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A Note on Prototypes, Convexity and Fuzzy Sets

Abstract. The work on prototypes in ontologies pioneered by Rosch [10] and elaborated by Lakoff [8] and Freund [3] is related to vagueness in the sense that the more remote an instance is from a prototype the fewer people agree that it is an example of that prototype. An intuitive example is the prototypical "mother", and it is observed that more specific instances like "single mother", "adoptive mother", "surrogate mother", etc., are less and less likely to be classified as "mothers" by experimental subjects. From a different direction Gärdenfors [4] provided a persuasive account of natural predicates to resolve paradoxes of induction like Goodman's "Grue" predicate [5]. Gärdenfors proposed that "quality dimensions" arising from human cognition and perception impose topologies on concepts such that the ones that appear "natural" to us are convex in these topologies. We show that these two cognitive principles — prototypes and predicate convexity — are equivalent to unimodal (convex) fuzzy characteristic functions for sets. Then we examine the case when the fuzzy set characteristic function is not convex, in particular when it is multi-modal. We argue that this is an indication that the fuzzy concept should really be regarded as a super concept in which the decomposed components are subconcepts in an ontological taxonomy.

Keywords: prototypes, convexity, fuzzy sets, ontology, induction.

1. Introduction

There are many ways to view the predicate vagueness problem. The fuzzy logic community regards most predicates outside the domain of formal mathematics as essentially inexact, i.e. the characteristic functions of the sets denoted by the predicates are not two valued, e.g. $\{0, 1\}$-valued but have a range in the interval $[0, 1]$ in the reals. A standard reference for fuzzy logic that more than covers what we need is Klir and Yuan [7]. The classical sets with two-valued characteristic functions are called *crisp* by this community, in contrast to their *fuzzy sets* that have fuzzy characteristic functions in which *degrees of membership* of candidates vary between 0 and 1. The well-known *Sorites Paradox*[1] is a common argument for the inherent fuzziness of many concepts in non-formal domains, and we shall return to this paradox later.

[1] The origin of this paradox is ancient. See http://plato.stanford.edu/entries/sorites-paradox/ for a detailed exposition.

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In this paper we suggest an alternative view, that it is sometimes viable to regard the fuzziness of a set as arising from an aggregation of the characteristic functions of a family of agents each of which has its own crisp interpretation of membership in the set. This in itself is not a particularly radical suggestion as it is one of several interpretations explored by Dubois, et al. [2]. The particular interpretation which is the basis of our view is that which they call the multiple agent concept approach. The key notion in that approach is that each agent has its own conception of which objects qualify for membership in some putative set A, and it regards A as *crisp*, and hence the two-valued characteristic function for A is agent-dependent. In this paper we propose two additional constraints on such functions. The constraints are derived from *cognitive science*, and to our knowledge their significance as yet another account of a common variety of fuzzy sets has not been recongized. In a sense our work may be regarded as a refinement of the multiagent approach in Dubois, et al. (op. cit.) when a cognitive component is added to the agents. The two cognitive constraints invoked in this paper arise from the notions of *prototypicality* and *convexity*, which we explain later. We assume, as in Dubois, et al., that there is a family of agents which make classical (crisp) membership decisions. Also, in line with their view we will aggregate agent decisions to form an overall decision on set membership.

We wish to focus on those fuzzy sets that have a *unimodal* (also known as convex in the fuzzy logic literature) characteristic function. So let us recall what is a unimodal function[2] on the reals.

Definition 1.1. A function $f : R \to R$ is unimodal if (i) there is some x (called a mode[3] of f) such that $\forall y \; f(y) \leq f(x)$, (ii) $\forall u \forall v \; u \leq v \leq x \Rightarrow f(u) \leq f(v)$, and (iii) $\forall u \forall v \; x \geq u \geq v \Rightarrow f(u) \geq f(v)$.

For such sets it turns out that the crisp set interpretation of each agent in the family must satisfy two well-known cognitive principles. Moreover the converse must hold, i.e., if a set is such that each agent in a family views the set as crisp and also satisfies these two cognitive principles, then the aggregated crisp characteristic functions will yield a unimodal fuzzy set charateristic function. We now summarize the two cognitive principles mentioned.

[2]Definition 1.1 is equivalent to the usual definition of a convex fuzzy function.

[3]If there is more than one mode, f has the same value on all of them.

1.1. Prototypes

Rosch [10] proposed that natural concepts exhibit the *prototype* property. A prototype is a most typical instance of the concept, but there are also other instances that are less prototypical. Lakoff [8] accepted this proposal and used it as a major theme in his book. A contemporary view of concepts which have prototypes is that of Freund [3]. Rosch noted from experimental evidence that a concept such as "mother" has instances that are "most typical" — and any competent user of language would agree that these instances are mothers. This is not to say that typical instances are immutable, for they may change with culture or over time. But at any point in time and in a given culture, almost all will agree to the typicality of these instances. For instance, a mother has to be a female and have children, raising them in a family with their father. A less typical mother has adoptive children only. Yet another less typical mother is a step-mother. An even less typical mother than both may be a surrogate mother. Commonsense reasoning often starts by thinking about the most typical instances — the prototypes — and then only considering less prototypical instances when necessary. This is the essence of *non-monotonic reasoning* [1]. Lakoff (op. cit.) took the Rosch thesis and developed it further to show that classical logic, with its insistence on mathematical sets as extensions for predicates, cannot give an adequate account of natural language semantics nor of cognition. The evidence is therefore quite strong that natural concepts formalized as predicates do not have classical extensions. It is interesting to note, that when subjects are asked about instances that satisfy a predicate they often differ. It may not be too far-fetched to imagine that these differences are responsible for the non-classicality of the predicate, but still be consistent with classicality for each subject.

1.2. Convexity

It was Hume [6] who first drew attention to the problems of induction. One of the problems which he described had to do with a common justification for an inductive inference, especially of the kind that is implicitly assumed in practice — that because each element of a time sequence of observations $x_1, x_2, \ldots, x_n$ (where n is large) has a common property π, we (inductively) infer that all future observations yet unseen will also enjoy property π. The informal justification for this is an appeal to *uniformity* of nature. But then this principle of uniformity itself appears to rely on a higher-order induction, that nature has so far been seen to be uniform. From this and similar argument Hume concluded that there is really no logical basis for induction.

Much later Goodman provided a contemporary challenge for induction that raised issues about choices of basic predicates for knowledge representation. This was his well-known *Grue Paradox*. A summary paraphrase of it is as follows. Suppose the repeated past observations that emeralds are green leads us to the inductive conclusion that all emeralds, including ones so far unseen, are green. Consider the predicate grue(x) which means that x is green before the year 2100 or x is blue thereafter. Then the two sentences $\forall x[emerald(x) \rightarrow grue(x)]$ and $\forall x[emerald(x) \rightarrow green(x)]$ are equally good as inductions for the present. If it is objected that "green" and "blue" are more basic predicates than "grue" as the latter seems to switch color at the year 2100, this objection may be answered by defining another predicate bleen(x) that means x is blue before the year 2100 or x is green thereafter. Then green(x) may be expressed as grue(x) up to the year 2100 or bleen(x) thereafter. So from a logical perspective there is no reason to favour green(x) over grue(x) in the induction. Nevertheless, it is intuitive that "green" and "blue" are fundamental to human perception whereas "grue" and "bleen" appear to be artifacts.

Gärdenfors [4] proposed a solution to the Grue Paradox using topological notions that overlayed concepts. He argued that the kind of concepts considered to be "natural" by humans have topological dimensions that arise from perceptual and cognitive bases that are grounded in our common physiologies, etc. For instance, the way we perceive colors is determined by our perceptual sensors and the way our brains process the received light signals. Experiments have shown that the "color circle" is indeed fundamental to all human color perception; while different languages may name contiguous sectors within this circle differently, the classifications are consistent. A partial depiction of the color circle is shown at the left end of the cylinder in figure 1, with labels on its circumference marking the approximate regions of the color named conventionally by the labels. Therefore, Gärdenfors (op. cit.) argued, this color circle has a natural topology in which the distance metric between two (color) points on an arc is the arc distance between them. There are other topological dimensions, hue or intensity, that adds to perceptual information leading to classifications like "light green" vs. "dark green", but for our discussion we do not need them. Gärdenfors then noted that relative to the arc topology that the sets of points on the arc that are classified as representing the same color are *convex* with respect to the arc topology. In this instance convexity can be defined as follows. Let a collection S of points on the circumference of the circle be such that if x and y are in S and z is a point in between x and y on the circumference then z must also be in S — then we call S convex (with respect to the arc topology). Then Gärdenfors further noted that Goodman's Grue Paradox resides in a setting that can be represented as shown in figure 1, showing that in addition the color circle there is another topological dimension, that of linear time which is another perceptual quality of humans[4]. The entire topology of the setting of the Grue Paradox is then the product topology of the color circle topology and the linear time topology. From figure 1 it is clear that while "green" (or "blue") are convex in this product topology, neither "grue" (dashed boundary) nor "bleen" (solid boundary) are. From this and other examples Gärdenfors proposed that the concepts (predicates) that are projectible, i.e., for which induction works, are those that have convex extensions in the topologies of their natural dimensions.

Induction over time is not the only problem posed by some predicates. The Sorites Paradox (op.cit) is one in which the induction is over some ordered sequence of values of the predicate argument. We recall one version of it. Suppose we consider the predicate $Heap(n)$ to mean "n grains of sand is a heap". Then supposedly noboby will regard just one grain of sand as a heap, so we have $\neg Heap(1)$. Now if $\neg Heap(n)$ then intuitively adding just one grain of sand will not result in a heap, so $\neg Heap(n+1)$. Then by induction it is never possible to have any heap of sand however large n gets. This is obviously absurd. Two solutions appear to be to give up induction, or to conclude that the predicate $Heap(n)$ does not have a classically defined truth value, e.g. it is a fuzzy predicate. There are numerous ways to interpret this fuzziness. Paoli's [9] treatment takes on the the standard fuzzy approach to expose difficulties, and proposes lattice-theoretic systems of truth degrees instead. Another is that of Yatabe and Inaoka [11] in which the interpretation of $Heap(\_)$ may not be fixed for all time or all arguments but can shift with either, so failing classical extensionality. They have some resemblance to our thesis that perhaps the fuzziness of a predicate may be attributed to differing crisp interpretations of it by different agents and it is to this we shall now proceed.

2. Agents Making Decisions About Set Membership

A decision problem is about membership in some set S. Given a candidate x, the problem is to decide $x \in S$ (or $x \notin S$). If computation is involved, it boils down to an effective procedure for realizing the characteristic function of S.

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[4]That some metaphysical traditions like ancient Hinduism hypothesize time to be ultimately cyclic does not vitiate this perception of linearity as the Grue Paradox assumes linear time.

Given a family F of agents not all of them may agree on the result of such a decision for some x. This may be due to a less than mathematically precise description of S. Many concepts, that can be formalized more or less as sets, are of that nature. This is not to say that the concept is vague for any agent. Indeed it is possible for each agent to have its own precise notion of what qualifies for membership in the set, and yet have disagreement between rational agents about the membership of a particular instance x. In particular it may be the case that each agent has its own *crisp* (classical, two-valued) interpretation of the extension of S.

The following definition is intended to simplify our presentation of the key points, and will be relaxed later in section 4.

DEFINITION 2.1. Suppose U is the set of all elements in the domain of discourse[5]. U is real-number representable if there is a bijection $\phi : U \rightarrow R$, where R is the set of real numbers.

The idea is to make elements of S (or $\bar{S}$) "concrete" so that an element x is described by a real number. This is independent of whether S is viewed as inherently fuzzy or as crisp. If x corresponds under this bijection to r the decision problem for $x \in S$ is equivalent to $r \in \phi(S)$. Therefore, how one interprets S determines how makes the decision $r \in \phi(S)$. If S is regarded as fuzzy, then it is possible for the grade of membership of x to be positive for both S and its complement $\bar{S}$, whence $\phi(S)$ and $\phi(\bar{S})$ will be fuzzy (possibly overlapping) subsets of the reals. Obviously this is not possible for agents that view S as crisp as in that case each agent i will have its own interpretation of S and $\bar{S}$ effectively (under ϕ) as *disjoint* subsets of R.

We now assume that U is real-number representable and drop reference to the map ϕ by conflating an element x with its real number representation. Further, we use the notation R_i to denote agent i's interpretation of S in the reals.

DEFINITION 2.2. S has the prototype property if there is an x such that every agent i in F decides positively $x \in S$ (equivalently $r \in R_i$). Such an x (or r) is called a prototype.

DEFINITION 2.3. S has the convexity property if for all agents i and all x, y and z in R such that $x < y < z$, if i decides positively for $x \in R_i$ and $z \in R_i$, then i also decides positively for $y \in R_i$.

[5]Candidates for membership in a set S and/or its complement $\bar{S}$.

Observation 1. *In definition 2.3 the fact that the values x, y and z were obtained via the function ϕ is not significant. Any other function ψ such that $\psi(u_1) \leq \psi(u_2)$ if and only if $\phi(u_1) \leq \phi(u_2)$ will suffice.*

Proposition 1. *If S has both the prototype and the convexity properties then*
(i) $\cap_{i \in F} R_i$ is non-empty
(ii) R_i is a connected interval in R for every agent i
(iii) The subset of all prototypes is a unique convex set, and is $\cap_{i \in F} R_i$.

Proof. Part (i) follows directly from definition 2.2 and part (ii) from definition 2.3. For (iii), if x and z are prototypes, then an arbitrary agent i decides postively for $x \in R_i$ and $z \in R_i$. Then for any y such that $x < y < z$, by convexity of R_i agent i must decide positively for $y \in R_i$. But i is arbitrary, so this holds for all agents. Moreover, suppose there is an interval $[s, t]$ of prototypes for some agent i. Let $[s', t']$ be the largest interval containing $[s, t]$ such that s' and t' are prototypes. Then by convexity of prototypes every number in this largest interval correspond to prototypes. By the prototype property all agents decide positively for this interval, and at least one agent decides negatively outside it. This largest interval is unique. ∎

For S with the prototype and convexity property, by virtue of Proposition 1 we may as well represent the characteristic function of an agent j as an interval I_j.

Definition 2.4. A set $\{I_j \mid j \in F\}$ of intervals is nested if there is an ordering $j_1, j_2, \ldots, j_n$ of them such that $I_{j_1} \subseteq I_{j_2} \ldots \subseteq I_{j_n}$.

There is no *a priori* reason to believe that such a nesting of agent intervals exists even though by Proposition 1 they must have non-empty intersection. However, given the prototype and convexity property we now argue that there is another family of agents respecting these properties but which have nested intervals. The next definition formalizes this by considering different families of agents which are *indistinguishable* with respect to the proportion which decide positively for $x \in S$ for all x. They are observationally identical if their internal structures are not accessible.

Definition 2.5. Two equal cardinality families F and F' of agents are *input-output (I-O) equivalent* if for each x the proportion of agents in F who decide that $x \in S$ is equal to the proportion of agents in F' who decide that $x \in S$. (Therefore the proportion which decide negatively also agree.)

If the two families are each finite but are not equal in cardinality, it may be the case that proportions are often unequal even though very close. There is a simple way to generalize definition 2.5 to recover its intention. Suppose $card(F) = N$ and $card(F') = M$. Then we increase the cardinality of F M-fold, and that of F' N-fold. Then, e.g., for a given decision $x \in S$ if originally K out of N in F decide positively, in the M-fold increased set this number becomes $K.M$; and similarly for F'. The only complication arises when in the increased set there is a proportion that cannot be "cancelled out" to a smaller proportion in which the denominator is equal to the original cardinality. However, this case can be treated as an interpolation.

Proposition 2. *Given a family F of agents and a set S, let each agent i have a characteristic function F_i for the set. If S has the convexity and prototype property then there is an I-O equivalent family F' of agents with charateristic functions that form a sequential nesting of intervals*[6].

The key idea behind Proposition 2 is illustrated by Figure 2. The same figure also shows the idea behind the next proposition.

Proof. By Proposition 1 the intervals I_j of the agent family F overlap. Denote explicitly the interval of agent j by $[a_j, b_j]$. There is a point c in the intersection $\cap_{j \in F}[a_j, b_j]$. Consider the intervals $[a_j, c]$ on the left side of $[a_j, b_j]$ and $[c, b_j]$ on the right side. Order the left side intervals in increasing (or non-decreasing) length into the sequence $[a_{j_1}, c] \subseteq [a_{j_2}, c] \subseteq \cdots \subseteq [a_{j_n}, c]$, and likewise the right side intervals into $[a_{k_1}, c] \subseteq [a_{k_2}, c] \subseteq \cdots \subseteq [a_{k_n}, c]$. It follows that the new intervals $[a_{j_1}, a_{k_1}] \subseteq [a_{j_2}, a_{k_2}] \subseteq \cdots \subseteq [a_{j_n}, a_{k_n}]$ form a nested interval sequence I'_l which are the intervals of the new agent family F'.

I-O equivalence between these two families is easily argued. For any x let x_m be the number of intervals in F that contain it. If $x \leq c$ then by construction x_m is also the number of intervals among $[a_{j_1}, c] \subseteq [a_{j_2}, c] \subseteq \cdots \subseteq [a_{j_n}, c]$ that contain x. Again by construction this is also the number of intervals in I' that contain x. The other case $c \geq x$ is similar. ■

The corollary below refers to the standard fuzzy set notion of an α cut (see Klir and Yuan, op. cit.). Suppose we pick a real number c with $0 \leq c \leq 1$. Then an α cut of a fuzzy function f is the set of all arguments x of f such that $f(x) \geq c$.

[6]It is convenient to call such a family of agents, as well as their individual crisp sets, "nested".

Corollary 1. *An n-member nested family with the prototype and covexity properties defines a class of unimodal fuzzy set characteristic functions such that these intervals are n of the α cuts of each of the functions.*

Corollary 1 and Proposition 2 together show the following.

Proposition 3. *The prototype and convexity properties imply a unimodal fuzzy set characteristic function.*

3. Unimodal Fuzzy Set Characteristic Functions

The converse of Propostion 3 is the following.

Proposition 4. *Let $G : R \to [0, 1]$ be a unimodal fuzzy set characteristic function for a fuzzy set S_f with mode at argument $c \in R$. Then for any $n \in N$ there is a set S with the prototype and convexity property and an n-agent family F whose characteristic functions for S are n α cuts of G.*

Proof. See figure 3 for the gist of the proof. More specifically let $c_1, c_2, \ldots, c_n$ be any n α cuts of G. By unimodality (i) each c_i is an interval, hence convex, (ii) (without loss of generality we may assume that) $c_1, c_2, \ldots, c_n$ are successively nested and hence their intersection is non-empty. Each interval can be regarded as an agent's crisp interpretation of the set. Intervals are convex, and their non-empty intersection are the prototypes. ■

4. Generalizations

Here we consider two generalizations. The first is that sometimes fuzzy charateristic functions are not unimodal. The second is that candidates for membership in a fuzzy set may not have a representation in R but in some n-fold product of R.

For the first case Propositions 3 and 4 imply the following observation.

Observation 2. *If a fuzzy set S has a bimodal characteristic function with modes at x and y, the set S can be regarded as the union of two sets S_1 and S_2, with x as a prototype for S_1 and y as a prototype for S_2.*

Observation 2 generalizes to multimodal functions, and has applications in ontology design where "basic" predicates are sought — S_1 and S_2 above are more basic than S. This simple idea has interesting ramifications in the selection of concepts that underlie ontology choice. Typically there are several alternative designs that are viable for a domain of application, each having its own proposed basic predicates that correspond to basic concepts. If it is possible to generate random instances of a chosen predicate, an experimental setting may be devised to ask subjects familiar with the domain to decide (classically) if the instances belong to the predicate, i.e. are in their (crisp) extension, or otherwise. Each subject is thus an agent. Using the approach outlined here, if the resulting aggregated membership function is multimodal, it suggests that the predicate is *not* in fact basic, and should be decomposed into its unimodal components. Each alternative design proposal can be subject to such empirical testing to expose the collection of the "true" basic concepts that they connote.

For the second case we refer to figure 4 when the representation of the domain U is a pair of reals in $R \times R$. In that figure we sketch the a (two-dimensional) unimodal characteristic function of a fuzzy set S. An α cut is shown as the dotted section. It should be clear from this how one can proceed to generalize the characterizations in the sections to such cases, including two-dimensionsional versions of the notions of prototypes and convexity for agents. For instance, in figure 4 the dotted section α cut is an agent's crisp view of the set S, and it is convex. The prototype is the point (x, y) which is the projection of the mode in the unimodal function. The further generalization to multi-dimensional representations is straightforward.

Finally, one can combine the two cases above to deal with multi-dimensional multi-modal functions in the obvious manner.

5. Conclusion

We have shown that it is possible to reconcile a fuzzy set S that has a unimodal characteristic function with crisp interpretations of S that satisfy the prototype and convexity properties. One way to summarize this view is to say that such a fuzzy set can be viewed *as if* it is being crisply interpreted by a family of agents which respect a prototype and also set convexity. We do not make any ontological claim beyond this.

There is no attempt here to see whether the alternative view of agent families, in which each agent uses classical inference, can provide a substrate for fuzzy inference when restricted to reasoning about unimodal fuzzy sets. This is a possible next step.

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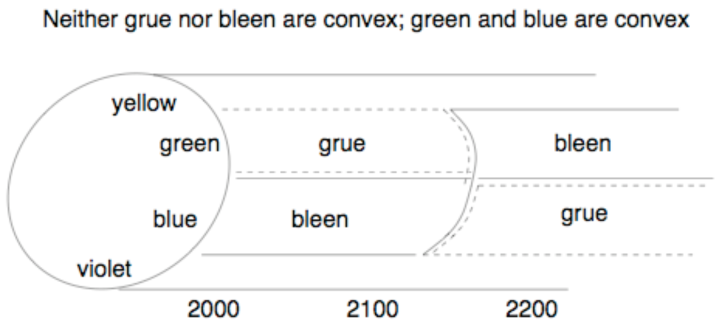

Figure 1. Topologies for the Grue Paradox

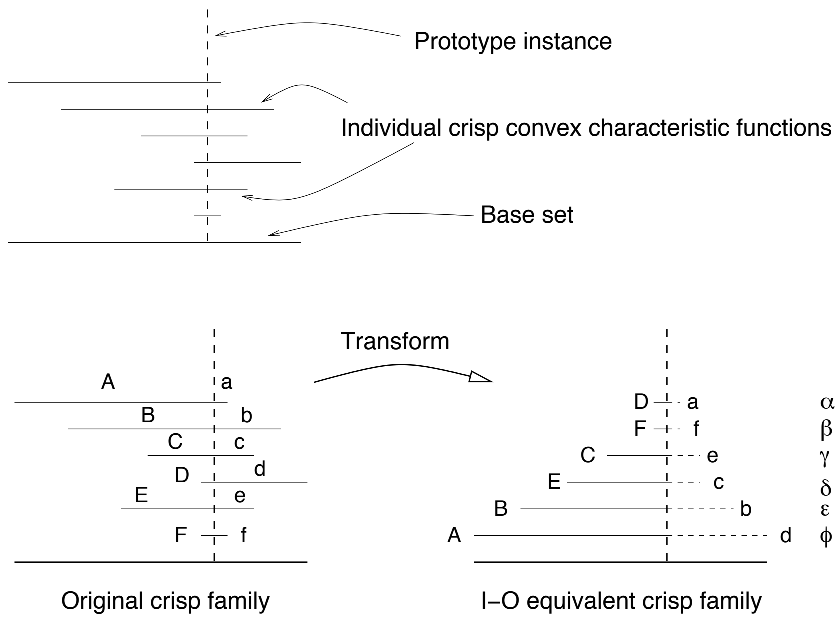

Figure 2. Equivalent I-O Systems

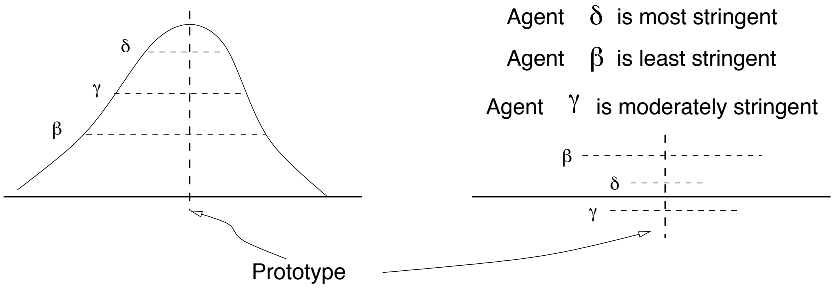

Figure 3. α cuts are agent crisp characteristic functions

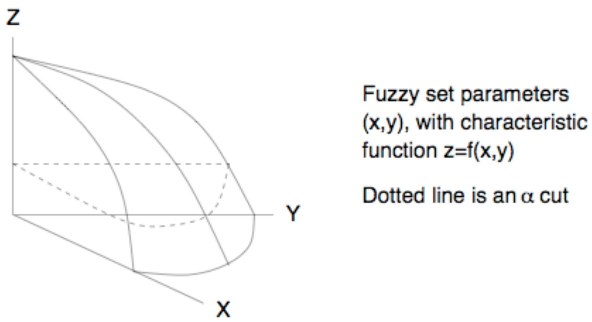

Figure 4. Multi-dimensional generalization

References

[1] Brewka, G., J. Dix, and K. Konolige, 'A Tutorial on Nonmonotonic Reasoning', in Brewka, Jantke, and Schmitt (eds.), *Proceedings of the 2nd International Workshop on Nonmonotonic and Inductive Logic, Lecture Notes in Artificial Intelligence*, Vol. 659, Springer Verlag, Berlin, 1991, pp. 1–88.

[2] Dubois, D., F. Esteva, L. Godo, and H. Prade, 'An Information-based discussion of vagueness', *Proceedings of the Tenth IEEE International Conference on Fuzzy Systems*, Melbourne, December 2001, pp. 781–784.

[3] Freund, M., 'On The Notion of Concept I', *Artificial Intelligence* (to appear).

[4] Gärdenfors, P., *Conceptual Spaces*, MIT Press, 2004, new edition.

[5] Goodman, N., *Fact, Fiction, and Forecast*, Cambridge, MA: Harvard UP, 1955.

[6] Hume, D., *An Enquiry Concerning Human Understanding*, Online edition: http://www.etext.leeds.ac.uk/hume/ehu/ehupbsb.htm#index-div3-N785031167

[7] Klir, G., and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, 1995.

[8] Lakoff, G., *Women, Fire and Dangerous Things*, University of Chicago Press, 1987.

[9] Paoli, F., 'A Really Fuzzy Approach to the Sorites Paradox', *Synthese* 134:363-387, 2003.

[10] Rosch, E., 'Cognitive reference points', *Cognitive Psychology* 7:532-547, 1975.

[11] Yatabe, S., and H. Inaoka, 'On Evan's Vague Object from Set Theoretic Viewpoint', *Journal of Philosophical Logic* 35:423-434, 2006.

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