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A new quantum mechanical formalism based on the probability representation of quantum states. The EPRB paradox resolution

Part I.Einstein's 1927 gedanken experiment revisited.

Foukzon, J., Potapov, A., Men'kova, E.

Abstract. Using new quantum mechanical formalism based on the probability representation of quantum states [14],[15],[16]-[18] the EPRB-paradox is considered. We find that the EPRB-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle. In Part I Einstein's 1927 gedanken experiment by using the probability representation of quantum states explained successfully.

Keywords.Probability representation of quantum states,EPRB-paradox,locality principle.

I.Introduction

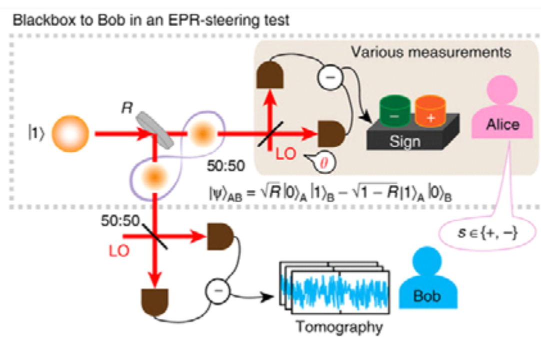
I.1. The EPR paradox

In 1935, Einstein, Podolsky and Rosen (EPR) originated the famous "EPR paradox" [1]. This argument concerns two spatially separated particles which have both perfectly correlated positions and momenta, as is predicted possible by quantum mechanics. The EPR paper spurred investigations into the nonlocality of quantum mechanics, leading to a direct challenge of the philosophies taken for granted by most physicists.The EPR conclusion was based on the assumption of local realism, and thus the EPR argument pinpoints a contradiction between local realism and the completeness of quantum mechanics.

I.2. Einstein's 1927 gedanken experiment

Einstein never accepted orthodox quantum mechanics because he did not believe that its nonlocal collapse of the wavefunction could be real. When he first made this argument in 1927 [2],he considered just a single particle. The particle's wavefunction was diffracted through a tiny hole so that it 'dispersed' over a large hemispherical area before encountering a screen of that shape covered in photographic film. Since the film only ever registers the particle at one point on the screen, orthodox quantum mechanics must postulate a 'peculiar mechanism of action at a distance, which prevents the wave... from producing an action in two places on the screen. That is, according to the theory, the detection at one point must instantaneously collapse the wavefunction to nothing at all other points.

Remark 1.2.1. It was only in 2010, nearly a century after Einstein's original proposal, that a scheme to rigorously test Einstein's 'spooky action at a distance [2],[3] using a single particle (a photon), as in his original conception, was conceived [4]. In this scheme, Einstein's 1927 gedankenexperiment is simplified so that the single photon is split into just two wavepackets, one sent to a laboratory supervised by Alice and the other to a distant laboratory supervised by Bob. However, there is a key difference, which enables demonstration of the nonlocal collapse experimentally: rather than simply detecting the presence or absence of the photon, homodyne detection is used. This gives Alice the power to make different measurements, and enables Bob to test (using tomography) whether Alice's measurement choice affects the way his conditioned state collapses, without having to trust anything outside his own laboratory.



Pic. Pic. 1.2.1.Simplified version of Einstein's original gedankenexperiment Adapted from [5].

Simplified version of Einstein's original gedankenexperiment [5]. A single photon is incident on a beam splitter of reflectivity R and then subjected to homodyne measurements at two spatially separated locations. Alice is trying to convince Bob that she can steer his portion of the single photon to different types of local quantum states by performing various measurements on her side. She does this by using different values of her LO phase θ, and extracting only the sign $s \in \{+,-\}$ of the quadrature she measures. Meanwhile, Bob scans his LO and performs full quantum-state tomography to reconstruct his local quantum state. He reconstructs unconditional and conditional local quantum states to test if his portion of the single photon has collapsed to different states according to Alice's LO setting θ, and result s see Pic. 1.2.1.

The key role of measurement choice by Alice in demonstrating 'spooky action at a distance' was introduced in the famous Einstein–Podolsky–Rosen (EPR) paper [1] of 1935. In its most general form, this phenomenon has been called EPR-steering, to acknowledge the contribution and terminology of Schrödinger [6], who talked of Alice 'steering' the state of Bob's quantum system. From a quantum information perspective, EPR-steering is equivalent to the task of entanglement verification when Bob (and his detectors) can be trusted but Alice (or her detectors) cannot. This is strictly harder than verifying entanglement with both parties trusted [7], but strictly easier than violating a Bell inequality [8], where neither party is trusted [7].

Remark 1.2.2. A recent experimental test of entanglement for a single photon via an entanglement witness has no efficiency loophole [9] however, it demonstrates a weaker form on nonlocality than EPR-steering. In [5], it was demonstrated experimentally that there exist Einstein's elusive 'spooky action at a distance' for a single particle without opening the efficiency loophole without claim to have closed the separation loophole. That is the one-sided device-independent verification of spatial-mode entanglement for a single photon.

I.3.The continuous variable EPR paradox.EPR-Reid's criteria

We remind that EPR treated the case of a non-factorizable pure state $|\psi\rangle$ which describes the results for measurements performed on two spatially separated systems at A and B (Fig.1.3.1). "Non-factorizable" means "entangled", that is, we cannot express $|\psi\rangle$ as a simple product $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, where $|\psi\rangle_A$ and $|\psi\rangle_B$ are quantum states for the results of measurements at A and B, respectively.

Fig.1.3.1.The original EPR gedanken experiment.Two particles move from the source into spatially separated regions A and B, and yet continue to have maximaly correlated positions: $x_A + x_0 = x_B$ and anti-correlated momenta: $p_A = -p_B$. Adapted from [10].

In the first part of their paper, EPR point out in a general way the problematic aspects of such entangled states. The key issue is that one can expand $|\psi\rangle$ in terms of more than one basis, that correspond to different experimental settings, which we parametrize by ϕ. Let us consider the state

$$|\psi\rangle = \int dx |\psi_x\rangle_{\phi,A} \otimes |u_x\rangle_{\phi,B}, \tag{1.3.1}$$

where the eigenvalue x could be continuous or discrete. The parameter setting ϕ at the detector B is used to define a particular orthogonal measurement basis $|u_x\rangle_{\phi,B}$. On measurement at B, this projects out a wave-function $|\psi_x\rangle_{\phi,A}$ at A, the process called “reduction of the wave packet”.

Remark 1.3.1. The locality assumption postulates no action-at-a-distance, so that measurements at a location B cannot immediately “disturb” the system at a spatially separated location A .

Remark 1.3.2. The problematic issue is that different choices of measurements ϕ at B will cause reduction of the wave packet at A in more than one possible way. EPR state that, “as a consequence of two different measurements” at B, the “second system may be left in states with two different wavefunctions”. Yet, “no real change can take place in the second system in consequence of anything that may be done to the first system”.

The problem was established by EPR by a specific example, shown in Fig.1.3.1. EPR considered two spatially separated subsystems, at A and B, each with two observables $\hat{x}$ and $\hat{p}$ where $\hat{x}$ and $\hat{p}$ are non-commuting quantum operators, with commutator

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = 2C \neq 0. \tag{1.3.2}$$

The results of the measurements $\hat{x}$ and $\hat{p}$ are denoted x and p respectively, and this convention we follow throughout the paper. We note that EPR assumed a continuous variable spectrum and considered wavefunction ψ defined in a position representation by

$$\psi(x,x^B) = \int e^{(ip/\hbar)(x-x^B-x_0)}dp, \tag{1.3.3}$$

where x_0 is an constant implying space-like separation. Here the pairs x and p refer to the results for position and momentum measurements at A, while x^B and p^B denote the position and momentum measurements at B. We leave off the superscript for system A, to emphasize the inherent asymmetry that exists in the EPR argument, where one system A is steered by the other, B.

Remark 1.3.3. According to canonical quantum mechanics, one can “predict with certainty” that a measurement $\hat{x}$ will give result $x^B + x_0$, if a measurement $\hat{x^B}$, with result x^B, was already performed at B. One may also “predict with certainty” the result of measurement $\hat{p}$, for a different choice of measurement at B. If the momentum at B is measured to be p, then the result for $\hat{p}$ is $-p$. These predictions are made “without disturbing the second system” at A, based on the assumption, implicit in the original EPR paper, of “locality”.

Remark 1.3.4.The locality assumption can be strengthened if the measurement events at A and B are causally separated (such that no signal can travel from one event to the other, unless faster than the speed of light)

Remark 1.3.5.The remainder of the EPR argument may be summarized as follows. Assuming local realism, one deduces that both the measurement outcomes, for x and p at A, are predetermined. The perfect correlation of x with $x^B + x_0$ implies the existence of an “element of reality” for the measurement $\hat{x}$. Similarly, the correlation of p with $-p^B$ implies an “element of reality” for $\hat{p}$. Although not mentioned by EPR, it will prove useful to mathematically represent the “elements of reality” for $\hat{x}$ and $\hat{p}$ by the respective variables μ_x^A and μ_p^A , whose “possible values are the predicted results of the measurement”

Remark 1.3.6. To continue the argument, *local realism* implies the existence of two elements of reality, μ_x^A and μ_p^A, that *simultaneously* predetermine, with absolute definiteness, the results for measurement x or p at A. These “elements of reality” for the localized subsystem A are not themselves consistent with quantum mechanics. Simultaneous determinacy for both the position and momentum is not possible for any quantum state. Hence, assuming the validity of local realism, one concludes quantum mechanics to be incomplete or even inconsistent!

Remark 1.3.7. We claim that any assumption of local realism is completely wrong. Such claim meant as minimum the weak postulate of nonlocality.

The weak postulate of nonlocality for continuous variables.

The Heisenberg uncertainty relations

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \geq 1 \tag{1.3.4}$$

cannot be violated in any cases:

(i) of course according to quantum mechanics, the Heisenberg uncertainty relations (1.3.4) cannot be violated if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured directly by measurements performed on the particle $\mathbf{A}$,

(ii) the Heisenberg uncertainty relations (1.3.4) cannot be violated even if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured indirectly, i.e. by using measurement on particle $\mathbf{B}$, as required in EPR gedanken experiment,

(iii) in any cases true coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** cannot be predicted samultaneously with a sufficiently small uncertainty $\Delta x^{\mathbf{A}}$ and $\Delta p^{\mathbf{A}}$ such that the Reid’s inequality [13]:

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} < 1 \tag{1.3.5}$$

based on local realism would be satisfied, i.e., alwais

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \nless 1. \tag{1.3.6}$$

We claim strictly stronger assumptions of nonlocality than mentioned above.

The strong postulate of nonlocality for continuous variables.

Let $|\psi_t^x\rangle_{\mathbf{A}}$ and $|\psi_t^x\rangle_{\mathbf{B}}$ be a state vector in x-representation at instant t of the particle **A** and

particle **B** correspondingly.

Let $|\psi_t^p\rangle_{\mathbf{A}}$ and $|\psi_t^p\rangle_{\mathbf{B}}$ be a state vector in p-representation at instant t of the particle **A** and

particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{B}}$ be a wave functions in x-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{B}}$ be a wave functions in p-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let

$\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}}, p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.

We claim that:

(i) whenever a measurement of the coordinate x of a particle **B** is performed at instant

t with result $\bar{x}^{\mathbf{B}} \in [x^{\mathbf{B}} - \varepsilon, x^{\mathbf{B}} + \varepsilon], \varepsilon \ll 1,$ then:

(a) according to quantum mechanics a state vector $|\psi_t^x\rangle_{\mathbf{B}}$ collapses at instant t to the state

vector

$$\left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)|\psi_t^x\rangle_{\mathbf{B}} \tag{1.3.7}$$

given by law (1.2.20), where $\widehat{L}^{\mathbf{B}}_{\mathbf{x}^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2-particle non projective Hilbert space $\mathbf{H}$, representing the localization of

particle $\mathbf{B}$ around the point $x^{\mathbf{B}}$, (see subsection II.2.),

(b) according postulate of nonlocality a state vector $|\psi_t^x\rangle_{\mathbf{A}}$ immediately collapses at instant t

to the state vector

$$\left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}(\delta,\varepsilon)|\psi_t^x\rangle_{\mathbf{A}} \tag{1.3.8}$$

given by law (1.2.20) and this is true independent of the distance in Minkovski spacetime

$M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi_t^x\rangle_{\mathbf{B}} \overset{\text{collapse}}{\to} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi_t^x\rangle_{\mathbf{A}} \overset{\text{collapse}}{\to} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}+x_0}\right\rangle_{\mathbf{A}} \tag{1.3.9}$$

(ii) under conditions given by Eq.(1.3.7)-Eq.(1.3.9) two-particle wave function $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}})$ collapses at instant t by law

$$\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}}) \overset{\text{collapse}}{\to} \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}}) \tag{1.3.10}$$

(iii) whenever a measurement of the momentum $p^{\mathbf{B}}$ of a particle **B** is performed at instant

t with result $\bar{p}^{\mathbf{B}} \in [p^{\mathbf{B}} - \varepsilon, p^{\mathbf{B}} + \varepsilon], \varepsilon \ll 1,$ then:

(a) according to quantum mechanics a state vector $|\psi_t^p\rangle_{\mathbf{B}}$ collapses at instant t to the state vector

$$\left|\psi^p_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)|\psi_t^p\rangle_{\mathbf{B}}, \tag{1.3.11}$$

where $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2-particle non projective Hilbert space $\mathbf{H}$, representing the localization of momentum of the particle $\mathbf{B}$ around the value $p^{\mathbf{B}}$. The localization operators $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ have been chosen to have the following form:

$$\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon) = \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{p} - p^{\mathbf{B}})^2\right] \tag{1.3.12}$$

where $\delta \in (0,1]$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

(b) according postulate of nonlocality a state vector $|\psi_t^p\rangle_{\mathbf{A}}$ immediately collapses at instant t to the state vector

$$\left|\psi^p_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}(\delta,\varepsilon)|\psi_t^p\rangle_{\mathbf{A}} \tag{1.3.13}$$

and this is true independent of the distance in Minkovski spacetime $M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi_t^p\rangle_{\mathbf{B}} \overset{\text{col}\,\textbf{lapse}}{\rightarrow} \left|\psi^p_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi_t^p\rangle_{\mathbf{A}} \overset{\text{col}\,\textbf{lapse}}{\rightarrow} \left|\psi^p_{t,\delta,\varepsilon,-p^{\mathbf{B}}}\right\rangle_{\mathbf{A}} \tag{1.3.14}$$

(iv) under conditions given by Eq.(1.3.11)-Eq.(1.3.13) two-particle wave function $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ collapses at instant t by law

$$\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}}) \overset{\text{col}\,\textbf{lapse}}{\rightarrow} \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}}). \tag{1.3.15}$$

Remark 1.3.8. Let $p_t^{\mathbf{A}}$ and $p_t^{\mathbf{B}}$ be the momentum at instant t of the particle **A** and particle **B** correspondingly. Note that whenever a measurement of the coordinate x of a particle **B** is performed at instant t with an accuracy $\varepsilon_{x^{\mathbf{B}}} \ll 1$ then:

(i) immediately after this measurement the momentum $p_t^{\mathbf{B}}$ at instant t changed according to quantum mechanics by the Heisenberg uncertainty relations (1.3.4);

(ii) immediately after this measurement the momentum $p_t^{\mathbf{A}}$ at instant t changed according to postulate of nonlocality by the Heisenberg uncertainty relations (1.3.4)

Remark 1.3.9. Let $x_t^{\mathbf{A}}$ and $x_t^{\mathbf{B}}$ be the coordinate at instant t of the particle **A** and

particle
B correspondingly. Note that whenever a measurement of the momentum p of a particle **B**
is performed at instant t with an accuracy $\varepsilon_{p^{\mathbf{B}}} \ll 1$ then:
(i) immediately after this measurement the coordinate $x_t^{\mathbf{B}}$ at instant t changed

according
to quantum mechanics by the Heisenberg uncertainty relations (1.3.4);
Remark 1.3.10. Schrödinger [6] pointed out that the EPR two-particle wavefunction in
Eq.(1.3.3) was verschränkten - which he later translated as entangled - i.e., not of the
separable form $\psi_A\psi_B$. Schrödinger considered as a possible resolution of the paradox that
this “entanglement” degrades as the particles separate spatially, so that EPR correlations
would not be physically realizable.
Definition 1.3.1. Quantum inseparability (entanglement) for a general mixed quantum state is defined as the failure of

$$\widehat{\rho} = \int d\lambda P(\lambda)\widehat{\rho}_{\lambda}^{\mathbf{A}} \otimes \widehat{\rho}_{\lambda}^{\mathbf{B}}, \tag{1.3.16}$$

where $\int d\lambda P(\lambda) = 1$ and $\widehat{\rho}$ is the density operator. Here λ is a discrete or continuous label for component states, and $\widehat{\rho}_{\lambda}^{\mathbf{A}}$ and $\widehat{\rho}_{\lambda}^{\mathbf{B}}$ correspond to density operators that are restricted to the Hilbert spaces $\mathbf{A}$ and B respectively.

Remark 1.3.11. The definition of inseparability extends beyond that of the EPR situation, in that one considers a whole spectrum of measurement choices, parametrized by θ for those performed on system A, and by ϕ for those performed on B. We use canonical notation $\hat{x}_{\theta}^{A}$ and $\hat{x}_{\phi}^{B}$ to describe all measurements at A and B. Denoting the eigenstates of $\hat{x}_{\theta}^{A}$ by $|x_{\theta}^{A}\rangle$, we define $P_Q(x_{\theta}^{A}|\theta,\lambda) = \langle x_{\theta}^{A}|\widehat{\rho}_{\lambda}^{A}|x_{\theta}^{A}\rangle$ and $P_Q(x_{\phi}^{B}|\phi,\lambda) = \langle x_{\phi}^{B}|\widehat{\rho}_{\lambda}^{B}|x_{\phi}^{B}\rangle$, which are the localized probabilities for observing results x_{θ}^{A} and x_{ϕ}^{B} respectively. The separability condition (1.3.9) then implies that joint probabilities $P(x_{\theta}^{A},x_{\phi}^{B})$ are given as [13]:

$$P(x_{\theta}^{A},x_{\phi}^{B}) = \int d\lambda P(\lambda)P_Q(x_{\theta}^{A}|\lambda)P_Q(x_{\phi}^{B}|\lambda). \tag{1.3.17}$$

Remark 1.3.12. We note the canonical restriction

$$\Delta^2(x^A|\lambda)\Delta^2(p^A|\lambda) \geq 1 \tag{1.3.18}$$

where $\Delta^2(x^{\mathbf{A}}|\lambda)$ and $\Delta^2(p^{\mathbf{A}}|\lambda)$ are the variances of $P_Q(x_{\theta}^{\mathbf{A}}|\theta,\lambda)$ for the choices θ corresponding to position x and momentum p, respectively. Thus

$$\Delta^2(x^{\mathbf{A}}|\lambda)\Delta^2(p^{\mathbf{A}}|\lambda) < 1 \tag{1.3.19}$$

is an EPR criterion, meaning that this would imply an EPR "paradox".

Remark 1.3.13.Note that the original EPR state of Eq. (1.3.3) is not separable. Suppose that, based on a result $x^{\mathbf{B}}$ for the measurement at $\mathbf{B}$, an estimate $x_{\mathbf{est}}(x_{\mathbf{B}})$ is made of the result x at $\mathbf{A}$. We may define the average error $\Delta_{\mathbf{inf}}x$ of this inference as the root mean square (RMS) of the deviation of the estimate from the actual value, so that [11]-[13]:

$$\Delta_{\mathbf{inf}}^2 x = \int dx dx^{\mathbf{B}} P_t(x,x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^B))^2 . \tag{1.3.20}$$

An inference variance $\Delta_{\mathbf{inf}}^2 p$ is defined similarly,i.e.

$$\Delta_{\mathbf{inf}}^2 p = \int dp dp^{\mathbf{B}} P_t(p,p^{\mathbf{B}})(p - p_{\mathbf{est}}(p^B))^2 . \tag{1.3.21}$$

Remark 1.3.14.Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) $P_t(x,x^{\mathbf{B}})$ is the joint probability of obtaining an outcome x at $\mathbf{A}$ and $x^{\mathbf{B}}$ at $\mathbf{B}$ at instant t is of the form

$$P_t(x,x^{\mathbf{B}}) \sim \left|\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})\right|^2, \tag{1.3.22}$$

(ii) $P_t(p,p^{\mathbf{B}})$ is the joint probability of obtaining an outcome p at $\mathbf{A}$ and $p^{\mathbf{B}}$ at $\mathbf{B}$ at instant t is of the form

$$P_t(p,p^{\mathbf{B}}) \sim \left|\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})\right|^2. \tag{1.3.23}$$

The best estimate, which minimizes $\Delta_{\mathbf{inf}}x$, is given by choosing $x_{\mathbf{est}}$ for each $x^{\mathbf{B}}$ to be the mean $\langle x|x^{\mathbf{B}}\rangle$ of the conditional distribution $P_t(x|x^{\mathbf{B}})$. This is seen upon noting that for each result $x^{\mathbf{B}}$, we can define the RMS error in each estimate as

$$\Delta_{\mathbf{inf}}^2(t,x|x^{\mathbf{B}}) = \int dx P_t(x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2 . \tag{1.3.24}$$

The average error in each inference is minimized for $x_{est} = \langle x|x^{\mathbf{B}}\rangle$, when each $\Delta_{\mathbf{inf}}^2(t,x|x^{\mathbf{B}})$ becomes the variance $\Delta^2(t,x|x^B)$ of $P_t(x|x^B)$. We thus define the minimum inference error $\Delta_{\mathbf{inf}}x$ for position, averaged over all possible values of $x^{\mathbf{B}}$, as

$$V_{\mathbf{A|B}}^x = \min(\Delta_{\mathbf{inf}}^2 x) = \int dx^B P_t(x^B)\Delta^2(t,x|x^B) , \tag{1.3.25}$$

where $P(x^{\mathbf{B}})$ is the probability density for a result $x^{\mathbf{B}}$ upon measurement of $\hat{x}^{\mathbf{B}}$. This minimized inference variance is the average of the individual variances for each outcome at $\mathbf{B}$. Similarly, we can define a minimum inference variance, $V_{A|B}^p$, for momentum,i.e.

$$V^{p}_{\mathbf{A|B}} = \min(\Delta^2_{\mathbf{inf}}p) = \int dp^B P_t(p^B)\Delta^2(t,p|p^B). \quad (1.3.26)$$

Remark 1.3.15.Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) according to local realism the conditional distributions densities $P_{\mathbf{loc}}(x|x^B)$ and

$P_{\mathbf{loc}}(p|p^B)$vare given by formulae

$$P_{\mathbf{loc}}(x|x^B) \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(x,x_{\mathbf{B}}) \quad (1.3.27)$$

and

$$P_{\mathbf{loc}}(p|p^B) \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}}). \quad (1.3.28)$$

(ii) distributions densities $P_{\mathbf{loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{loc}}(t,p^B)$ are given by formulae

$$P_{\mathbf{loc}}(t,x^B) = \int dx P_{\mathbf{loc}}(t,x|x^B) \quad (1.3.29)$$

and

$$P_{\mathbf{loc}}(t,p^B) = \int dp P_{\mathbf{loc}}(t,p|p^B). \quad (1.3.30)$$

Remark 1.3.16.Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) according to postulates of nonlocality the conditional distributions densities

$P_{\mathbf{n.loc}}(t,x|x^B)$

and $P_{\mathbf{n.loc}}(t,p|p^B)$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x|x^B) = \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(x,x_{\mathbf{B}}) \quad (1.3.31)$$

and

$$P_{\mathbf{n.loc}}(t,p|p^B) \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(p,p_{\mathbf{B}}), \quad (1.3.32)$$

see Eq.(1.3.10) and Eq.(1.3.15) respectively.

(ii) distributions $P_{\mathbf{n.loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{n.loc}}(t,p^B)$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x^B) = \int dx P_{\mathbf{n.loc}}(t,x|x^B) \quad (1.3.33)$$

and

$$P_{\mathbf{n.loc}}(t,p|^B) = \int dp P_{\mathbf{n.loc}}(t,p|p^B) \quad (1.3.34)$$

Thus we can define corresponding RMS errors as

$$\begin{aligned}\Delta^2_{\mathbf{loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2 \\ \Delta^2_{\mathbf{loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \tag{1.3.35}$$

and

$$\begin{aligned}\Delta^2_{\mathbf{n.loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2, \\ \Delta^2_{\mathbf{n.loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \tag{1.3.36}$$

respectively.We thus define the minimum inference error $\Delta_{\mathbf{inf}}x$ for position, averaged over all possible values of $x^{\mathbf{B}}$ and $p^{\mathbf{B}}$ as

$$\begin{aligned}\min(\Delta^2_{\mathbf{loc.inf}}x) &= \int dx^B P_{\mathbf{loc}}(t,x^B)\Delta^2_{\mathbf{loc.}}(t,x|x^B)\,, \\ \min(\Delta^2_{\mathbf{loc.inf}}p) &= \int dp^B P_{\mathbf{loc}}(t,p^B)\Delta^2_{\mathbf{loc.}}(t,p|p^B)\end{aligned} \tag{1.3.37}$$

and

$$\begin{aligned}\min(\Delta^2_{\mathbf{n.loc.inf}}x) &= \int dx^B P_{\mathbf{n.loc}}(t,x^B)\Delta^2_{\mathbf{n.loc.}}(t,x|x^B)\,, \\ \min(\Delta^2_{\mathbf{n.loc.inf}}p) &= \int dp^B P_{\mathbf{n.loc}}(t,p^B)\Delta^2_{\mathbf{n.loc.}}(t,p|p^B).\end{aligned} \tag{1.3.38}$$

respectively.From Eq.(1.3.37) and Eq.(1.3.38) we obtain the EPR-nonlocality criteria

$$\begin{aligned}&\left|\min\Delta^2_{\mathbf{loc.inf.}}\,x - \min\,\Delta^2_{\mathbf{n.loc.inf.}}\,x\right| = \\ &\left|\int dx^B[P_{\mathbf{loc}}(t,x^B)\Delta^2_{\mathbf{loc.}}(t,x|x^B) - P_{\mathbf{n.loc}}(t,x^B)\Delta^2_{\mathbf{n.loc.}}(t,x|x^B)]\right| > 0, \\ &\left|\min\Delta^2_{\mathbf{loc.inf.}}\,p - \min\Delta^2_{\mathbf{n.loc.inf.}}\,p\right| = \\ &\left|\int dp^B[P_{\mathbf{loc}}(t,p^B)\Delta^2_{\mathbf{loc.}}(t,p|p^B) - P_{\mathbf{n.loc}}(t,p^B)\Delta^2_{\mathbf{n.loc.}}(t,p|p^B)]\right| > 0\end{aligned} \tag{1.3.39}$$

and

$$\left|\left(\min\Delta^2_{\mathbf{loc.inf.}}\,x\right)\left(\min\Delta^2_{\mathbf{loc.inf.}}\,p\right) - \left(\min\,\Delta^2_{\mathbf{n.loc.inf.}}\,x\right)((\Delta^2_{\mathbf{n.loc.inf}}p))\right| > 0. \tag{1.3.40}$$

I.4.The EPR-Bohm paradox. Reid's criteria for EPR-Bohm paradox.

Bohm [26]-[27] considered two spatially-separated spin-$1/2$ particles at A and B produced in an entangled singlet state (often referred to as the "EPR-Bohm state" or the "Bell-state"):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle_A\left|-\frac{1}{2}\right\rangle_B - \left|-\frac{1}{2}\right\rangle_A\left|\frac{1}{2}\right\rangle_B\right) \tag{1.4.1}$$

Here $|\pm\frac{1}{2}\rangle_A$ are eigenstates of the spin operator $\hat{J}^A_z$, and we use $\hat{J}^A_z$, $\hat{J}^A_x$, $\hat{J}^A_y$ to define the spin-components measured at location A. The spin-eigenstates and measurements at

B are defined similarly. By considering different quantization axes, one obtains different but equivalent expansions of $|\psi\rangle$ in Eq. (1.4.1), just as EPR suggested.

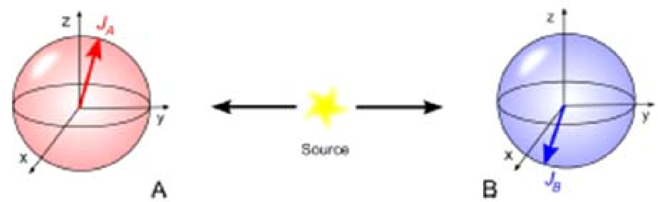


Fig.1.4.1.The Bohm gedanken EPR experiment.Two spin-$1/2$ partiles prepared in a singlet state from the source into spatially separated regions A and B,and give anti-correlated outomes for J_θ^A and J^B,where θ is x,y or z.Adapted from [13].

Bohm's paradox is based on the existence, for Eq. (1.9.1), of a maximum anti-correlation between not only $\widehat{J}_z^A$ and $\widehat{J}_z^B$, but $\widehat{J}_y^A$ and $\widehat{J}_y^B$, and also $\widehat{J}_x^A$ and $\widehat{J}_x^B$. An assumption of local realism would lead to the conclusion that the three spin components of particle A were simultaneously predetermined, with absolute definiteness. Since no such quantum description exists, this is the situation of an EPR paradox.

Remark 1.4.1.Bohm's paradox is based on the existence, for Eq. (1.4.1), of a maximum
anti-correlation between not only $\widehat{J}_z^A$ and $\widehat{J}_z^B$, but $\widehat{J}_y^A$ and $\widehat{J}_y^B$, and also $\widehat{J}_x^A$ and $\widehat{J}_x^B$.

Remark 1.4.2. Note that an assumption of local realism would lead to the conclusion that
the three spin components of particle A were simultaneously predetermined, with absolute
definiteness. Since no such quantum description exists, this is the situation of an EPR
paradox.

Remark 1.4.3.Criteria sufficient to demonstrate Bohm's EPR paradox can be derived
using Reid's canonical inferred uncertainty approach [13]. Using the Heisenberg spin
uncertainty relation

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq |\langle J_z^{\mathbf{A}} \rangle|/2, \tag{1.4.2}$$

one obtains the following canonical spin-EPR criterion that is useful for the Bell state given by Eq. (1.4.1)

$$\Delta_{\mathbf{inf}} J_x^{\mathbf{A}} \Delta_{\mathbf{inf}} J_y^{\mathbf{A}} < \frac{1}{2} \sum_{J_z^B} P(J_z^{\mathbf{B}}) \left| \langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}} \right|. \tag{1.4.3}$$

Here $\langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}}$ is the mean of the conditional distribution $P(J_z^{\mathbf{A}}|J_z^{\mathbf{B}})$.Calculations for Eq.(1.4.1) including the effect of detection efficiency η reveals this EPR criterion to be satisfied for $\eta > 0.62$.The concept of spin-EPR has been experementally tested in the continuum limit with purely optical systems for states with $\langle J_z^A \rangle \neq 0$.In this case the EPR criterion linked closely to definition of spin squeesing

$$\Delta_{\mathbf{inf}} J_x^A \Delta_{\mathbf{inf}} J_y^A < |\langle J_z^A \rangle|. \tag{1.4.4}$$

Remark 1.4.4.We claim that any assumption of local realism is completely wrong.The

three spin components of particle A were simultaneously predetermined, does not with

absolute definiteness but only with uncertainties which requred by Heisenberg spin uncertainty relations (1.4.5). Such claim meant as minimum the weak postulate of nonlocality.

I.4.1.The weak postulate of nonlocality.

The Heisenberg spin uncertainty relations

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq |\langle J_z^{\mathbf{A}} \rangle|/2, \Delta J_x^{\mathbf{A}} \Delta J_z^{\mathbf{A}} \geq |\langle J_y^{\mathbf{A}} \rangle|/2, \Delta J_z^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq |\langle J_x^{\mathbf{A}} \rangle|/2 \tag{1.4.5}$$

does not violated in any cases:

(i) if the three spin components of the particle **A** are measured directly by measurements

performed on the particle **A**

(ii) and even if some spin components of the particle **A** are measured indirectly as required in Bohm gedanken EPR experiment.

Think of the following situation: a particle with zero spin decays into two particles (**A** and **B**), each with $1/2$-spin. Due to the fact that spin angular momentum must be conserved during the decay, if initially the total spin angular momentum was zero, then after the decaying process it must still be zero. Therefore, particles **A** and **B** have opposite spin.Take as an example the dissociation of an exited hydrogen molecule into two hydrogen atoms. If the decaying mechanism does not change total angular momentum, then the spins on the hydrogen atoms will be anti-correlated.

Remark 1.4.5.Whenever a measurement of the spin of **A** is found to be positive with respect of the z-axis (we shall note this state as $|\uparrow\rangle_z$, then,under local realism,we could infer that the spin of the **B** particle must be negative $|\downarrow\rangle_z$, and this is true independent of the distance that separates the particles. The spin of these particles are then entangled.

Remark 1.4.6. We claim again that any assumption of local realism is completely wrong.

I.4.2.The strong postulate of nonlocality.

Let $|\psi_t\rangle_{\mathbf{A}}$ and $|\psi_t\rangle_{\mathbf{B}}$ a state at instant t of the particle **A** and particle **B** correspondingly.

Let $|\uparrow\rangle_{z,\mathbf{A/B}}$ be eigenstates of the spin operator $\mathbf{S}^z_{\mathbf{A/B}}$:

$$\mathbf{S}^z_{\mathbf{A/B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.4.6)$$

We claim that:

(i) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and particle **A** is found in the state $|\uparrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,A}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), then a state $|\psi_{t_1}\rangle_B$ immediately collapses at instant t_1 to the state $|\downarrow\rangle_{z,B}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), and this is true independent of the distance in Minkovski spacetime that separates the particles:

$$|\psi_{t_1}\rangle_A \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\uparrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\downarrow\rangle_{z,B} \quad (1.4.7)$$

(ii) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and particle **A** is found in the state $|\downarrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state $|\downarrow\rangle_{z,A}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), then a state $|\psi_{t_1}\rangle_B$ immediately $|\psi_t\rangle_B$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,B}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), and this is true independent of the distance in Minkovski spacetime that separates the particles:

$$|\psi_{t_1}\rangle_A \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\downarrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\uparrow\rangle_{z,B} \quad (1.4.8)$$

Note that,we can not predict which spin will be positive (or negative) with respect of the z-axis, so the state that describes the spins of the particles could be for instance the spin singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \tag{1.4.9}$$

We have a probability of 50% for the spin of particle **A** to be positive (and the spin of **B** negative) and a probability of 50% of it being the other way around.

Remark 1.4.7. So far we have assumed that we are performing a measurement along the z-axis, but measurements are not restricted to this particular election, we could measure for instance the spin of particle **A** along the $\mathbf{a}$-axis and the spin of **B** along the $\mathbf{b}$-axis. Let's see what happens if we decide to measure the spin along the x-axis: $\mathbf{a} = \mathbf{b} = x$. As it known for $1/2$-spins, the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$ can be represented by the 2×2 hermitian matrix

$$\mathbf{S}^x_{\mathbf{A/B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.4.10}$$

By performing a change of basis we can rewrite the state $|\psi\rangle$ in terms of the eigenstates of the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$:

$$|u\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle), |v\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle - |\uparrow\rangle), \tag{1.4.11}$$

and using Eq. (1.4.10), we can rewrite the state $|\psi\rangle$ as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|vu\rangle - |uv\rangle). \tag{1.4.12}$$

The strong postulate of nonlocality in this case takes the form similarly mentioned above.Just like before, by choosing to measure the spin of **A** along the x-axis we can determine it's value and infer the value of the spin of particle $\widetilde{\mathbf{B}}$ $\left(\widetilde{\mathbf{B}} \neq \mathbf{B}\right)$ in the state $|\psi\rangle_{x,\widetilde{\mathbf{B}}} = |u\rangle_{x,\mathbf{B}} \neq |\psi\rangle_{x,\mathbf{B}}$ without the need to measure it (and vice versa).

Furthermore, it turns out that this is the case independent of the election of the axis we choose to measure! (Provided that $\mathbf{a} = \mathbf{b} = v$).

This is exactly the same situation such that a simple choice of the axis along which to measure the spin **A** allow us to establish the value of the spin of **B** along this same axis without the need to measure it. And this is also the case (as we already saw) for physical properties described by non-commuting operators ($\mathbf{S}^x$ and $\mathbf{S}^z$ do not commute).

II.A new quantum mechanical formalism based on the probability representation of quantum states.

II.1.Generalized Postulates for Continuous Valued Observables.

Suppose we have an n-dimensional physical quantum system.

I.Then we claim the following:

Q.I.1. Any given n-dimensional quantum system is identified by a set $\mathbf{Q}$:

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \mathfrak{I}, \mathfrak{R}, \mathcal{L}_{2,1}, \mathfrak{I}^*(\mathbf{H}), \mathbf{G}, |\psi_t\rangle\rangle$$

where:

(i) $\mathbf{H}$ that is some infinite-dimensional complex Hilbert space,

(ii) $\mathfrak{I} = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,

(iii) $\mathfrak{R} = (\mathbb{R}^n, \Sigma)$ that is measurable space,

(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of complex valued random variables $X : \Omega \to \mathbb{C}^n$ such

that

$$\int_{\Omega} \|X(\omega)\| d\mathbf{P} < \infty, \int_{\Omega} \|X(\omega)\|^2 d\mathbf{P} < \infty \tag{2.1.1}$$

(v) $\mathbf{G} : \mathbf{C}^*(\mathbf{H}) \times \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_{\Omega}\left(\mathbf{G}\left[\widehat{Q}, |\psi\rangle\right](\omega)\right)d\mathbf{P} = \mathbf{E}_{\Omega}\left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right),$$
$$\mathbf{G}\left[\widehat{\mathbf{1}}, |\psi\rangle\right](\omega) = 1 \tag{2.1.2}$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian adjoint operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$ such that

$\widehat{Q} \in \mathfrak{I}^*(\mathbf{H}) \subsetneq C^*(\mathbf{H})$, where $C^*(\mathbf{H})$ is C^*- algebra of the Hermitian adjoint operators in $\mathbf{H}$

and $\mathfrak{I}^*(\mathbf{H})$ is commutative subalgebra of $C^*(\mathbf{H})$.

(vi) $|\psi_t\rangle$ is an continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which representedthe evolution of the quantum system $\mathbf{Q}$.

Q.I.2. For any $|\psi_1\rangle, |\psi_2\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$ such that

$$\left\langle \psi_1 \middle| \widehat{Q} \middle| \psi_2 \right\rangle = \left\langle \psi_2 \middle| \widehat{Q} \middle| \psi_1 \right\rangle = 0 \tag{2.1.3}$$

the equality holds

$$\mathbf{G}\left[\widehat{Q}(|\psi_1\rangle + |\psi_2\rangle)\right](\omega) = \mathbf{G}\left[\widehat{Q}|\psi_1\rangle\right](\omega) + \mathbf{G}\left[\widehat{Q}|\psi_2\rangle\right](\omega). \tag{2.1.4}$$

Definition 2.1.1. A random variable $X : \Omega \to E$ is a measurable function from the set of

possible outcomes Ω to some set E.

Definition 2.1.2. Given a probability space $\mathfrak{I} = (\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $\mathfrak{R} = (\mathbb{R}^n, \Sigma)$, any $\mathbb{R}^n$-valued stochastic process that is a collection of $\mathbb{R}^n$-valued random

variables on Ω, indexed by a totally ordered set T ("time"). That is, a stochastic process

$X_t(\omega)$ is a collection $\{X_t(\omega) | t \in T\}$, where each $X_t(\omega)$ is an $\mathbb{R}^n$-valued random variable on Ω. The space $\mathbb{R}^n$ is then called the state space of the process.

Q.I.3. Suppose that the evolution of the quantum system is represented by continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$. Then any process of continuous measurements on measuring observable $\widehat{Q}$ for the system in state $|\psi_t\rangle$ one can to describe by an continuous $\mathbb{R}^n$-valued stochastic process

$$X_t(\omega) = X_t\left(\omega; \left|\widehat{Q}\psi_t\right\rangle\right) \triangleq X_{\left|\widehat{Q}\psi_t\right\rangle}(\omega)$$

given on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $(\mathbb{R}^n, \Sigma)$.

Remark 2.1.1.We assume now for short but without loss of generality that $n = 1$.

Remark 2.1.2. Let $X(\omega)$ be random variable $X(\omega) \in \mathcal{L}_{2,1}(\Omega)$ such that
$X(\omega) = \mathbf{G}[|\psi\rangle](\omega)$,
then we denote such random variable by $X_{|\psi\rangle}(\omega)$. The probability density of random variable $X_{|\psi\rangle}(\omega)$ we denote by $p_{|\psi\rangle}(q), q \in \mathbb{R}$.

Definition 2.1.3. The *classical pure states* correspond to vectors $\mathbf{v} \in \mathbf{H}$ of norm $\|\mathbf{v}\| \equiv 1$.

Thus the set of all classical pure states corresponds to the unit sphere $\mathbf{S}^\infty \subset \mathbf{H}$ in a Hilbert space $\mathbf{H}$.

Definition 2.1.4. The projective Hilbert space $P(\mathbf{H})$ of a complex Hilbert space $\mathbf{H}$ is the set of equivalence classes $[\mathbf{v}]$ of vectors $\mathbf{v}$ in $\mathbf{H}$, with $\mathbf{v} \neq \mathbf{0}$, for the equivalence relation given by $\mathbf{v} \sim_P \mathbf{w} \Leftrightarrow \mathbf{v} = \lambda \mathbf{w}$ for some non-zero complex number $\lambda \in \mathbb{C}$. The equivalence classes for the relation $\sim_P$ are also called rays or projective rays.

Remark 2.1.3.The physical significance of the projective Hilbert space $P(\mathbf{H})$ is that in canonical quantum theory, the states $|\psi\rangle$ and $\lambda|\psi\rangle$ represent the same physical state of the quantum system, for any $\lambda \neq 0$. It is conventional to choose a state $|\psi\rangle$ from the ray $[|\psi\rangle]$ so that it has unit norm $\sqrt{\langle\psi|\psi\rangle} = 1$.

Remark 2.1.4. In contrast with canonical quantum theory we have used instead contrary to $\sim_P$ equivalence relation $\sim_Q$, a Hilbert space $\mathbf{H}$, see Definition 2.1.7.

Definition 2.1.5.The *non-classical pure states* correspond to the vectors $\mathbf{v} \in \mathbf{H}$ of a norm $\|\mathbf{v}\| \neq 1$. Thus the set of all non-classical pure states corresponds to the set $\mathbf{H} \backslash \mathbf{S}^\infty \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}$.

Suppose we have an observable Q of a quantum system that is found through an exhaustive series of measurements, to have a set $\Im$ of values $q \in \Im$ such that $\Im = \cup_{i=1}^{m} (\theta_1^i, \theta_2^i), m \geq 2,\ (\theta_1^i, \theta_2^i) \cap \left(\theta_1^j, \theta_2^j\right) = \varnothing, i \neq j.$Note that in practice any observable Q is measured to an accuracy δq determined by the measuring device. We represent now by $|q\rangle$ the idealized state of the system in the limit $\delta q \to 0,$ for which the observable definitely has the value q.

II.**Then we claim the following**:

Q.**II**.**1**.The states $\{|q\rangle : q \in \Im\}$ form a complete set of δ-function normalized basis states

for the state space $\mathbf{H}_\Im$ of the system.That the states $\{|q\rangle : q \in \Im\}$ form a complete set of

basis states means that any state $|\psi[\Im]\rangle \in \mathbf{H}_\Im$ of the system can be expressed as:

$$|\psi[\Im]\rangle = \int_\Im c_{\psi[\Im]}(q) dq, \tag{2.1.5}$$

where $\mathrm{supp}(c_{\psi[\Im]}(q)) \subseteq \Im$ and while δ-function normalized means that $\langle q|q'\rangle = \delta(q - q')$

from which follows $c_{\psi[\Im]}(q) = \langle q|\psi[\Im]\rangle$ so that

$$|\psi[\Im]\rangle = \int_\Im |q\rangle\langle q|\psi[\Im]\rangle dq. \tag{2.1.6}$$

The completeness condition can then be written as

$$\int_\Im |q\rangle\langle q| dq = \hat{\mathbf{1}}_{\mathbf{H}_\Im}. \tag{2.1.7}$$

Q.**II**.**2**.For the system in state $|\psi[\Im]\rangle$ the probability $P(q, q + dq; |\psi[\Im]\rangle)$ of obtaining the

result $q \in \Im$ lying in the range $(q, q + dq) \subset \Im$ on measuring observable Q is given by

$$P(q, q + dq; |\psi[\Im]\rangle) = p_{|\psi[\Im]\rangle}(q) dq \tag{2.1.8}$$

for any $|\psi[\Im]\rangle \in \mathbf{H}_\Im$.

Remark 2.**1**.**5**. Note that in general case $p_{|\psi[\Im]\rangle}(q) \neq |c_{\psi[\Im]}(q)|^2$.

Q.**II**.**3**.The observable $Q_\Im$ is represented by a Hermitian operator $\widehat{Q}_\Im : \mathbf{H}_\Im \to \mathbf{H}_\Im$ whose eigenvalues are the possible results $\{q : q \in \Im\}$, of a measurement of $Q_\Im$, and the associated eigenstates are the states $\{|q\rangle : q \in \Im\}$,i.e. $\widehat{Q}_\Im|q\rangle = q|q\rangle, q \in \Im$.

Remark 2.**1**.**6**. Note that the spectral decomposition of the operator $\widehat{Q}_\Im$ is then

$$\widehat{Q}_\Im = \int_\Im q|q\rangle\langle q| dq. \tag{2.1.9}$$

Definition 2.**1**.**6**. A connected set in $\mathbb{R}$ is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Definition 2.**1**.**7**. The *well localized pure states* $|\psi[\Theta]\rangle$ *with a support* $\Theta = (\theta_1, \theta_2)$ correspond to vectors of the norm 1 and such that: $\mathrm{supp}(c_{\psi[\Theta]}(q)) = \Theta$ is a connected set in $\mathbb{R}$ Thus the set of all well localized pure states corresponds to the unit sphere

$\mathbf{S}^{\infty}_{\Theta} \subsetneqq \mathbf{S}^{\infty} \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}_{\Theta} \subsetneqq \mathbf{H}$.

Suppose we have an observable Q_{Θ} of a system that is found through an exhaustive series of measurements, to have a continuous range of values $q : \theta_1 < q < \theta_2$.

III.Then we claim the following:

Q.III.1.For the system in well localized pure statestate $|\psi[\Theta]\rangle$ such that:

(i) $|\psi[\Theta]\rangle \in \mathbf{S}^{\infty}_{\Theta}$ and

(ii) supp$(c_{\psi[\Theta]}(q)) \triangleq \{q|c_{\psi[\Theta]}(q) \neq 0\}$ is a connected set in $\mathbb{R}$, then the probability $P(q,q+dq;|\psi[\Theta]\rangle)$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring observable Q_{Θ} is given by

$$P(q,q+dq;|\psi[\Theta]\rangle) = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq. \tag{2.1.10}$$

Q.III.2. $p_{|\psi[\Theta]\rangle}(q)dq = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq.$

Q.III.3.Let $|\psi[\Theta_1]\rangle$ and $|\psi[\Theta_2]\rangle$ be well localized pure states with $\Theta_1 = (\theta^1_1,\theta^1_2)$ and $\Theta_2 = (\theta^2_1,\theta^2_2)$ correspondingly. Let $X_1(\omega) = X_{|\psi[\Theta_1]\rangle}(\omega)$ and $X_2(\omega) = X_{|\psi[\Theta_2]\rangle}(\omega)$ correspondingly. Assume that $\overline{\Theta}_1 \cap \overline{\Theta}_2 = \varnothing$ (here the closure of $\Theta_i, i = 1,2$ is denoted by

$\overline{\Theta}_i, i = 1,2$) then random variables $X_1(\omega)$ and $X_2(\omega)$ are independent.

Q.III.4. If the system is in well localized pure state $|\psi[\Theta]\rangle$ the state $|\psi[\Theta]\rangle$ described by a

wave function $\psi(q,\Theta) = \langle q||\psi[\Theta]\rangle$ and the value of observable Q_{Θ} is measured once each

on many identically prepared system, the average value of all the measurements will be

$$\langle Q_{\Theta}\rangle = \frac{\int_{\Theta} q|\psi(q,\Theta)|^2 dq}{\int_{\Theta} |\psi(q,\Theta)|^2 dq}. \tag{2.1.11}$$

The completeness condition can then be written as $\int_{\Theta} |q\rangle\langle q| dq = \widehat{\mathbf{1}}_{\mathbf{H}_{\Theta}}$. Completeness means that for any state $|\psi[\Theta]\rangle \in \mathbf{S}^{\infty}_{\Theta}$ it must be the case that $\int_{\Theta} |\langle q|\psi[\Theta]\rangle|^2 dq \neq 0$, i.e. there must be a non-zero probability to get some result on measuring observable Q_{Θ}.

Q.III.5.(**von Neumann measurement postulate**) Assume that

(i) $|\psi\rangle \in \mathbf{S}^{\infty}_{\Theta}$ and (ii) supp$(c_{\psi}(q)) = \Theta$ is a connected set in $\mathbb{R}$. Then if on performing a measurement of Q_{Θ} with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the system will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}}. \tag{2.1.12}$$

IV.**We claim the following**:

Q.**IV**.**1** For the system in state $|\psi^a[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_\Theta,$ where: (i)

$|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty, |a| \neq 1,$

(ii) supp$(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and (iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq$

$$\mathbf{G}\left[\widehat{Q}_\Theta|\psi^a[\Theta]\rangle\right] = |a|^2\mathbf{G}\left[\widehat{Q}_\Theta|\psi[\Theta]\rangle\right]. \tag{2.1.13}$$

Q.**IV**.**2**. Assume that the system in state $|\psi^a[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_\Theta,$ where (i) $|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty,$

$|a| \neq 1,$(ii) supp$(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and (iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq.$

Then if the system is in state $|\psi^a[\Theta]\rangle$ described by a wave function $\psi^a(q;\Theta) = \langle q||\psi^a[\Theta]\rangle$ and the value of observable Q_Θ is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_\Theta\rangle = \int_\Theta q|\psi^a(q;\Theta)|^2 dq. \tag{2.1.14}$$

Q.**IV**.**3**. The probability $P(q,q+dq;|\psi^a[\Theta]\rangle)dq$ of obtaining the result q lying in the range

$(q,q+dq)$ on measuring Q_Θ is

$$P(q,q+dq;|\psi^a[\Theta]\rangle)dq = |a|^{-2}|c_{\psi[\Theta]}(q|a|^{-2})|^2 dq. \tag{2.1.15}$$

Remark 2.**1**.**7**.Note that Q.IV.3 immediately folows from Q.IV.1 and Q.III.2.

Q.**IV**.**4**. (**Generalized von Neumann measurement postulate**) If on performing a measurement of observable Q_Θ with an accuracy $\delta q,$ the result is obtained in the range $(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q)$, then the system immediately after measurement will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi^a[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi^a[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} = \frac{a\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} \in \mathbf{H}_\Theta. \tag{2.1.16}$$

Q.V.1. Let $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle = |\psi_1^{a_1}[\Theta_1]\rangle + |\psi_2^{a_2}[\Theta_2]\rangle \in \mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2} \subsetneqq \mathbf{H}$, where
(i) $|\psi_i^{a_i}[\Theta_i]\rangle = a_i|\psi_i[\Theta_i]\rangle \in \mathbf{H}_{\Theta_i}, |\psi_i\rangle = |\psi_i[\Theta_i]\rangle \in \mathbf{S}_{\Theta_i}^{\infty}, |a_i| \neq 1, i = 1,2;$
(ii) supp$(c_{\psi_i[\Theta_i]}(q)), i = 1,2$ is a connected sets in $\mathbb{R}$;
(iii) $\left(\text{supp}(c_{\psi_1[\Theta_1]}(q))\right) \cap \left(\text{supp}(c_{\psi_2[\Theta_2]}(q))\right) = \varnothing$ and
(iv) $|\psi_i[\Theta_i]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi_i[\Theta_i]}(q)|q\rangle dq, i = 1,2.$

Then if the system is in a state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$ described by a wave function $\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2) = \langle q||\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle, q \in \Theta_1 \cup \Theta_2$ and the value of observable Q_{Θ_1,Θ_2} is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_{\Theta_1,\Theta_2}\rangle = \int_{\Theta_1\cup\Theta_2} q|\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2)|^2 dq. \tag{2.1.17}$$

Q.V. 2. The probability of getting a result q with an accuracy δq such that $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in supp(c_{\psi_1}(q))$ or $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in supp(c_{\psi_2}(q))$ given by

$$\int_{|q-q'|\leq\delta q/2}\left[\left(|\langle q'||\psi_1^{a_1}[\Theta_1]\rangle|^2\right) * \left(|\langle q'||\psi_2^{a_2}[\Theta_2]\rangle|^2\right)\right]dq'. \tag{2.1.18}$$

Remark 2.1.8. Note that Q.IV.3 immediately folows from Q.III.3.

Q.V. 3. Assume that the system is initially in the state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$. If on performing a measurement of Q_{Θ_1,Θ_2} with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the state of the system immediately after measurement given by

$$\frac{\widehat{P}(q_i,\delta q)|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle}{\sqrt{\langle\psi|\widehat{P}(q_i,\delta q)|\psi\rangle}} =$$

$$\frac{\int_{|q_i-q'|\leq\delta q/2}(|q'\rangle\langle q'||\psi_1^{a_1}[\Theta_1]\rangle+|q'\rangle\langle q'||\psi_2^{a_2}[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2+|\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} = \tag{2.1.19}$$

$$\frac{\int_{|q_i-q'|\leq\delta q/2}(a_1|q'\rangle\langle q'||\psi_1[\Theta_1]\rangle+a_2|q'\rangle\langle q'||\psi_2[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2+|\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} \in \mathbf{H}_{\Theta_i},$$

$$q_i \in \Theta_i, i = 1,2.$$

Definition 2.1.8. Let $\mathbf{H}_{1,2}$ be $\mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2}$.

Definition 2.1.9. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a Q-equivalent: $|\psi^a\rangle \sim_Q |\psi_a\rangle$ iff

$$P(q,q+dq;|\psi^a\rangle) = |a|^{-2}|c_\psi(q|a|^{-2})|^2 dq = P(qq+dq;|\psi_a\rangle)dq \tag{2.1.20}$$

Q.V.For any state $|\psi^a\rangle = a|\psi\rangle$,where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_Q |\psi_a\rangle$.

Definition 2.1.10. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a $\widehat{Q}$-equivalent ($|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$) iff: $\langle\psi^a|\widehat{Q}|\psi^a\rangle = \langle\psi_a|\widehat{Q}|\psi_a\rangle$.

Q.VI.For any state $|\psi^a\rangle = a|\psi\rangle$,where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$

II.2.The nonclassical collapse models with spontaneous localizations based on generalized measurement postulates

The nonclassical collapse models attempt to overcome the difficulties that standard quantum mechanics meets in accounting for the measurement (or macro-objectification) problem, an attempt based on the consideration of nonlinear and nonlocal stochastic modifications of the Schroedinger equation.The proposed new nonlocal dynamics is characterized by the feature of not contradicting any known fact about microsystems and of accounting, on the basis of a unique, universal dynamical principle, for wavepacket reduction and for the classical behavior of macroscopic systems.

II.2.1.Quantum Mechanics with Nonclassical Spontaneous Localizations is based on the following assumptions:

(**1**) Each particle of a system of n distinguishable particles experiences, with a mean rate λ_i, a sudden spontaneous localization process.

(**2**) In the time interval between two successive spontaneous processes the system evolves according to the usual Schrödinger equation.

(**3**) Let $|\psi\rangle_{cl}$ be the classical pure state correspond to an vector $|\psi\rangle_{cl} \in \mathbf{S}^{\infty} \subsetneqq \mathbf{H}$ in a non projective Hilbert space $\mathbf{H}$, see Subsection II.1, Def 2.1.1-2.1.2. Then the sudden spontaneous process is a localization given by:

$$|\psi\rangle_{cl} \overset{\delta,\varepsilon\text{-}\mathbf{localization}}{\longrightarrow} \frac{|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}}{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|}, \mathbf{x} \in \mathbb{R}^3, \delta \in (0,1], \varepsilon \ll 1, \tag{2.2.1}$$

where

$$|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} = \widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{cl}. \tag{2.2.2}$$

Here $\widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^{i}_{\mathbf{x}}(\delta,\varepsilon)$ in the n-particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle i around the point $\mathbf{x}$.

Definition 2.2.1. Such localization as mentioned above is called δ,ε-localization or δ,ε-collapse of the state $|\psi\rangle_{cl}$.

(**4**) The probability density $p_i(\mathbf{x},\delta,\varepsilon)$ for the occurrence of a localization at point $\mathbf{x}$ is assumed to be

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2}{\iiint\limits_{\mathbb{R}^3} \| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 d^3x}. \tag{2.2.3}$$

(**5**) Let $|\psi\rangle_{n.cl}$ be the nonclassical pure state correspond to an vector $|\psi^{\zeta}\rangle = \zeta|\psi\rangle \in \mathbf{H}\backslash\mathbf{S}^{\infty}$,

where $|\psi\rangle \in \mathbf{S}^{\infty}, |\zeta| \neq 1$, see subsection II.1, Def.2.1.10. Then the sudden spontaneous

process is a localization given by:

$$|\psi\rangle_{n.cl} \overset{\delta,\varepsilon-\mathbf{localization}}{\longrightarrow} \frac{\zeta|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl}}{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl} \|}, \mathbf{x} \in \mathbb{R}^3, \tag{2.2.4}$$

where

$$|\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl} = \widehat{L}^i_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{n.cl}. \quad (2.2.5)$$

Definition 2.2.2. Such localization as mentioned above is called δ,ε-localization or δ,ε-collapse of the state $|\psi\rangle_{n.cl}$.

(**6**) The probability density $p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at point $\mathbf{x} \in \mathbb{R}^3$ in acordance to postulate Q.IV.3 (see Subsection II.1, Eq.(2.1.15)) is assumed to be

$$p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,) = \frac{|\zeta|^{-6}\left\| \left|\psi^i_{\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}}\right\rangle_{n.cl}\right\|^2}{\iiint\limits_{\mathbb{R}^3} \||\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2 d^3x}. \quad (2.2.6)$$

(**7**) The localization operators $\widehat{L}^i_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^i_{\mathbf{x}}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x})^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| > \varepsilon. \end{cases} \quad (2.2.7)$$

Here $\delta \in (0,1]$ $\int d^3x[L^i_{\mathbf{x}}(\delta,\varepsilon)]^2 = 1$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

Remark 2.2.1. In one dimension case it follows that

$$\widehat{L}^i_x(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x)^2\right] \text{ iff } |q_i - x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x| > \varepsilon. \end{cases} \quad (2.2.8)$$

Remark 2.2.2. Note that from Eq.(2.2.3) and Eq.(2.2.7) follows that a probability density $p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization inside sphere $S(\mathbf{x},\varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon\}$ is given by

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\||\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2}{\Omega(\delta,\varepsilon)}, \Omega(\delta,\varepsilon) = \iiint\limits_{\mathbb{R}^3} \||\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2 d^3x,$$

$$\||\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i - \mathbf{x}\|\leq\varepsilon} d^3q_i\psi^i(\mathbf{q}_i)\exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x})^2\right], \quad (2.2.9)$$

$$\psi^i(q_i) = \langle q_i \| \psi^i\rangle_{cl},$$

and therefore

$$p_i(\mathbf{x},\varepsilon) = \lim_{\delta\to 0} p_i(\mathbf{x},\delta,\varepsilon) =$$

$$\lim_{\delta\to 0} \Omega^{-1}(\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int_{\|\mathbf{q}_i-\mathbf{x}\|\leq\varepsilon} d^3q_i\psi^i(\mathbf{q}_i)\exp\left[-\frac{1}{\delta}(\mathbf{q}_i-\mathbf{x})^2\right] = \psi^i(\mathbf{x}). \quad (2.2.10)$$

Remark 2.2.3. In one dimension case it follows that a probability density $p_i(x,\delta,\varepsilon,)$ for the occurrence of a localization inside interval $[x-\varepsilon, x+\varepsilon]$ is given by

$$p_i(x,\delta,\varepsilon) = \| |\psi^i_{\delta,\varepsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int_{|q_i-x|\leq\varepsilon} d^3q_i\psi^i(q_i)\exp\left[-\frac{1}{\delta}(q_i-x)^2\right],$$
$$\psi^i(q_i) = \langle q_i \| \psi^i\rangle_{cl}, \quad (2.2.11)$$

and therefore

$$p_i(x,\varepsilon) = \lim_{\delta\to 0} p_i(x,\delta,\varepsilon) =$$
$$= \lim_{\delta\to 0}\Omega^{-1}(\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int_{|q_i-x|\leq\varepsilon} dq_i\psi^i(q_i)\exp\left[-\frac{1}{\delta}(q_i-x)^2\right] = \psi^i(x). \quad (2.2.12)$$

II.2.2.The generalization of nonclassical collapse models

(**1**) Let $|\psi_t\rangle_{cl}, t\in[0,T]$ be the classical pure states correspond to an vector-function $|\psi_t\rangle_{cl} : [0,T]\times\mathbf{S}^\infty \to \mathbf{S}^\infty$ such that $|\psi_t\rangle_{cl} \in \mathbf{S}^\infty \subsetneq \mathbf{H}, t\in[0,T]$, where is a non projective Hilbert space $\mathbf{H}$, see Subsection II.1.1, Def. 2.1.1-2.1.2.Then the sudden spontaneous process is a localization along classical trajectory $\mathbf{x}_t : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{cl} \overset{\delta,\varepsilon,\mathbf{x}_t-\textbf{localization}}{\longrightarrow} \frac{|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}}{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|},$$
$$\delta\in(0,1], \varepsilon \ll 1, \mathbf{x}_t\in\mathbb{R}^3, t\in[0,T], \quad (2.2.13)$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{cl}. \quad (2.2.14)$$

Here $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^i_{\mathbf{x}_t}(\delta,\varepsilon)$ in the n-particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle i at each instant $t\in[0,T]$ around the point $\mathbf{x}_t$.

Definition 2.2.3. Such localization as mentioned above is called $\delta,\varepsilon,\mathbf{x}_t$-localization or
$\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi_t\rangle_{cl}$.

(**2**) The probability density $p_i(t,\mathbf{x}_t,\delta,\varepsilon)$ for the occurrence of a localization at point $\mathbf{x}_t$ at instant t is assumed to be

$$p_i(t,\mathbf{x}_t,\delta,\varepsilon) = \frac{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2}{\Omega(t,\delta,\varepsilon)}, \Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 d^3x. \qquad (2.2.15)$$

(**3**) Let $|\psi_t\rangle_{n.cl}$ be the nonclassical pure state correspond to an vector-function $\left|\psi^\zeta_t\right\rangle = \zeta|\psi_t\rangle \in \mathbf{H}\backslash\mathbf{S}^\infty$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |\zeta| \neq 1, t \in [0,T]$ see Subsection **II**.1, Def.2.1.10.

Then the sudden spontaneous process is a localization along classical trajectory $\mathbf{x}_t : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{n.cl} \overset{\delta,\varepsilon,\mathbf{x}_t-\mathbf{localization}}{\longrightarrow} \frac{\zeta|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl}}{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl} \|}, \qquad (2.2.16)$$
$$\mathbf{x}_t \in \mathbb{R}^3, t \in [0,T]$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{n.cl}. \qquad (2.2.17)$$

Definition 2.2.4. Such localization as mentioned above is called $\delta,\varepsilon,\mathbf{x}_t$-localization or $\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi\rangle_{n.cl}$.

(**4**) The probability density $p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at point $\mathbf{x}_t \in \mathbb{R}^3$ at instant $t \in [0,T]$ in acordance to postulate Q.IV.3 (see Subsection **II**.1., Eq.(2.1.14)) is assumed to be

$$p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,) = \frac{|\zeta|^{-6} \left\| \left|\psi^i_{t,\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}_t}\right\rangle_{n.cl} \right\|^2}{\Omega(t,\delta,\varepsilon)}, \Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 d^3x. \qquad (2.2.18)$$

(12) The localization operators $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x}_t)^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| > \varepsilon. \end{cases} \qquad (2.2.19)$$

Here $\delta \in (0,1]$ and $\lim_{\delta\to 0} \pi_\delta = \pi$.

Remark 2.2.4. In one dimension case it follows that

$$\widehat{L}^i_{x_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x_t)^2\right] \text{ iff } |q_i - x_t| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x_t| > \varepsilon. \end{cases} \qquad (2.2.20)$$

Remark 2.2.5. Note that from Eq.(2.2.18) and Eq.(2.2.19) follows that a probability density $p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at instant t inside sphere $S(\mathbf{x}_t,\varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon\}$ is given by

$$p_i(t,\mathbf{x}_t,\delta,\varepsilon) = \frac{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2}{\Omega(t,\delta,\varepsilon)}$$

$$\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i-\mathbf{x}_t\|\leq\varepsilon} d^3q_i\psi^i_t(\mathbf{q}_i)\exp\left[-\frac{1}{\delta}(\mathbf{q}_i-\mathbf{x}_t)^2\right], \qquad (2.2.21)$$

$$\psi^i_t(q_i) = \langle q_i \| \psi^i_t\rangle_{cl},$$

and therefore

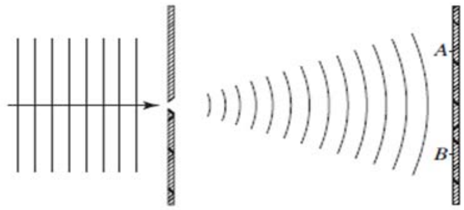
$$p_i(t,x,\varepsilon) = \lim_{\delta\to 0} p_i(t,x,\delta,\varepsilon) =$$

$$= \lim_{\delta\to 0}\Omega^{-1}(t,\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int\limits_{|q_i-x|\leq\varepsilon} dq_i\psi^i(q_i)\exp\left[-\frac{1}{\delta}(q_i-x_t)^2\right] = \psi^i(x_t). \qquad (2.2.22)$$

III.Einstein's 1927 gedanken experiment revisited.

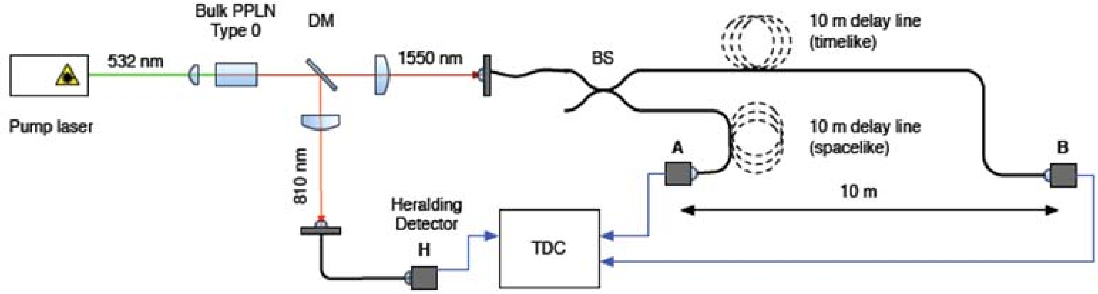
III.1.Single-photon space-like antibunching.

During the famous 5th Solvay conference in 1927, Einstein [2] considered a single particle which, after diffraction in a pin-hole encounters a “detection plate” (e.g. in the case of photons, a photographic plate), see Fig 2.2.1. We simplify this thought experiment, though keeping the essence, by replacing the “detection plate” by two detectors. Einstein noted that there is no question that only one of them can detect the particle, otherwise energy would not be conserved. However, he was deeply concerned about the situation in which the two detectors are space-like separated, as this prevents - according to relativity - any possible coordination among the detectors: “It seems to me,” Einstein continued, “that this difficulty cannot be overcome unless the description of the process in terms of the Schrödinger wave is supplemented by some detailed specification of the localization of the particle during its propagation. I think M. de Broglie is right in searching in this direction.”



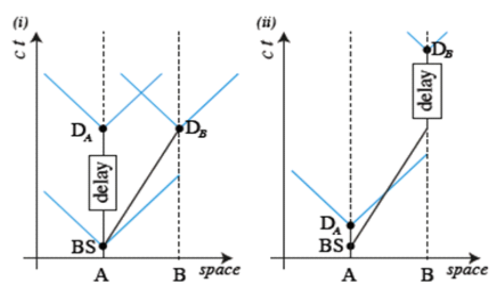
Pic. 3.1.1.Einstein's 1927 gedanken experiment.
A and B are points on the photographic plate,
for which the events of detection can be space-
like separated from each other.Adapted from [2].

But what happened to Einstein's original "Gedanken experiment"? This simple - with today's technology - experiment had been done originally by T. Guerreiro, B. Sanguinetti, H. Zbinden N. Gisin, and A. Suarez, see [19].This experiment consists in verifying that when a single photon is thrown at a beam splitter, it is detected in only one arm, i.e. the probability $\mathbf{P}_{A\wedge B}$ of getting a coincidence between the two detectors A and B is much smaller than the product of the probabilities of detection on each side $\mathbf{P}_A \times \mathbf{P}_B$, as would be expected in the case of uncorrelated events.The experimental setup is shown in Fig. 3.1.2 and consists of a source of heralded single photons which is coupled into a single mode fiber and injected into a fiber beamsplitter (BS). Each of the two outputs of the beamsplitter goes to a single photon detector (IDQ ID200), detector A being close to the source and detector B being separated by a distance of approximately 10 meters.



Pic. 3.1.2.Experimental setup: photon pairs a regenerated by Spontaneous Parametric Down Conversion at the wavelengths of 1550 nm and 810 nm. These pairs are split by a dichroic mirror (DM), and the 810 nm photon is sent to detector D,used to herald the presence of the1550 nm photon which follows to the beam splitter (BS). Arbitrary electronic delays were applied beforeTDC to ensure the coincidence peak swould remain on scale. Adapted from [19].

If we ensure that the fiber lengths before each detector are equal by inserting a 10 m (50ns) fiber delay loop before detector A, the detections will happen simultaneously in some reference frame, thus being space-like separated (a signal would take 33 ns to travel between the two detectors at the speed of light; simultaneity of detection is guaranteed to within 1ns by the matched length of fiber both before and inside the detectors). It is also possible to make the detections time-like separated by removing the 10m delay line from detector A and adding it to detector B.

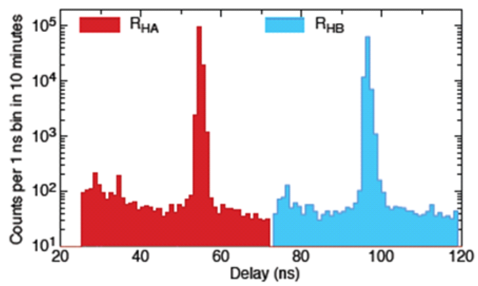


Pic. 3.1.3.[19]. Spacetime diagrams for spacelike (i) and timelike (ii) configurations. A and B represent the locations of the detectors. Adapted from [19].

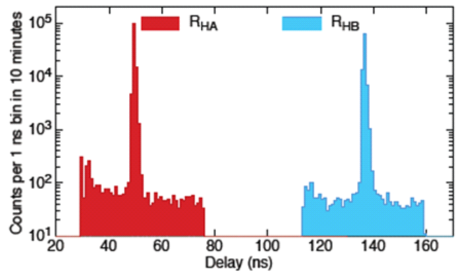
First one measure the probabilities of detecting a photon at detector A or at detector

B given that a heralding photon has been detected at H. We denote R_{HA} the total number of coincident counts at detector H and detector A during the time of measurement, and $R_{H(A)}$ the total number of counts at detector H alone during the same measurement; R_{HB} and $R_{H(B)}$ denote similar quantities for the measurement with H and B. Next we measure the probability of detectors A and B clicking at the same time, again given a heralding signal. R_{HAB} denotes the number of triple coincident counts at the detectors H, A and B, and $R_{H(AB)}$ the total number of counts at detector H alone during the same measurement. All these quantities are measured directly for both a space-like configuration and a time-like configuration.

Next one measure the probability of detectors A and B clicking at the same time, again given a heralding signal. R_{HAB} denotes the number of triple coincident counts at the detectors H, A and B, and $R_{H(AB)}$ the total number of counts at detector H alone during the same measurement. All these quantities are measured directly for both a space-like configuration and a time-like configuration.



Pic. 3.1.4.Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with spacelike separation,measured in a window of 1ns during a time period of 10 minutes. $R_{HA} = 9.49 \times 104/10$ min, $R_{HB} = 6.39 \times 104/10$ min.The noise is on average:$R_N = 50/10$ min. Adapted from [19].



Pic. 3.1.5.Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with timelike separation,measured in a window of 1ns during a time period of 10 minutes. $R_{HA} = 9.90 \times 104/10$ min, $R_{HB} = 6.22 \times 104/10$ min. Adapted from [19].

The raw TDC data is shown in Figures 3.1.4-3.1.5 and the results are summarized in Table 3.1.1.

Spacelike separation

R_{HA}	$R_{H(A)}$	$P_A^{SL} = R_{HA}/R_{H(A)}$
$(94.8 \pm 0.3) \cdot 10^3$	$(5570 \pm 2) \cdot 10^3$	$(1.703 \pm 0.006) \cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{SL} = R_{HB}/R_{H(B)}$
$(63.8 \pm 0.2) \cdot 10^3$	$(5860 \pm 2) \cdot 10^3$	$(1.090 \pm 0.004) \cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{SL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(17145 \pm 4) \cdot 10^3$	$(2.3 \pm 1.2) \cdot 10^{-7}$
R_{HN}	$R_{H(N)}$	$P_N^{SL} = R_{HN}/R_{H(N)}$
50 ± 7	$(5500 \pm 2) \cdot 10^3$	$(9.0 \pm 1.3) \cdot 10^{-6}$

Timelike separation

R_{HA}	$R_{H(A)}$	$P_A^{TL} = R_{HA}/R_{H(A)}$
$(99.0 \pm 0.3) \cdot 10^3$	$(6130 \pm 2) \cdot 10^3$	$(1.616 \pm 0.005) \cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{TL} = R_{HB}/R_{H(B)}$
$(62.2 \pm 0.2) \cdot 10^3$	$(6100 \pm 2) \cdot 10^3$	$(1.019 \pm 0.004) \cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{TL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(18345 \pm 4) \cdot 10^3$	$(2.2 \pm 1.1) \cdot 10^{-7}$

Table 3.1.1.Summary of results.Values obtained for the different counting rates and corresponding probabilities defined inthetext,measured with spacelike and timelike separation. Adapted from [19].

The number of counts given by detector noise and twophoton events can be estimated by looking at the counts away from the peak. As an example, for the space-like configuration (Figure 3.1.4.) in a window of 1ns the noise rate is on average $R_{HN} = 50/10$ for a 10 minutes integration time [19]. This corresponds to a noise probability $\mathbf{P}_N = 9\cdot 10^{-6}(1.3\cdot 10^{-6})$. From the values in Table 3.1.1 one derives the following probability values for spacelike separation:

$$\begin{aligned}&\mathbf{P}_A^{SL}\bullet P_B^{SL} = 1.86 \pm 0.01\cdot 10^{-4},\\&\mathbf{P}_{A\wedge B}^{SL} = 0.002 \pm 0.001\cdot 10^{-4}.\end{aligned} \quad (3.1.1)$$

For timelike separation one derives the values:

$$\begin{aligned}&\mathbf{P}_A^{TL}\cdot P_B^{TL} = 1.65 \pm 0.01\cdot 10^{-4},\\&\mathbf{P}_{A\wedge B}^{TL} = 0.002 \pm 0.001\cdot 10^{-4}.\end{aligned} \quad (3.1.2)$$

For the probability $\mathbf{P}_N^{SL}$ that A and B detect photons coming from different pairs (noise) one derives the value:

$$\begin{aligned}&\mathbf{P}_N^{SL}(1,1) = \mathbf{P}_N^{SL}\cdot\mathbf{P}_A^{SL} + \mathbf{P}_N^{SL}\cdot\mathbf{P}_B^{SL} \approx\\&0.0025 \pm 0.0026\cdot 10^{-4}\end{aligned} \quad (3.1.3)$$

III.2.The measure algebra of physical events in Minkowski space-time.

Definition 3.2.1.[20].A measure algebra $\mathcal{F} = (\mathbf{B},\mathbf{P})$ with a probability measure $\mathbf{P}$, is a

Boolean algebra $\mathbf{B}$ with a countably additive probability measure.

Definition 3.2.2.(**i**) A measure algebra of physical events $\mathcal{F}^{ph} = (\mathbf{B},\mathbf{P})$ with a probability measure $\mathbf{P}$, is an Boolean algebra of physical events $\mathbf{B}$ with an countably additive probability measure.

(**ii**) A Boolean algebra of physical events can be formally defined as a set $\mathbf{B}$ of elements

$a,b,\ldots$ with the following properties:

1. $\mathbf{B}$ has two binary operations, $\wedge$ (logical AND, or "wedge") and $\vee$ (logical OR, or "vee"),

which satisfy:

the idempotent laws:(1) $a\wedge a = a\vee a = a$,

the commutative laws:(2) $a\wedge b = b\wedge a$,(3) $a\vee b = b\vee a$,

and the associative laws: (4) $a\wedge(b\wedge c) = (a\wedge b)\wedge c$,(5) $a\vee(b\vee c) = (a\vee b)\vee c$.

2. The operations satisfy the absorption law:

(6) $a\wedge(a\vee b) = a\vee(a\wedge b) = a$.

3. The operations are mutually distributive

(7) $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$,

(8) $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$.

4. $\mathbf{B}$ contains universal bounds $\mathbf{0}$ and $\mathbf{1}$ which satisfy
(9) $\mathbf{0} \wedge a = \mathbf{0}$
(10) $\mathbf{0} \vee a = a$
(11) $\mathbf{1} \wedge a = a$
(12) $\mathbf{1} \vee a = \mathbf{1}$.
5. $\mathbf{B}$ has a unary operation $\neg a$ (or a') of complementation (logical negation), which obeys the laws:
(13) $a \wedge \neg a = \mathbf{0}$
(14) $a \vee \neg a = \mathbf{1}$
All properties of negation including the laws below follow from the above two laws alone.
6. Double negation law: $\neg(\neg a) = a$
7.De Morgan's laws: (i) $\neg a \wedge \neg b = \neg(a \vee b)$, (ii) $\neg a \vee \neg b = \neg(a \wedge b)$.
8.Operations composed from the basic operations include the following importent examples:
The first operation, $a \to b$ (logical material implication):
(i) $a \to b \triangleq \neg a \vee b$.
The second operation, $a \oplus b$, is called exclusive. It excludes the possibility of both a and b
(ii) $a \oplus b \triangleq (a \vee b) \wedge \neg(a \wedge b)$.
The third operation, the complement of exclusive or, is equivalence or Boolean equality:
(iii) $a \equiv b \triangleq \neg(a \oplus b)$
9. $\mathbf{B}$ has a unary predicate $\mathbf{Occ}(a)$, which meant that event a has occurred, and which obeys the laws:
(i) $\mathbf{Occ}(a \wedge b) \Leftrightarrow \mathbf{Occ}(a) \wedge \mathbf{Occ}(b)$,
(ii) $\mathbf{Occ}(a \vee b) \Leftrightarrow \mathbf{Occ}(a) \vee \mathbf{Occ}(b)$,
(iii) $\mathbf{Occ}(\neg a) \Leftrightarrow \neg\mathbf{Occ}(a)$.
Remark 3.2.1. A probability measure $\mathbf{P}$ on a measure space (Ω, Σ) gives a probability measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null sets.
Definition 3.2.3.(i) Let $\mathbf{B}$ be a Boolean algebra of physical events. A Boolean algebra $\mathbf{B}_{M_4}$ of physical events in Minkowski spacetime $M_4 = \mathbb{R}^{1,3}$ that is cartesian product $\mathbf{B}_{M_4} = \mathbf{B} \times M_4$.
(ii) Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime. A measure algebra of physical events $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ in Minkowski spacetime that is a Boolean

algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$.

(iii) Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski spacetime and let

$\mathcal{F}_{M_4}^{ph}$ be an measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P}$. We denote such

physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc.,where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc.

(iv) We will be write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ etc., instead $\mathbf{Occ}(A(\mathbf{x})), \mathbf{Occ}(B(\mathbf{x})), \ldots$ etc.

Definition 3.2.4. Let $\mathbf{Aut}_{\mathbf{P}}(\mathbf{B}_{M_4})$ be a set of the all measure-preserving automorphism of

$\mathbf{B}_{M_4}$. This is a group,being a subgroup of the group $\mathbf{Aut}(\mathbf{B}_{M_4})$ of all Boolean automorphism

of $\mathbf{B}_{M_4}$. Let $P_+^{\uparrow}$ be Poincaré group.

Remark 3.2.2. We assume now that: any element $\Theta = (\Lambda, a) \in P_+^{\uparrow}$ induced an element

$\widetilde{\Theta} \in \mathbf{Aut}_{\mathbf{P}}(\mathbf{B}_{M_4})$ by formula $\widetilde{\Theta} = \Theta[A(\mathbf{x})] = A(\Lambda\mathbf{x} + \mathbf{a}) \in \mathbf{B}_{M_4}$.

Definition 3.2.5. Given two events A and B from the algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ the

conditional probability of A given B is defined as the quotient of the probability of the joint

of events A and B, and the probability of B :

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \wedge B)}{\mathbf{P}(B)} = \frac{\mathbf{P}_{A\wedge B}}{\mathbf{P}_B} = \mathbf{P}_{A|B}, \tag{3.2.1}$$

where $\mathbf{P}(B) \neq 0$.

Definition 3.2.6. (i) Events A and B from the algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ are defined to be statistically independent or uncorrelated iff

$$\mathbf{P}_{A\wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B, \tag{3.2.2}$$

where $\mathbf{P}_B \neq 0$, then this is equivalent to the statement that $\mathbf{P}_{A|B} = \mathbf{P}_A$. Similarly, if $\mathbf{P}_A$ is not zero, then $\mathbf{P}_{B|A} = \mathbf{P}_B$ is also equivalent.

(ii) Events A and B from the algebra $\mathcal{F} = (\mathbf{B}_{M_4}, \mathbf{P})$ are defined to be statistically almost independent or almost uncorrelated iff

$$\begin{gathered}\mathbf{P}_{A\wedge B} \approx \mathbf{P}_A \cdot \mathbf{P}_B, \\ \mathbf{P}_{A\wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B - \delta(A,B), 0 < \delta(A,B) \ll \mathbf{P}_A \cdot \mathbf{P}_B.\end{gathered} \tag{3.2.3}$$

Remark 3.2.3. Note that

$$\mathbf{P}_{A\vee B} = \mathbf{P}_A + \mathbf{P}_B - \mathbf{P}_{A\wedge B}. \tag{3.2.4}$$

Although mathematically equivalent, this may be preferred philosophically; under major probability interpretations such as the subjective theory, conditional probability is considered a primitive entity. Further, this "multiplication axiom" introduces a symmetry with the summation axiom for mutually exclusive events, i.e.

$$\mathbf{P}_{A\vee B} = \mathbf{P}_A + \mathbf{P}_B - \mathbf{P}_{A\wedge B}. \tag{3.2.5}$$

Definition 3.2.7. (i) Events $A_1,A_2,\ldots,A_n \in \mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ are said to be exactly mutually exclusive if the occurrence of any one of them implies the non-occurrence of the remaining $n-1$ events. Therefore, two mutually exclusive events cannot both occur. Formally said, the conjunction of each two of them is $\mathbf{0}$ (the null event): $A \wedge B = \mathbf{0}$. In consequence, exactly mutually exclusive events A and B have the property:

$$\mathbf{P}(A \wedge B) = 0. \tag{3.2.6}$$

(ii) Events $A_1,A_2,\ldots,A_n \in \mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ are said to be almost mutually exclusive if $A_1,A_2,\ldots,A_n$ have the property:

$$\begin{gathered}\mathbf{P}(A_1 \wedge A_2 \wedge\ldots\wedge A_n) \approx 0,\\ \mathbf{P}(A_1 \wedge A_2 \wedge\ldots\wedge A_n) \ll \mathbf{P}(A_1)\cdot\mathbf{P}(A_2)\cdots\mathbf{P}(A_n).\end{gathered} \tag{3.2.7}$$

In consequence, almost mutually exclusive events A and B have the property:

$$\begin{gathered}\mathbf{P}(A \wedge B) \approx 0,\\ \mathbf{P}(A \wedge B) \ll \mathbf{P}(A)\cdot\mathbf{P}(B).\end{gathered} \tag{3.2.8}$$

Remark 3.2.4. Let A^{ph},B^{ph} be events such that detectors A,B detect photon at an instants t_1 and t_2 correspondingly. Note that (3.1.1) and (3.1.2) show that whether the separation between the detectors is timelike or spacelike, the number of coincidences is three orders of magnitude smaller than what would be expected had the events been statistically almost uncorrelated, i.e., $\mathbf{P}_{A\wedge B} \approx \mathbf{P}_A\cdot\mathbf{P}_B,$ see Def.3.2.6 (ii).

Remark 3.2.5. Let A^{ph},B^{ph} be events such that detectors **A**,**B** detect photon at an instants

t_1 and t_2 correspondingly. Note that:

(i) from Eq.(3.1.1) follows probability value for spacelike separation:

$$\mathbf{P}^{SL}_{A^{ph}\wedge B^{ph}} = 0.002 \pm 0.001\cdot 10^{-4} \neq 0, \tag{3.2.9}$$

(ii) from Eq.(3.1.2) follows probability value for timelike separation:

$$\mathbf{P}^{TL}_{A^{ph}\wedge B^{ph}} = 0.002 \pm 0.001\cdot 10^{-4} \neq 0. \tag{3.2.10}$$

Therefore in both cases the property (3.2.6) are violated, i.e. $\mathbf{P}_{A^{ph}\wedge B^{ph}} \neq 0$ but however in both cases the property (3.2.8) is satisfied

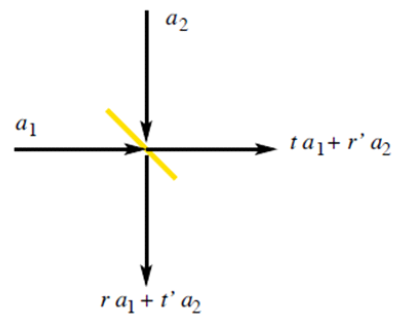
$$\begin{gathered}0.002 \pm 0.001\cdot 10^{-4} = \mathbf{P}^{SL}_{A^{ph}\wedge B^{ph}} \ll \mathbf{P}^{SL}_{A^{ph}}\cdot\mathbf{P}^{SL}_{B^{ph}} = 1.86 \pm 0.01\cdot 10^{-4},\\ 0.002 \pm 0.001\cdot 10^{-4} = \mathbf{P}^{TL}_{A^{ph}\wedge B^{ph}} \ll \mathbf{P}^{TL}_{A^{ph}}\cdot\mathbf{P}^{TL}_{B^{ph}} = 1.65 \pm 0.01\cdot 10^{-4}\end{gathered} \tag{3.2.11}$$

and therefore in both cases the events A^{ph},B^{ph} are almost mutually exclusive events.

Beamsplitter transformation.

A beamsplitter is the most simple way to mix two modes, see Figure 3.2.1. From classical electrodynamics, one gets the following amplitudes for the outgoing modes:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}} \mapsto \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{\mathbf{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}}. \tag{3.2.12}$$



Pic. 3.2.1.Mixing of two modes by a beam splitter.

The recipe for quantization is now: 'replace the classical amplitudes by annihilation operators'. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$[A_i(out), A_j(out)] = \delta_{ij} \tag{3.2.13}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t'|^2 + |r'|^2 = 1.$ We are now looking for an unitary operator $\mathbf{S}$ [the $\mathbf{S}$-matrix] that implements this beamsplitter transformation in the following sense:

$$A_i = \mathbf{S}^\dagger a_i \mathbf{S}, i = 1,2. \tag{3.2.14}$$

Let us start from the general transformation (summation over double indices)

$$a_i \mapsto A_i = B_{ij} a_i, \vec{a} \mapsto \vec{A} = \mathbf{B}\vec{a} \tag{3.2.15}$$

where we have introduced matrix and vector notation. Using this $\mathbf{S}$-matrix one can also compute the transformation of the states: $|out\rangle = \mathbf{S}|in\rangle$. For the unitary transformation, we make the ansatz

$$\mathbf{S}(\theta) = \exp(i\theta J_{ki} a_k^\dagger a_i) \tag{3.2.16}$$

with J_{kl} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\theta) = \mathbf{S}^\dagger(\theta) a_i \mathbf{S}(\boldsymbol{\theta}) = B_{ij} a_j. \tag{3.2.17}$$

We compute this 'operator conjugation' by using a differential equation:

$$\frac{dA_i(\theta)}{d\theta} = iJ_{ki}A_i(\theta). \quad (3.2.18)$$

This is a system of linear differential equations with constant coefficients, so that one obtains a solution

$$\vec{A}(\theta) = \exp(i\theta\mathbf{J}). \quad (3.2.19)$$

We thus conclude that the so-called generator $\mathbf{J}$ of the beam splitter matrix is fixed by equatuon

$$\mathbf{B} = \exp(i\theta\mathbf{J}). \quad (3.2.20)$$

If the transformation $\mathbf{B}$ is part of a continuous group and depends on θ as a parameter, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\mathbf{B} = 1 + i\theta\mathbf{J} + \dots \quad (3.2.21)$$

Equation (3.2.21) explains the name generator for the matrix $\mathbf{J}$: it actually generates a subgroup of matrices $\mathbf{B} = \mathbf{B}(\theta)$ parametrized by the angle θ. The unitary transformation we are looking for is thus determined via the same generator $\mathbf{J}$. For the two-mode beam splitter, an admissible transformation is given by

$$\mathbf{B}(\theta) = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \quad (3.2.22)$$

The factor i is just put for convenience so that the reflection amplitudes are the same for both sides, $r = r'$, as expected by symmetry. Expanding for small θ, the generator is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \quad (3.2.23)$$

and so that the unitary operator for this beamsplitter is

$$S(\theta) = \exp[i\theta(a_1^\dagger a_2 + a_2^\dagger a_1)]. \quad (3.2.24)$$

Therefore, the effective Hamiltonian of the beam splitter is given by

$$\widehat{H}_{\mathbf{eff}} = a_1^\dagger a_2 + a_2^\dagger a_1. \quad (3.2.25)$$

Splitting a two-photon state

Let us consider two single photon states $|in\rangle = |1,1\rangle$ incident on the beam splitter such that mentioned above.Then

$$|\psi(\theta)\rangle = |out\rangle = \mathbf{S}|in\rangle = \mathbf{S}a_1^\dagger S^\dagger S a_2^\dagger S^\dagger S|0,0\rangle =$$
$$(a_1^\dagger\cos\theta + ia_2^\dagger\sin\theta)(a_2^\dagger\cos\theta + ia_1^\dagger\sin\theta)|0,0\rangle = \qquad (3.2.26)$$
$$(|2,0\rangle - |0,2\rangle)\frac{\sin\theta}{\sqrt{2}} + |1,1\rangle\cos\theta.$$

Let $\mathbf{H}$ be a complex Hilbert space such that

$$\forall\theta[|\psi(\theta)\rangle_{cl} \in \mathbf{H}],$$
$$\forall\theta\forall\delta(\delta \in (0,1])\forall\varepsilon(\varepsilon \in (0,1])\left[|\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\rangle_{cl} \in \mathbf{H}\right], \qquad (3.2.27)$$
$$|\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\rangle_{cl} = L^i_{\mathbf{x}}(\delta,\varepsilon)|\psi(\theta)\rangle_{cl}.$$

By postulate Q.I.1 (see section II.1) quantum system wih Hamiltonian given by Eq.(3.2.25) is identified with a set $\Xi \triangleq \left\langle \mathbf{H}, \widehat{H}_{\mathbf{eff}}, \Im, \Re, \mathcal{L}_{2,1}(\Omega), \mathbf{G}, |\psi_t\rangle \right\rangle$, where

(i) $\mathbf{H}$ that is a complex Hilbert space defined above,
(ii) $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,
(iii) $\Re = (\mathbb{R}^n, \Sigma)$ that is measurable space ,

(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of random variables $X : \Omega \to \mathbb{R}^n$ such that

$$\int_\Omega \|X(\omega)\|d\mathbf{P} < \infty, \int_\Omega \|X(\omega)\|^2 d\mathbf{P} < \infty, \qquad (3.2.28)$$

(v) $\mathbf{G} : \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_\Omega \left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right)d\mathbf{P} = \mathbf{E}_\Omega\left[\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right] \qquad (3.2.29)$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$,
(vi) $|\psi_t\rangle$ is an continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which representedthe canonical evolution of the quantum system Ξ.

Remark 3.2.6. Note that $\Im^{ph}_{M_4} = \mathcal{F} \times M_4 = (\Omega, \Sigma, \mathbf{P}) \times M_4$,where $\mathcal{F}$ is a probability measure
algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null sets, see
Remark 3.2.1.

Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski spacetime M_4 and let
$\widetilde{\mathcal{F}}_{M_4}$ be an measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$ with a probability measure $\widetilde{\mathbf{P}}$, see Definition 3.2.2 (vii).

We assume now that there exist subalgebra $\mathcal{F}^\#_{M_4} \subsetneq \widetilde{\mathcal{F}}_{M_4}$ and isomorphism $\lambda[\cdot] : \mathcal{F}^\#_{M_4} \mapsto \Im^{ph}_{M_4}$ such that for any event $A(\mathbf{x}) \in \mathcal{F}^\#_{M_4}, \mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ (see Definition 3.2.2):

$$\lambda[A(\mathbf{x})] = \lambda[A](\mathbf{x}),$$
$$\widetilde{\mathbf{P}}(A(\mathbf{x})) = \mathbf{P}(\lambda[A](\mathbf{x})) \triangleq \mathbf{P}(A_\lambda(\mathbf{x})). \quad (3.2.30)$$

Proposition 3.2.1. Suppose that A and B are events in measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$.

Then following properties is satisfied:

$$\begin{aligned} &1.\ \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) > \mathbf{P}(A)\mathbf{P}(B) \\ &2.\ \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) < \mathbf{P}(A)\mathbf{P}(B) \\ &3.\ \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) = \mathbf{P}(A)\mathbf{P}(B) \end{aligned} \quad (3.2.31)$$

Proposition 3.2.2. Suppose that A and B are events in measure algebra $\mathfrak{I}_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Then following properties is satisfied:

$$\begin{aligned} &1.\ \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) > \mathbf{P}(A)\mathbf{P}(B) \\ &2.\ \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) < \mathbf{P}(A)\mathbf{P}(B) \\ &3.\ \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) \end{aligned} \quad (3.2.32)$$

Definition 3.2.8. In case (1), A and B are said to be positively correlated.

Intuitively, the occurrence of either event means that the other event is more likely.

In case (2), A and B are said to be negatively correlated.

Intuitively, the occurrence of either event means that the other event is less likely.

In case (3), A and B are said to be uncorrelated or independent.

Intuitively, the occurrence of either event does not change the probability of the other event.

Remark 3.2.7. Suppose that A and B are events in measure algebra $\mathfrak{I}_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Note from the result above that if $A \subseteq B$ or $B \subseteq A$ then A and B are positively correlated. If A and B are disjoint then A and B are negatively correlated.

Proposition 3.2.3. Suppose that A and B are events in measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$. Then:

(i) A and B have the same correlation (positive, negative, or zero) as $\neg A$ and $\neg B$.

(ii) A and B have the opposite correlation as A and $\neg B$ (that is, positive-negative, negative-positive, or zero-zero).

Proposition 3.2.4. Suppose that A and B are events in measure algebra $\mathfrak{I}_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Then:

(i) A and B have the same correlation (positive, negative, or zero) as A^c and B^c.

(ii) A and B have the opposite correlation as A and B^c (that is, positive-negative, negative-positive, or zero-zero).

Definition 3.2.9.Let $A(\mathbf{x}_1) = A(t_1,\mathbf{r}_1)$ and $B(\mathbf{x}_2) = B(t_2,\mathbf{r}_2)$ be an events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ which occurs at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly.

Let $\mathbf{x}_{1,2}$ be a vector: $\mathbf{x}_{1,2} = \{c(t_1 - t_2),\mathbf{r}_1 - \mathbf{r}_2\} = (ct_{1,2},\mathbf{r}_{1,2}), t_{1,2} = t_1 - t_2, \mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2$.

Vectors $\mathbf{x}_{1,2} = (ct_{1,2},\mathbf{r}_{1,2})$ are classified according to the sign of $c^2t^2_{1,2} - \mathbf{r}^2_{1,2}$. A vector is (i) timelike if $c^2t^2_{1,2} > \mathbf{r}^2_{1,2}$, (ii) spacelike if $c^2t^2_{1,2} < \mathbf{r}^2_{1,2}$, and null or lightlike if (iii) $c^2t^2_{1,2} = \mathbf{r}^2_{1,2}$.

Pairs of events $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4}$ are classified according to the sign of $c^2t^2_{1,2} - \mathbf{r}^2_{1,2}$:

(i) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is timelike separated if $c^2t^2_{1,2} > \mathbf{r}^2_{1,2}$, and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}}$

(ii) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is spacelike separated if $c^2t^2_{1,2} < \mathbf{r}^2_{1,2}$, and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$

(iii) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is null or lightlike separated if $c^2t^2_{1,2} = \mathbf{r}^2_{1,2}$. and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{l.l.s.}}$

Definition 3.2.10.(i) Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}$ be a set of the all timelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}}$ which are corresponding to a given vector $\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$,i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | c^2t^2_{1,2} > \mathbf{r}^2_{1,2}\}. \quad (3.2.33(a))$$

(ii) Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of the all spacelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$ which is corresponding to a given vector

$\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$,i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | c^2t^2_{1,2} < \mathbf{r}^2_{1,2}\}. \quad (3.2.33(b))$$

Remark 3.2.8. Let $[\mathcal{F}^{\#}_{M_4},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of the all pairs $\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\}$ which is corresponding to a given vector $\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\} \in M_4 \times M_4,\ \mathbf{r}_1 \neq \mathbf{r}_2$,i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | 0 < \mathbf{r}^2_{1,2}\}, \quad (3.2.34)$$

$$\mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2.$$

Such pairs obviously is spacelike separated. Note that

$$\forall t \forall \mathbf{r}_1 \forall \mathbf{r}_2 (\mathbf{r}_1 \neq \mathbf{r}_2) \left\{ [\mathcal{F}^{\#}_{M_4}, \{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} \neq \varnothing \right\}. \quad (3.2.35)$$

Definition 3.2.11. Let $A^{t_1} \triangleq A(\mathbf{x}_1) = A(t_1,x_A)$ and $B^{t_2} \triangleq B(\mathbf{x}_2) = B(t_2,x_B)$ be a symbols such that A^{t_1} and B^{t_2} represent there is detection events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly, where symbols x_A and x_B represent the locations of the detectors A and B correspondingly (see Pic. 3.1.3). We assume that

$$\{A^{t_1}, B^{t_2}\} \in [\mathcal{F}^{\#}_{M_4}, \{(t_1,x_A),(t_2,x_B)\}]_{\mathbf{s.l.s.}}. \quad (3.2.36)$$

Remark 3.2.9. We assume now without loss of generality that $t_1 = t_2 = t$, note that such assuption valid by properties: $A(\Lambda\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ and $B(\Lambda\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$, required above,see Remark 3.2.2.

III.3.Einstein's 1927 gedanken experiment explained.

In classical case considered by A. Einstein in his 1927 gedanken experiment, by postulates of canonical QM, both events $A^t \in \mathcal{F}^{\#}_{M_4}$ and $B^t \in \mathcal{F}^{\#}_{M_4}$ cannot occur simultaneously, i.e. that is mutually exclusive events with a probability $= 1$, and therefore $A^t \wedge B^t = \mathbf{0}$. Such exactly mutually exclusive events have the property:

$$\widetilde{\mathbf{P}}(A^t \wedge B^t) = 0, \quad (3.3.1)$$

see Definition 3.2.6.

We remind that the probability density $p^{\mathbf{ph}}(x,\delta,\varepsilon)$ for the occurrence of a photon localization at point x is assumed to be

$$p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left| \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \right\rangle_{cl} \right\|^2, \quad \delta \in (0,1], \varepsilon \in (0,1], \quad (3.3.2)$$

where

$$\left| \psi^{\mathbf{ph}}_{\delta,\varepsilon,x} \right\rangle_{cl} = L_x(\delta,\varepsilon) |\psi^{\mathbf{ph}}(\theta)\rangle_{cl}. \quad (3.3.3)$$

and where the localization operators $L_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}_x(\widehat{q},\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}-x)^2\right] \text{ iff } |\widehat{q}-x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |\widehat{q}-x| > \varepsilon. \end{cases} \quad (3.3.4)$$

see subsection **II**.2.1.

Remark 3.3.1. Note that: (i) from (3.2.27) follows that $\left| \psi^{\mathbf{ph}}_{\delta,\varepsilon,\mathbf{x}}(\theta) \right\rangle_{cl} \in \mathbf{H}$,

(ii) from (3.3.3) and (3.3.4) where $\delta \ll 1$ follows that

$$p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta)\right\rangle_{cl} \right\|^2 = \int dq \left\langle \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \right| |q\rangle\langle q| \left| \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \right\rangle =$$

$$\int dq \left\langle \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \right| |q\rangle\langle q| \left| \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \right\rangle =$$

$$\int dq L_x^2(q,\delta,\varepsilon)\langle\psi^{\mathbf{ph}}(\theta)||q\rangle\langle q||\psi^{\mathbf{ph}}(\theta)\rangle = \|\langle x||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\|^2 + O(\delta) \asymp \tag{3.3.5}$$

$$\asymp \|\langle x||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\|^2,$$

$$\delta \ll 1, \varepsilon \in (0,1],$$

From postulate Q.I.3 follows that there exist unique random variable $X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ given on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and a measurable space $(\mathbb{R}^n,\Sigma)$ by formula

$$X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) \triangleq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) = \mathbf{G}\left[\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right] \tag{3.3.6}$$

The probability density of random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$ we denote by $p_{|\psi^{\mathbf{ph}}(\theta)\rangle}(q), q \in \mathbb{R}$.

Remark 3.3.2. From postulate Q.II.2 (see subsection II.1) follows that for the system in state $\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle$ the probability $P\left(q,q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring observable $\widehat{q}$ given by

$$P\left(q,q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) = p_{|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}}(q)dq = \left|c_{|\psi^{\mathbf{ph}}(\theta)_{cl}\rangle}(q)\right|^2 = \left|\langle q|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right|^2 \tag{3.3.7}$$

Now we go to explain Einstein's 1927 gedanken experiment. Let $A^{ph}(t,x_A)$ and $B^{ph}(t,x_A)$ be events such that detectors A,B detect photon at an instant t correspondingly. By properties (3.2.31) we obtain

$$\begin{gathered}\mathbf{P}\left(A^{ph}_{\lambda}(t,x_A)\right) \triangleq \mathbf{P}(\lambda[A^{ph}](t,x_A)) = \widetilde{\mathbf{P}}(A^{ph}(t,x_A)),\\ \mathbf{P}\left(B^{ph}_{\lambda}(t,x_B)\right) \triangleq \mathbf{P}(\lambda[B^{ph}](t,x_B)) = \widetilde{\mathbf{P}}(B^{ph}(t,x_B)).\end{gathered} \tag{3.3.8}$$

Note that

$$\begin{gathered}A^t \triangleq A^{ph}_{\lambda}(t,x_A) = \left\{\omega | x_A - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_A - \epsilon\right\},\\ B^t \triangleq B^{ph}_{\lambda}(t,x_B) = \left\{\omega | x_B - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_B - \epsilon\right\},\\ \epsilon \in (0,\gamma], \gamma \ll 1,\end{gathered} \tag{3.3.9}$$

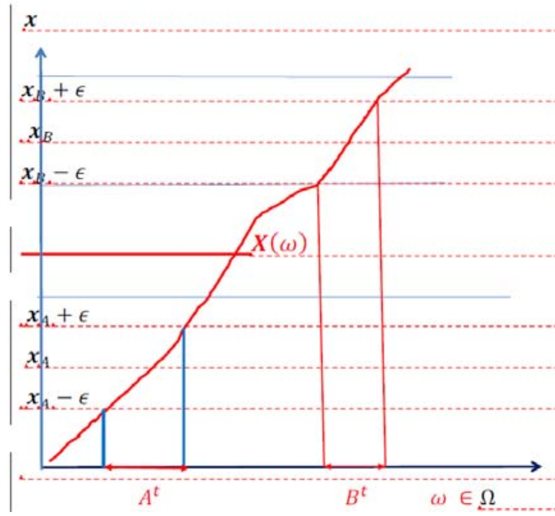
where a small parameter $\epsilon \ll |x_A - x_B|$ dependent on measuring device. Thus by general definition of random variable one obtains directly

$$A^{ph}_{\lambda}(t,x_A) \cap B^{ph}_{\lambda}(t,x_B) = \varnothing \tag{3.3.10}$$

and therefore

$$\mathbf{P}\left(A^{ph}_{\lambda}(t,x_A) \cap B^{ph}_{\lambda}(t,x_B)\right) = 0 \tag{3.3.11}$$

The property (3.3.11) follows directly from (3.3.8).



Pic. 3.3.1. The plot of the random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$.

$$A^t = A^{ph}_{\lambda}(t,x_A), B^t = B^{ph}_{\lambda}(t,x_B), A^t \cap B^t = \varnothing.$$

Remark 3.3.3. Let $[\mathcal{F}^{\#}_{M_4},\{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}}$ be a set of the all pairs $\{A(t,x_A),B(t,x_B)\}$ which is corresponding to a given vector $\{(t,x_A,0,0),(t,x_B,0,0)\} \in M_4 \times M_4$, $x_A \neq x_B$, i.e.,

$$\begin{gathered}[\mathcal{F}^{\#}_{M_4},\{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} = \\ \{\{A(t,x_A),B(t,x_B)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | 0 < (x_A - x_B)^2\}.\end{gathered} \tag{3.3.12}$$

Such pairs obviously is spacelike separated. Note that

$$\forall t \forall x_A \forall x_B (x_A \neq x_B)\left\{[\mathcal{F}^{\#}_{M_4},\{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} \neq \varnothing\right\}. \tag{3.3.13}$$

Now we go to explain non zero result $\widetilde{\mathbf{P}}(A^t \wedge B^t) \neq 0$ given above by (3.1.1) and (3.1.2):

$$\begin{gathered}\widetilde{\mathbf{P}}^{TL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}, \mathbf{P}^{TL}_{A^t} \cdot P^{TL}_{B^t} = 1.65 \pm 0.01 \cdot 10^{-4}, \\ \widetilde{\mathbf{P}}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}. \mathbf{P}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}.\end{gathered} \tag{3.3.14}$$

We consider this problem in general case.

Remark 3.3.4. Note that: (i) a probability density $p(x,\delta_A,\epsilon,)$ for the occurrence of a localization inside interval $[x-\epsilon,x+\epsilon]$ in arm with detector **A** (see Pic. 3.1.2) is given by formula

$$p(x,\delta_A,\epsilon) = \frac{\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_A,\epsilon)}, \tag{3.3.15}$$

where

$$\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_A \pi_{\delta_A}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_A}(q-x)^2\right],$$
$$\psi(q) = \langle q \| \psi \rangle, \tag{3.3.16}$$
$$\Delta(\delta_A,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 dx,$$

and where parametr δ_A depend on arm with detector **A**.

(ii) a probability density $p(x,\delta_B,\epsilon,)$ for the occurrence of a localization inside interval $[x-\epsilon,x+\epsilon]$ in arm with detector **B** (see Pic. 3.1.2) is given by formula

$$p(x,\delta_B,\epsilon) = \frac{\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_B,\epsilon)}, \tag{3.3.17}$$

where

$$\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_B \pi_{\delta_B}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_B}(q-x)^2\right],$$
$$\psi(q) = \langle q \| \psi \rangle, \tag{3.3.18}$$
$$\Delta(\delta_B,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 dx,$$

and where parametr δ_B depend on arm with detector **B**.

Remark 3.3.5.Note that parametr δ in formula (3.3.18) of course depend on measurement device and there no exist two equivalet devices such that $\delta_A = \delta_B$.

We assume now that

$$\begin{gathered} \delta_A \simeq \delta_B \ll 1, \\ 0 < |\delta_A - \delta_B|, \\ \int_{-\infty}^{\infty} x|\psi(x)|^2 dx < \infty, \\ \int_{-\infty}^{\infty} x\left[|\psi(x)|^2\right]'' dx < \infty. \end{gathered} \tag{3.3.19}$$

From Eq.(3.3.16) and (3.3.19) by using laplace approximation, we obtain:

$$\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta_A\pi_{\delta_A}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq|\psi(q)|^2 \exp\left[-\frac{1}{\delta_A}(q-x)^2\right] \approx$$

$$\approx |\psi(x)|^2 + \delta_A O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_A c_1^A\left[|\psi(x)|^2\right]'', \qquad (3.3.20)$$

$$\Delta(\delta_A,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2 dx = 1 + c_2^A\delta_A, c_2^A = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).$$

From Eq.(3.3.18) and (3.3.19) by using laplace approximation, we obtain:

$$\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta_B\pi_{\delta_B}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq|\psi(q)|^2 \exp\left[-\frac{1}{\delta_B}(q-x)^2\right] \approx$$

$$|\psi(x)|^2 + \delta_B O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_B c_1^B\left[|\psi(x)|^2\right]'', \qquad (3.3.21)$$

$$\Delta(\delta_B,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2 dx = 1 + c_2^B\delta_B, c_2^B = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).$$

From Eq.(3.3.15) and Eq.(3.3.17) we obtain

$$p(x,\delta_A,\epsilon) = \frac{\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2}{\Delta(\delta_A,\epsilon)} = \frac{|\psi(x)|^2 + \delta_A c_1^A\left[|\psi(x)|^2\right]''}{1 + c_2^A\delta_A,}. \qquad (3.3.22.a)$$

From Eq.(2.2.54) and Eq.(2.2.57) we obtain

$$p(x,\delta_B,\epsilon) = \frac{\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2}{\Delta(\delta_B,\epsilon)} = \frac{|\psi(x)|^2 + \delta_B c_1^B\left[|\psi(x)|^2\right]''}{1 + c_2^B\delta_B,}. \qquad (3.3.22.b)$$

Definition 3.3.1.We define now signed measures $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t)$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t)$ by formulae

$$\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t) = \int\limits_{A^t} xp(x,\delta_A,\epsilon)d\mu(x),$$

$$\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t) = \int\limits_{A^t} xp(x,\delta_B,\epsilon)d\mu(x), \qquad (3.3.23)$$

where $A^t \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure and $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on a set $[a,b]$.

Definition 3.3.2.Let $\mathbf{P}(\omega)$ be a probability measure on a measurable space $\{[a,b],B([a,b])\}$. We assume now that $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle} \ll \mathbf{P}$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle} \ll \mathbf{P}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$.By Radon-Nicodym theorem we obtain for any $A^t \in \Sigma_{a,b}$:

$$
\begin{aligned}
\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(A^t) &= \int_{A^t} X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega)d\mathbf{P}(\omega),\\
X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) &= \frac{d\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}}{d\mathbf{P}},\\
\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(A^t) &= \int_{A^t} X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega)d\mathbf{P}(\omega),\\
X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) &= \frac{d\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}}{d\mathbf{P}}.
\end{aligned}
\tag{3.3.24}
$$

We write below for a short

$$
X_1(\omega) \triangleq X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega), X_2(\omega) \triangleq X_{\left|\psi_{\delta_B,\epsilon,x}\right\rangle}(\omega). \tag{3.3.25}
$$

Remark 3.3.6. We assume now without loss of generality that

$$
X_2(\omega) - X_1(\omega) \geq 0 \text{ a.s.} \tag{3.3.26}
$$

see Pic. 3.3.1.

Let us consider now the quantity

$$
\eta_{1,2} = \int_{\Omega} |X_1(\omega) - X_2(\omega)| d\mathbf{P}(\omega) = \int_{\Omega} [X_2(\omega) - X_1(\omega)] d\mathbf{P}(\omega). \tag{3.3.27}
$$

We assume now that

$$
\int_{-\infty}^{\infty} x|\psi(x)|^2 dx < \infty, \int_{-\infty}^{\infty} \left[x|\psi(x)|^2\right]'' dx < \infty, \tag{3.3.28}
$$

From Eq.(3.3.27) by using Eq.(3.3.21) and Eq.(3.3.22) we obtain

$$\eta_{1,2} =$$

$$\int_{\mathbb{R}} p(x,\delta_B,\epsilon)dx - \int_{\mathbb{R}} p(x,\delta_A,\epsilon)dx = \frac{1}{1+c_2^B\delta_B}\int_{\mathbb{R}}\Big[x|\psi(x)|^2 + \delta_B c_1^B\big[x|\psi(x)|^2\big]''\Big]dx -$$

$$-\frac{1}{1+c_2^A\delta_A}\int_{\mathbb{R}}\Big[x|\psi(x)|^2 + \delta_A c_1^A\big[x|\psi(x)|^2\big]''\Big]dx \simeq$$

$$(1-c_2^B\delta_B)\int_{\mathbb{R}}\Big[x|\psi(x)|^2 + \delta_B c_1^B\big[x|\psi(x)|^2\big]''\Big]dx -$$

$$-(1-c_2^A\delta_A)\int_{\mathbb{R}}\Big[x|\psi(x)|^2 + \delta_A c_1^A\big[x|\psi(x)|^2\big]''\Big]dx = \qquad (3.3.29)$$

$$\delta_B c_1^B\int_{\mathbb{R}}\big[x|\psi(x)|^2\big]''dx - c_2^B\delta_B\int_{\mathbb{R}}x|\psi(x)|^2dx - \delta_B^2 c_1^B c_2^B\int_{\mathbb{R}}\big[x|\psi(x)|^2\big]''dx -$$

$$-\delta_A c_1^A\int_{\mathbb{R}}\big[x|\psi(x)|^2\big]''dx + c_2^A\delta_A\int_{\mathbb{R}}x|\psi(x)|^2dx + \delta_A^2 c_1^A c_2^A\int_{\mathbb{R}}\big[x|\psi(x)|^2\big]''dx =$$

$$(\delta_B c_1^B - \delta_A c_1^A - \delta_B^2 c_1^B c_2^B + \delta_A^2 c_1^A c_2^A)\int_{\mathbb{R}}\big[|\psi(x)|^2\big]''dx + (c_2^A\delta_A - c_2^B\delta_B)\int_{\mathbb{R}}|\psi(x)|^2dx \simeq$$

$$\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A),$$

where

$$\alpha_1 = \int_{\mathbb{R}} x|\psi(x)|^2dx, \alpha_2 = \int_{\mathbb{R}}\big[x|\psi(x)|^2\big]''dx. \qquad (3.3.30)$$

Lemma 3.3.1. Let $(\Omega,\Sigma,\mathbf{P})$ be a measure space, and let f be an real-valued measurable function defined on Ω. Then for any real number $t > 0$:

$$\mathbf{P}\{\omega \in \Omega \| f(\omega)| \geq t\} \leq \frac{1}{t}\int_{|f(\omega)|\geq t}\big|f(\omega)\big|d\mathbf{P}(\omega). \qquad (3.3.31)$$

From inequality (3.3.31) and Eq.(3.3.29) we obtain

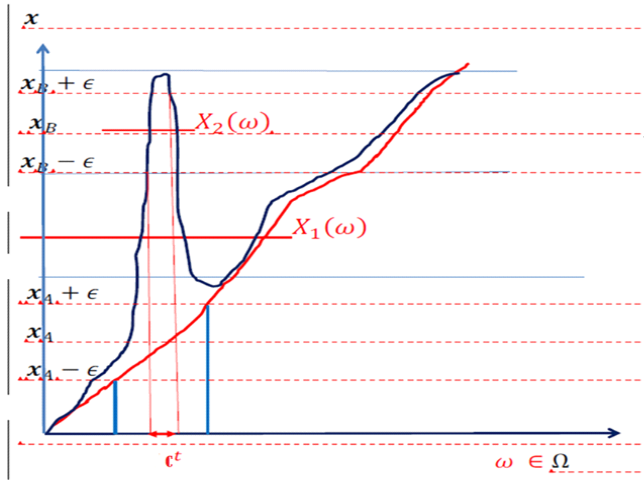
$$\mathbf{P}\Big\{\omega \in \Omega : \ |X_1(\omega) - X_2(\omega)| \geq t\Big\} \leq \frac{1}{t}\int_{|X_1(\omega)-X_2(\omega)|\geq t}[|X_1(\omega) - X_2(\omega)|]d\mathbf{P}(\omega)$$

$$< \frac{1}{t}\int_{\Omega}[X_1(\omega) - X_2(\omega)]d\mathbf{P}(\omega) = \frac{\eta_{1,2}}{t} \simeq \frac{\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{t}. \qquad (3.3.32)$$

We define now

$$A^t = A_\lambda^{ph}(t,x_A) = \{\omega | x_A - \epsilon \leq X_1(\omega) \leq x_A - \epsilon\},$$
$$B^t = B_\lambda^{ph}(t,x_B) = \{\omega | x_B - \epsilon \leq X_2(\omega) \leq x_B - \epsilon\}, \qquad (3.3.33)$$

and chose in (3.3.31) number $t = x_B - x_A \gg 1$.



Pic. 3.3.1.The plot of the random variables $X_1(\omega)$ and $X_2(\omega)$.

$$A^t = A^{ph}_\lambda(t,x_A), B^t = B^{ph}_\lambda(t,x_B), A^t \cap B^t \subseteq C^t.$$

Note that

$$\mathbf{P}(A^t \cap B^t) \leq \mathbf{P}(C^t), \qquad (3.3.34)$$

see Pic. 3.3.1. From Eq.(3.3.32)-Eq.(3.3.34) follows that

$$\mathbf{P}(A^t \cap B^t) < \frac{\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{x_B - x_A} \ll 1. \qquad (3.3.35)$$

IV.Conclusions

Einstein's 1927 gedanken experiment revised and by using the probability representation of quantum states explained successfully.

References

[1] A. Einstein et al., Phys. Rev. 47, 777 (1935).

[2] Bacciagaluppi, G. & Valentini, A. Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference 487 (Cambridge University Press,2009).

[3] Born, M. Natural Philosophy of Cause and Chance 109 (Oxford University Press, 1949).

[4] Jones,S.J. & Wiseman,H. M. Nonlocality of a single photon: paths to an Einstein-Podolsky-Rosen-steering experiment. Phys. Rev. A 84, 012110 (2011)

[5] M.Fuwa,S.Takeda, M. Zwierz, H. M. Wiseman,A. Furusawa,Experimental proof of nonlocal wavefunction collapse for a single particle using homodyne measurements. Nature Communications 6, Article number: 6665 doi:10.1038/ncomms7665

http://www.nature.com/ncomms/2015/150324/ncomms7665/full/ncomms7665.html
[6] Schrödinger, E. Discussion of probability relations between separated systems. Proc.Cambridge Philos. Soc. 31, 553 (1935).
[7] Terhal, B. M. Bell inequalities and the separability criterion. Phys. Lett. A 271, 319–326 (2000)
[8] Bell, J. S. On the Einstein Podolsky Rosen paradox. Physics 1, 195–200 (1964).
[9] Mittelstaedt, P. "The Problem of Decoherence and the EPR-Paradox." Decoherence: Theoretical, Experimental, and Conceptual Problems, Edited by P. Blanchard, D. Giulini, E. Joos, C. Kiefer, I.-O. Stamatescu, Lecture Notes in Physics, Published by Springer Berlin Heidelberg ISBN: 978-3-540-66899-2 DOI: 10.1007/3-540-46657-6
[10] E.G.Cavalcanti, S.J.Jones, H. M. Wiseman, M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox,Phys. Rev. A 80, 032112 (2009) http://arxiv.org/abs/0907.1109v2
[11] M. D. Reid,The Einstein-Podolsky-Rosen Paradox and Entanglement 1: Signatures of EPR correlations for continuous variables. http://arxiv.org/abs/quant-ph/0112038v1
[12] E. G. Cavalcanti, P. D. Drummond, H. A. Bachor, M. D. Reid,Spin entanglement, decoherence and Bohm's EPR paradox,Optics Express, Vol. 17, Issue 21, pp. 18693-18702 (2009) http://arxiv.org/abs/0711.3798
[13] M. D. Reid, P. D. Drummond, E. G. Cavalcanti, W. P. Bowen, P. K. Lam, H. A. Bachor, U. L. Andersen, G. Leuchs The Einstein-Podolsky-Rosen paradox: from concepts to applications,Rev.Mod. Phys. 81, 1727 (2009) http://arxiv.org/abs/0806.0270
[14] Foukzon, J., Potapov, A., Men'kova, E. and Podosenov, S. (2017) Schrödinger's Cat Paradox Resolution Using GRW Collapse Model: Von Neumann Measurement Postulate Revisited. Journal of Applied Mathematics and Physics, 5, 494-521. doi: 10.4236/jamp.2017.52044.
[15] Foukzon, J., Potapov, A., Men'kova, E. and Podosenov, S. Schrödinger's Cat Paradox Resolution: A New Quantum Mechanical Formalism Based on the Probability Representation of Continuous Observables Paperback 84 pages,Publisher: LAP LAMBERT Academic Publishing (December 19, 2017).ISBN-10: 3659935433;ISBN-13: 978-3659935435
[16] S. L. Adler, Where is quantum theory headed? Journal of Physics: Conference

Series 504 (2014) 012002 doi:10.1088/1742-6596/504/1/012002
[17] V. Man’ko, Classical propagators of quadratic quantum systems, Theoretical and Mathematical Physics,November 1999, Volume 121, Issue 2, pp. 1496-1505 DOI: http://dx.doi.org/10.4213/tmf809 http://link.springer.com/article/10.1007/BF02557220#page-1
[18] S.Mancini,V.I.Man’ko,P.Tombesi.Phys.Lett.A.1996.V.213.P.1
[19] T. Guerreiro, B.Sanguinetti, H. Zbinden, N. Gisin, A. Suarez, Single-photon space-like antibunching. arXiv:1204.1712v1 [quant-ph] 8 Apr 2012
[20] D.H.Fremlin, Measure Theory,Volume 3: Measure Algebras, May 31, 2002.
[21] M. Gondran, A. Gondran, Measurement in the de Broglie-Bohm interpretation: Double-slit, Stern-Gerlach and EPR-B. https://arxiv.org/abs/1309.4757
[22] P. M. Walker, L. Tinkler, M. Durska, D. M. Whittaker, I. J. Luxmoore, B. Royall, D. N. Krizhanovskii, M. S.Skolnick, I. Farrer, and D. A. Ritchie, Applied Physics Letters 102, 012109 (2013).
[23] P. M. Walker, L. Tinkler, D. V. Skryabin, A. Yulin, B. Royall, I. Farrer, D. A. Ritchie, M.S. Skolnick, and D. N. Krizhanovskii, Nat Commun 6, 8317 (2015).
[24] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S.M. Girvin, and R. J.Schoelkopf, Nature 431, 162 (2004).
[25] D. I. Schuster, A. A. Houck, J. A. Schreier, A. Wallraff, J. M. Gambetta, A. Blais, L. Frunzio, J. Majer, B. Johnson, M. H. Devoret, et al., Nature 445, 515 (2007).
[26] Y. Yin, Y. Chen, D. Sank, P. J. J. O’Malley, T. C. White, R. Barends, J. Kelly, E. Lucero, M. Mariantoni, A. Megrant, et al., Phys. Rev. Lett. 110, 107001 (2013).
[27] R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, et al., Nature 508, 500 (2014).
[28] Y. Sherkunov, David M. Whittaker, and Vladimir Fal’ko, Position-momentum-entangled photon pairs in nonlinear waveguides and transmission lines,Phys. Rev. A 93, 043842 http://arxiv.org/abs/1511.03514
[29] D. Bohm, Quantum Theory (New York, Prentice-Hall, 1951).
[30] D.,Bohm,Y.,Aharonov, Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky,
[31] M. Gondran, A. Gondran, "A complete analysis of the Stern-Gerlach experiment using Pauli spinors", quantph/05 1276 (2005).
[32] John C. Howell, Ryan S. Bennink, Sean J. Bentley, and R. W. Boyd Realization of the Einstein-Podolsky-Rosen Paradox Using Momentum and

Position-Entangled Photons from Spontaneous Parametric Down Conversion
http://www.pas.rochester.edu/~howell/mysite2/EPR%20PRL.pdf
[33] Z.Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Realization of the
Einstein-Podolsky-Rosen paradox for continuous variables. Phys. Rev. Lett. 68,
3663
[34] J.Clauser, M.Horne, A.Shimony, R.Holt, (1969). "Proposed Experiment to Test
Local
Hidden-Variable Theories". Physical Review Letters. 23 (15): 880–884.