)\,\hat{\theta}^{\mu\nu}(x)\eta_{\nu}, \quad (6.1a)$$

and

$$\hat{M}^{\mu\nu} = \int d^{4}x\,\delta(\eta x - \tau)\,\{x^{\mu}\hat{\theta}^{\nu\lambda}(x) - x^{\nu}\hat{\theta}^{\mu\lambda}(x)\}\eta_{\lambda}, \quad (6.1b)$$

for arbitrary dimensionless, time-like unit 4-vector, η^{μ}, and parameter, τ, which, together, parameterize a space-like hyperplane.

This means the total generators are not merely time independent but fully space-like hyperplane independent. While time independence is a notion requiring reference to a coordinate system, hyperplane independence is a coordinate system independent notion. From a covariant perspective one might have expected this result immediately since <u>any</u> space functional relative to one inertial Minkowski coordinate system <u>is</u> a *space-like functional over a hyperplane* relative to any other inertial Minkowski coordinate system.

The time dependence of the *partial* unmodified generators of sections **4** and **5** and of *partial* generators for interacting subsystems of the total closed system means that the space functional character of those generators is essential. They can not be <u>equivalently</u> regarded as space-like functionals over an arbitrary hyperplane. But under a passive transformation between inertial Minkowski coordinate systems one would expect a space functional relative to the first coordinate system to be related to a space-like functional relative to the second coordinate system. So the non-covariant character of time dependent *partial* generators would seem to be due to the lack of their <u>generalization</u> to space-like functionals over arbitrary hyperplanes. Implement the generalization and covariant transformation rules for the *partial* generators will follow.

Some details of the implementation are spelled out in **Appendix II**. But assuming it has been carried out, the results are captured in the following.

First we have the covariant expression of the total generators in terms of the hyperplane dependent *partial* generators and interaction terms:

$$\hat{P}^{\mu} = \hat{P}_1{}^{\mu}(\eta,\tau) + \hat{P}_2{}^{\mu}(\eta,\tau) + \eta^{\mu}\hat{V}(\eta,\tau), \tag{6.2a}$$

$$\hat{M}^{\mu\nu} = \hat{M}_1{}^{\mu\nu}(\eta,\tau) + \hat{M}_2{}^{\mu\nu}(\eta,\tau) + \eta^{\nu}\hat{W}^{\mu}(\eta,\tau) - \eta^{\mu}\hat{W}^{\nu}(\eta,\tau), \tag{6.2b}$$

where,

$$\eta_{\mu}\hat{W}^{\mu}(\eta,\tau) = 0. \tag{6.2c}$$

Note that when expressed relative to the (η, τ) hyperplane it is the generators that produce space-like translations and rotations within the hyperplane, i.e., leaving the hyperplane invariant, that are not modified by interaction terms.

This is the covariant generalization of the result expressed in equations (5.3,4).

The interaction terms, $\hat{V}(\eta, \tau)$ and $\hat{W}^{\mu}(\eta, \tau)$, may, themselves, contain terms involving only the dynamical variables for one of the subsystems. Such terms are of the character discussed in **4** and **5** and constitute modifications of the interactions within that subsystem. The remaining terms in $\hat{V}(\eta, \tau)$ and $\hat{W}^{\mu}(\eta, \tau)$, which multiplicatively involve dynamical variables of both subsystems, describe interactions between the subsystems.

Second, we have the covariant transformation equations,

$$\hat{U}(\Lambda, a)^{\dagger} \hat{P}_n{}^{\mu}(\eta', \tau')\hat{U}(\Lambda, a) = \Lambda^{\mu}{}_{\nu}\hat{P}_n{}^{\nu}(\eta, \tau), \tag{6.3a}$$

and,

$$\hat{U}(\Lambda, a)^{\dagger} \hat{M}_n{}^{\mu\nu}(\eta', \tau')\hat{U}(\Lambda, a)$$

$$= \Lambda^{\mu}{}_{\lambda}\Lambda^{\nu}{}_{\rho}\hat{M}_n{}^{\lambda\rho}(\eta, \tau) + a^{\mu}\Lambda^{\nu}{}_{\rho}\hat{P}_n{}^{\rho}(\eta, \tau) - a^{\nu}\Lambda^{\mu}{}_{\lambda}\hat{P}_n{}^{\lambda}(\eta, \tau), \tag{6.3b}$$

where,

$$\tau' = \tau + a_{\mu}\Lambda^{\mu}{}_{\nu}\eta^{\nu}, \quad \eta'^{\mu} := \Lambda^{\mu}{}_{\nu}\eta^{\nu}, \tag{6.3c}$$

and the subscript, $n = 1, 2$. Remember that the unitary operators appearing in (6.3) are constructed from the total generators in accordance with (2.5).

Third, we have the *partial* generator Lie algebra of equal hyperplane commutation relations,

$$[\hat{P}_m{}^{\mu}(\eta, \tau), \hat{P}_n{}^{\nu}(\eta, \tau)] = 0, \tag{6.4a}$$

$$[\hat{M}_m{}^{\mu\nu}(\eta, \tau), \hat{P}_n{}^{\lambda}(\eta, \tau)] = \delta_{mn}\, i\hbar(\eta^{\nu\lambda}\hat{P}_n{}^{\mu}(\eta, \tau) - \eta^{\mu\lambda}\hat{P}_n{}^{\nu}(\eta, \tau)), \tag{6.4b}$$

and

$$[\hat{M}_m{}^{\mu\nu}(\eta,\tau), \hat{M}_n{}^{\lambda\rho}(\eta,\tau)]$$

$$= \delta_{mn}\, i\hbar(\eta^{\mu\lambda}\hat{M}_n{}^{\nu\rho}(\eta,\tau) - \eta^{\nu\lambda}\hat{M}_n{}^{\mu\rho}(\eta,\tau)$$

$$+ \eta^{\nu\rho}\hat{M}_n{}^{\mu\lambda}(\eta,\tau) - \eta^{\mu\rho}\hat{M}_n{}^{\nu\lambda}(\eta,\tau)), \qquad (6.4c)$$

where m, n = 1, 2.

Finally, by considering infinitesimal transformations in (6.3), we obtain the Heisenberg-like evolution equations for the hyperplane dependent *partial* generators,

$$[\hat{P}_m{}^{\mu}(\eta,\tau), \hat{P}^{\nu}] = i\hbar\, \eta^{\nu} \frac{\partial \hat{P}_m{}^{\mu}(\eta,\tau)}{\partial\tau}, \qquad (6.5a)$$

$$[\hat{M}_m{}^{\mu\nu}(\eta,\tau), \hat{P}^{\lambda}] = i\hbar\{ \eta^{\nu\lambda}\hat{P}_m{}^{\mu}(\eta,\tau) - \eta^{\mu\lambda}\hat{P}_m{}^{\nu}(\eta,\tau)$$

$$+ \eta^{\lambda} \frac{\partial \hat{M}_m{}^{\mu\nu}(\eta,\tau)}{\partial\tau}\}, \qquad (6.5b)$$

$$[\hat{P}_m{}^{\mu}(\eta,\tau), \hat{M}^{\nu\lambda}] = i\hbar\{ \eta^{\mu\nu}\hat{P}_m{}^{\lambda}(\eta,\tau) - \eta^{\mu\lambda}\hat{P}_m{}^{\nu}(\eta,\tau)$$

$$+ \eta^{\nu} \frac{\partial \hat{P}_m{}^{\mu}(\eta,\tau)}{\partial\eta_{\lambda}} - \eta^{\lambda} \frac{\partial \hat{P}_m{}^{\mu}(\eta,\tau)}{\partial\eta_{\nu}}\}, \qquad (6.5c)$$

and

$$[\hat{M}_m{}^{\mu\nu}(\eta,\tau), \hat{M}^{\lambda\rho}]$$

$$= i\hbar\, \{ \eta^{\mu\lambda}\hat{M}_m{}^{\nu\rho}(\eta,\tau) - \eta^{\nu\lambda}\hat{M}_m{}^{\mu\rho}(\eta,\tau)$$

$$+ \eta^{\nu\rho}\hat{M}_m{}^{\mu\lambda}(\eta,\tau) - \eta^{\mu\rho}\hat{M}_m{}^{\nu\lambda}(\eta,\tau)$$

$$+ \eta^{\lambda} \frac{\partial \hat{M}_m{}^{\mu\nu}(\eta,\tau)}{\partial\eta_{\rho}} - \eta^{\rho} \frac{\partial \hat{M}_m{}^{\mu\nu}(\eta,\tau)}{\partial\eta_{\lambda}}\}. \qquad (6.5d)$$

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Appendix I: Consider two Lagrangian densities, $\hat{L}(\mathrm{x})$ and $\hat{L}_{(0)}(\mathrm{x})$, involving the same number and kinds of fields. If these Lagrangian densities give rise to different E-L field equations then the fields that satisfy those equations and their canonical momenta will be different. Denote fields and their canonical momenta satisfying the E-L equations from $\hat{L}(\mathrm{x})$ by $\hat{\varphi}^{\mathrm{n}}{}_{\mathrm{A}}(\mathrm{x})$ and $\hat{\pi}^{\mathrm{n,A}}(\mathrm{x})$, respectively. For $\hat{L}_{(0)}(\mathrm{x})$ the corresponding fields and momenta will be denoted by $\hat{\varphi}_{(0)}{}^{\mathrm{n}}{}_{\mathrm{A}}(\mathrm{x})$ and $\hat{\pi}_{(0)}{}^{\mathrm{n,A}}(\mathrm{x})$.

Let $\hat{\mathrm{G}}^{\mathrm{m}}$ denote a generic IHLG generator for the fields, $\hat{\varphi}^{\mathrm{n}}{}_{\mathrm{A}}(\mathrm{x})$, constructed in the canonical way expressed by (4.8). Let $\hat{\mathrm{G}}_{(0)}{}^{\mathrm{m}}$ denote the corresponding generator for the fields, $\hat{\varphi}_{(0)}{}^{\mathrm{n}}{}_{\mathrm{A}}(\mathrm{x})$. These generators are space functionals of the corresponding fields and their time derivatives <u>or</u>, equivalently, of the corresponding fields and their canonical momenta. For the first set of generators, let $G_L{}^{1,\mathrm{m}}$ and $G_L{}^{2,\mathrm{m}}$ denote the <u>forms</u> of the functional dependencies for the *first way* and *second way* of expressing the generators as space functionals, respectively. Similarly, for the second set of generators, let $G_{L(0)}{}^{1,\mathrm{m}}$ and $G_{L(0)}{}^{2,\mathrm{m}}$ play the same role. Thus,

$$\hat{\mathrm{G}}^{\mathrm{m}} = G_L{}^{1,\mathrm{m}}[\hat{\varphi}(\mathrm{x}^0), \partial^0\hat{\varphi}(\mathrm{x}^0)] = G_L{}^{2,\mathrm{m}}[\hat{\varphi}(\mathrm{x}^0), \hat{\pi}(\mathrm{x}^0)] \ , \qquad \text{(AI.1)}$$

and

$$\hat{\mathrm{G}}_{(0)}{}^{\mathrm{m}} = G_{L(0)}{}^{1,\mathrm{m}}[\hat{\varphi}_{(0)}(\mathrm{x}^0), \partial^0\hat{\varphi}_{(0)}(\mathrm{x}^0)] = G_{L(0)}{}^{2,\mathrm{m}}[\hat{\varphi}_{(0)}(\mathrm{x}^0), \hat{\pi}_{(0)}(\mathrm{x}^0)] \ . \quad \text{(AI.2)}$$

Both sets of generators are time independent and both sets satisfy the Lie algebra of the IHLG.

Now, using this notation, the so-called 'unmodified' generators of the text, constructed in the *second way*, occurring in the equations (5.2) and some of which are time dependent, are defined by,

$$\hat{G}_{(0)}{}^{m}(x^0) := G_{L(0)}{}^{2,m}[\hat{\varphi}(x^0), \hat{\pi}(x^0)]$$

$$\neq_{\text{(possibly)}} G_{L(0)}{}^{1,m}[\hat{\varphi}(x^0), \partial^0\hat{\varphi}(x^0)] \ . \tag{AI.3}$$

Three points should be noted: (1) The functional forms in (AI.3) are those associated with the Lagrangian density, $\hat{L}_{(0)}(x)$, while the fields over which the functionals are constructed are those associated with the Lagrangian density, $\hat{L}(x)$. (2) The inequality in (AI.3) holds only if the forms of derivative couplings in $\hat{L}(x)$ differ from those in $\hat{L}_{(0)}(x)$. (3) For values of the superscript, m, denoting space translation or rotation generators, the middle functional in (AI.3) is identical with the rightmost functional in (AI.1). This is a consequence of the *second way* form for those generators being independent of the form of the Lagrangian density, as indicated by (5.3,4).

Appendix II: Using the notation of **Appendix I** write the transformation rules for the full IHLG generators for the total closed system as,

$$\hat{U}(\Lambda, a)^{\dagger}\hat{G}^{m}\hat{U}(\Lambda, a) = C^{m}{}_{n}(\Lambda, a)\hat{G}^{n} , \tag{AII.1}$$

and the Lie algebra equations for the generators as,

$$[\hat{G}^{m}, \hat{G}^{n}] = i\hbar\, c^{mn}{}_{p}\hat{G}^{p} . \tag{AII.2}$$

These equations recapitulate equations (2.6) and (2.7), respectively, in the notation of **Appendix I**. Next, express each generator as the sum of the corresponding *partial* generators for hypothetical subsystems, 1 and 2, of which the total system is composed, plus possible interaction terms. Thus,

$$\hat{G}^{m} = \hat{G}_{1}{}^{m}(x^0) + \hat{G}_{2}{}^{m}(x^0) + \hat{V}^{m}(x^0) . \tag{AII.3}$$

All of these generators are to be regarded as space functionals, at the time x^0,of the basic fields and their canonical momenta. The *partial* generators, $\hat{G}_{1}{}^{m}(x^0)$, are such functionals of only the fields and momenta that are dynamical variables for subsystem 1 while the $\hat{G}_{2}{}^{m}(x^0)$ are such functionals of only the dynamical variables for subsystem 2. The structure of the functionals is such as to enable the *partial* generators to satisfy the equal time commutation relations,

$$[\hat{G}_1{}^m(x^0), \hat{G}_1{}^n(x^0)] = i\hbar\, c^{mn}{}_p \hat{G}_1{}^p(x^0)\,, \tag{AII.4a}$$

$$[\hat{G}_2{}^m(x^0), \hat{G}_2{}^n(x^0)] = i\hbar\, c^{mn}{}_p \hat{G}_2{}^p(x^0)\,, \tag{AII.4b}$$

and

$$[\hat{G}_1{}^m(x^0), \hat{G}_2{}^n(x^0)] = 0\,. \tag{AII.4c}$$

This guarantees that with the vanishing of all of the interaction terms, $\hat{V}^m(x^0)$, the then time independent *partial* generators could serve as total generators for the then closed subsystems. Note, in particular, that, while in accordance with the discussion of section **5** the interaction terms always vanish for the spatial translation and rotation generators, still the spatial translation and rotation generators for the individual subsystems will not be separately time independent unless all interaction between the subsystems ceases.

The interaction terms, $\hat{V}^m(x^0)$, may, themselves, contain terms involving only the dynamical variables for one of the subsystems. Such terms are of the character discussed in **4** and **5** and constitute modifications of the interactions within that subsystem. The remaining terms in $\hat{V}^m(x^0)$, which multiplicatively involve dynamical variables of both subsystems, describe interactions between the subsystems.

Now consider the *partial* generators, $\hat{G}_1{}^m(x^0)$, as explicit space functionals,

$$\hat{G}_1{}^m(x^0) = \int d^3x\, g_1{}^m(\, x\,; \hat{\varphi}_1{}^n{}_A(x), \partial^j \hat{\varphi}_1{}^n{}_A(x), \hat{\pi}_1{}^{n,A}(x)\,)\,. \tag{AII.5}$$

To generalize these generators to the larger family of space-like functionals which then transform covariantly among themselves under the full IHLG we take the following steps: (1) Note the transformation,

$$\hat{U}(\Lambda, a)^\dagger D'_{\eta'}{}^\mu\, \hat{\varphi}_1{}^n{}_A(x')\hat{U}(\Lambda, a) = \Lambda^\mu{}_\nu C_A{}^B(\Lambda) D_\eta{}^\nu \hat{\varphi}_1{}^n{}_B(x)\,, \tag{AII.6}$$

where,

$$D_\eta^{\ \mu} := \partial^\mu - \eta^\mu \eta\partial \ , \qquad \text{(AII.7a)}$$

and,

$$x'^\mu \ = \Lambda^\mu{}_\nu x^\nu + a^\mu , \ \ \eta'^\mu := \Lambda^\mu{}_\nu \eta^\nu . \qquad \text{(AII.7b)}$$

For the case, $\eta^\mu = (1, \mathbf{0})$, we have, $D_\eta^{\ \mu} \ = (0, \partial^j)$.

(2) Note the transformation,

$$\hat{U}(\Lambda, a)^\dagger \hat{\pi}_1^{\ n,A}(\eta', x')\hat{U}(\Lambda, a) = (C^{-1})_B{}^A(\Lambda)\hat{\pi}_1^{\ n,B}(\eta, x) \ , \qquad \text{(AII.8)}$$

where,

$$\hat{\pi}_1^{\ n,A}(\eta , x) := \eta^\mu \ \frac{\partial \hat{L}(x)}{\partial(\partial^\mu \hat{\varphi}_1^{\ n}{}_A(x))} \quad , \qquad \text{(AII.9a)}$$

and,

$$\hat{\pi}_1^{\ n,A}((1, \mathbf{0}), x) = \hat{\pi}_1^{\ n,A}(x) \ . \qquad \text{(AII.9b)}$$

(3) Define the function,

$$g_1^{\ m}(\eta, x ; \hat{\varphi}_1^{\ n}{}_A(x), D_\eta^{\ \mu}\hat{\varphi}_1^{\ n}{}_A(x), \hat{\pi}_1^{\ n,A}(\eta, x))$$

$$:= C^m{}_n(\Lambda, a)\hat{U}(\Lambda, a)g_1^{\ n}(x ; \hat{\varphi}_1^{\ n}{}_A(x), \partial^j\hat{\varphi}_1^{\ n}{}_A(x), \hat{\pi}_1^{\ n,A}(x))\hat{U}(\Lambda, a)^\dagger, \quad \text{(AII.10)}$$

where, $\eta^\mu = \Lambda^\mu{}_0$. As a consequence, the transformation rule,

$$\hat{U}(\Lambda, a)^\dagger g_1^{\ m}(\eta', x' ; \hat{\varphi}_1^{\ n}{}_A(x'), D'_{\eta'}{}^{\mu} \hat{\varphi}_1^{\ n}{}_A(x'), \hat{\pi}_1^{\ n,A}(\eta', x'))\hat{U}(\Lambda, a)$$

$$= C^m{}_n(\Lambda, a) \, g_1^{\ n}(\eta, x ; \hat{\varphi}_1^{\ n}{}_A(x), D_\eta^{\ \mu}\hat{\varphi}_1^{\ n}{}_A(x), \hat{\pi}_1^{\ n,A}(\eta, x)) \ , \qquad \text{(AII.11)}$$

holds.

(4) Finally, introduce the generalized *partial* generators,

$$\hat{G}_1^{\ m}(\eta, \tau)$$

$$:= \int d^4x \ \delta(\eta x - \tau) \, g_1^{\ m}(\eta, x ; \hat{\varphi}_1^{\ n}{}_A(x), D_\eta^{\ \mu}\hat{\varphi}_1^{\ n}{}_A(x), \hat{\pi}_1^{\ n,A}(\eta, x)) \ , \qquad \text{(AII.12)}$$

which are now space-like functionals over the (η, τ) space-like hyperplane and which satisfy the transformation rules,

$$\hat{U}(\Lambda, a)^{\dagger}\hat{G}_1{}^m(\eta', \tau')\hat{U}(\Lambda, a) = C^m{}_n(\Lambda, a)\hat{G}_1{}^n(\eta, \tau)\,, \qquad \text{(AII.13)}$$

and the equal hyperplane commutation relations of the Lie algebra,

$$[\hat{G}_1{}^m(\eta, \tau)\boldsymbol{,}\, \hat{G}_1{}^n(\eta, \tau)\boldsymbol{]} = i\hbar\, c^{mn}{}_p\hat{G}_1{}^p(\eta, \tau)\,, \qquad \text{(AII.14)}$$

and the identification, $\hat{G}_1{}^m(\eta = (1, \mathbf{0}), \tau = x^0) = \hat{G}_1{}^m(x^0)$. (AII.15)

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