Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series. Part I.

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This is an article about foundation of paralogical nonstandard analysis and its applications to the continuous function without a derivative presented by absolutely convergent trigonometrical series and another famous problems of trigonometrical and orthogonal series. In part 1 of the present work, using the methods of paralogical nonstandard analysis, we shall obtain a general criterion for that there is no function which would be the almost everywhere finite derivative function for the following continuous function:

$$\Im(x)=\sum_{n=1}^{\infty}\frac{\exp(i\cdot x\cdot\pi\cdot\omega_1(n))}{\omega_2(n)},\ \omega_1(n):N\to N,\ \omega_2(n):N\to R,\ \sum_{n=1}^{\infty}|\,\omega_2^{-1}(n)\,|<\infty .$$

Part I

Foundation of paralogical nonstandard analysis and application to problems of non differentiable functions presented by absolutely convergent trigonometrical series. Part I.

I. Introduction and results

According to Weierstrass [1], in a talk to the Royal Academy of Sciences in Berlin on 18 July 1872, Riemann introduced the function

$$\Re(x)=\sum_{n=1}^{\infty}\frac{\sin(\pi\cdot n^2x)}{n^2}\qquad(1.1)$$

in order to warn that continuous functions need not have a derivative. Not succeeding in verifying that $\Re(x)$ is nowhere differentiable, Weierstrass proved this property instead for the series

$$W(x) = \sum_{n=1}^{\infty} b^n \cos(a^n t), \quad 0 < b < 1, \qquad (1.2)$$

with suitably chosen positive numbers a and b. This appeared first in print in Du-Bois-Reymond [2]. According to Butzer and Stark [3], there are no other known sources which confirm Riemann's role in the story. Hardy [4, pp. 322-323] proved that Riemann's function $\Re(x)$ is not differentiable in any irrational point x and also not differentiable in a large class of rational $x \in Q$. With a completely elementary but long proof, Gerver [5] succeeded in 1970 in showing that at every rational point $r = p/q$ with p and q both odd, f(x) is differentiable, and has derivative equal to $-1/2$ at r. Furthermore he showed that at all other rational points the function is not differentiable. Other,shorter proofs were given by Smith [6], Quefelec [7], Mohr [8], Itatsu [9], Luther [10] and Holschneider and Tchamitchian [11]. For previous reviews on Riemann's function, see Neuenschwander [12] and Segal [13]; the literature list of [3] contains many further references abaut the Riemann's function $\Re(x)$. For $2 < \beta < 4$, in [14] directed analyze the behavior, near the points $x = p\pi / q$ of

$$G_{3,\beta}(x) = \sum_{n=1}^{\infty} n^{-\beta} \exp(ixn^3), \qquad (1.3)$$

considered as a function of x,and expand this series into a constant term, a term on the order of $(x - p\pi/q)^{(\beta-1)/3}$, a term linear in $x - p\pi/q$, a ``chirp" term on the order of $(x - p\pi/q)^{(2\beta-1)/4}$, and an error term on the order of $(x - p\pi/q)^{\beta/2}$. At every such rational point, the left and right derivatives are either both finite (and equal) or both infinite, in contrast with the quadratic series, where the derivative is often finite on one side and infinite on the other. However, in the cubic series, again in contrast with the quadratic case, the chirp term generally has a different set of frequencies and amplitudes on the right and left sides. Finally, we show that almost every irrational point can be closely approximated, in a suitable Diophantine sense, by rational points where the cubic series has an infinite derivative. This implies that when

$$\beta \le (\sqrt{97} - 1)/4 = 2.212.., \qquad (1.4)$$

both the real and imaginary parts of the cubic series are differentiable almost nowhere. At the same time it is necessary to note that in spite of a big progress obtained in the considered studies area, any general absence criterions of the finite almost everywhere derivate for absolutely convergent trigonometrical series was not obtained.

In the present work, using the methods of paralogical nonstandard analysis, we shall obtain the general absence criterion of the almost everywhere finite derivative function for the following continuous function

$$\Im(x) = \sum_{n=1}^{\infty} \frac{\exp(i \cdot x \cdot \pi \cdot \omega_1(n))}{\omega_2(n)}, \tag{1.5}$$

$\omega_1(n): N \to N$, $\omega_2(n): N \to R$, $\sum_{n=1}^{\infty} |\omega_2^{-1}(n)| < \infty$. It is shown that by the execution of condition

$$\sum_{n=1}^{\infty} \left(\frac{\omega_1(n)}{\omega_2(n)} \right)^2 = \infty, \tag{1.6}$$

function $\Im(x)$ does not have a finite derivate on a quantity of a positive measure. Particularly we shall reinforce the foregoing Gerver's result by showing that inequality (1.4) is possible to change by inequality $\beta \leq 3.5$, at least for a quantity of points of a positive measure.

II. Paralogical sets. Strong paraconsistens set theory $ZFC^{\#}$

A set theory is paraconsistent if it is inconsistent but nontrivial, i.e., at least one contradiction is derived but still there are formulas that are not theorems. Thus the anderlying logic of paraconsistent set theory must be a paraconsistent set theory must be a paraconsistent logic, i.e. in which there is a symbol of negation $\neg$, such that from a formula A and its negation $\neg A$, it is not possible in general to obtain any formula B whatsoever. For the first time paraconsistent set theory was made N.C.A. da Costa, in 1963, [15] , the same work in which he presented his infinite hierarchy of paraconsistent logics. Further attempts can be found Arruda [16], Assenjo and Tamburino [17], and Goodman [18]. Except for da Costa's, Assenjo and Tamburino's set theories the others are proved to be nontrivial. Those, paraconsistent set theory already proven to be nontrivial may be called *weak* paraconsistent set theory for because of her underlying special logic, a lot of the basic results of classical set theories system ZFC are not valid in them. The others, supposing that they are nontrivial, may by called *almost strong,* paraconsistent set theory for almost all results of classical set theories system ZFC are valid in them. The others, supposing that they are nontrivial, may by called *strong* paraconsistens set theory for all results of classical set theories system ZFC are valid in them. Arruda [19] proved that da Costa's formulation of the axiom schema of abstraction for the systems $NF_n, 1 \le n < \omega$, leads to the trivialization of the systems, NF_ω with da Costa's formulation of the axiom schema of abstraction, leads to the paradox of identity: $\forall x \forall y (x = y)$, NF_ω after correction by Arruda's special syntactical development, may be considered as a almost strong paraconsistent set theory. But many ears the method for strong paraconsistent set theory production was not found.

Let us call a set X-strong paralogical set if it may exist in strong paraconsistent set theory but not in a classical set theory ZFC. Let us call a set X-paralogical set if for certain relation $\Re(\circ)$ such $\Re(X)$ and $\neg\Re(X)$. For example the all set X such $X \in X$ and $X \notin X$ is a paralogical set, thus Rassel's set $\Re_0 \stackrel{\Delta}{=} \hat{X}\neg(X \in X)$, is a paralogical set, thus Rassel's n-order set $\Re_n \stackrel{\Delta}{=} \hat{X}\neg^{(n)}(X \in X)$, where $\neg^{(n)} A$ is defined as $\neg A \wedge A^{(n)}$, $A^{(n)} \stackrel{\Delta}{=} A^1 \wedge A^2 ... A^n$, $A^{n+1} \stackrel{\Delta}{=} (A^n)^0$, $A^0 = A^1 = \neg(A \wedge \neg A)$, is a paralogical set, thus the universal set $V \stackrel{\Delta}{=} \hat{x}(x = x)$, is a paralogical set.

Theorem I. X **strong paralogical set** $\Leftrightarrow$ X **is a paralogical set.**

Let us call the abstract relation $\Re(\circ)$-**paralogical relation** if for certain object X we have $\Re(X)$ and $\neg\Re(X)$. Let us call the abstract relation $\Re(\circ)$ auto-logical if $\Re(\Re(\circ))$ is true i.e. $\Re(\circ) \in \Re(\circ)$, and let us call the abstract relation $\Re(\circ)$ hetero-logical if $\Re(\Re(\circ))$ is false i.e. $\Re(\circ) \notin \Re(\circ)$. Thus Grelling's relation $Gr(\circ) \stackrel{\Delta}{=} \hat{\Re}(\circ)[\Re(\circ) \notin \Re(\circ)]$, is a paralogical relation. (Grelling's paradox), n-order Grelling's relation $Gr_n(\circ) \stackrel{\Delta}{=} \hat{\Re}(\circ)[\neg^{(n)}[\Re(\circ) \in \Re(\circ)]]$, is a paralogical relation

(n-order Grelling's paradox).

Let us call the proposition $A-$this is a **paralogical proposition,** if for certain objects $X, \Re(\circ): A \Leftrightarrow \Re(X)$, and $\Re(X) \wedge \neg\Re(X)$.

Definition I. $V\Re \stackrel{\Delta}{=} \hat{\Re}(X)[\Re(X) \Leftrightarrow \Re(X)]$.**.**

Definition II. $VP\Re \stackrel{\Delta}{=} \hat{\Re}(X)[\Re(X)$ **is a paralogical proposition,**

$X \in V, \Re(X) \in V\Re]$**, i.e.** $VP\Re$ **is a set of the all paralogical propositions.**

Definition III. Let us call the paraconsistent logic $\Im-$this is a **strong paraconsistent logic,** if any paralogical proposition don't invalidate logic $\Im$.

The postulates of propositional **strong paraconsistent logic** $L^{\#}$ are the following:

I. Logical postulates:

(1) $A \Rightarrow (B \Rightarrow A),$

(2) $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)),$

(3) $A \Rightarrow (B \Rightarrow A \wedge B),$

(4) $A \wedge B \Rightarrow A,$

(5) $A \wedge B \Rightarrow B,$

(6) $A \Rightarrow (A \vee B),$

(7) $B \Rightarrow (A \vee B),$

(8) $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C)),$

(9) $A \vee \neg A,$ if and only if $A \notin VP\mathfrak{R}$.

(10) $B \Rightarrow (\neg B \Rightarrow A).$

II. Weak modus ponens :

(1) $A, A \Rightarrow B \models_{L^{\#}} B,$ if $A \notin VP\mathfrak{R}$ (but not: if and only if),

(2) $A, A \Rightarrow B \not\models_{L^{\#}} B,$ if $A \in VP\mathfrak{R}$ and $B \in V\mathfrak{R} \setminus VP\mathfrak{R}$,

(3) $B, B \Rightarrow (\neg B \Rightarrow A) \not\models_{L^{\#}} \neg B \Rightarrow A,$ if $B \in VP\mathfrak{R}.$

Definition IV. $ZFC^{\#} \equiv ZFC + (\exists V) + (L^{\#})$.

Theorem II. $ZFC^{\#}$ **is a strong paraconsistens set theory.**

Theorem III. ConsisZFC $\Leftrightarrow$ ParaConsisZFC $^{\#}$.

III. Paralogical ultrafilters and paralogical nonstandard extensions

The starting point of classical nonstandard analysis is the construction and use, of an ordered field ${}^{\bullet}R$ which is a proper extension of the usual ordered field R of real numbers, and which satisfies ***almost*** all the properties of R. We refer to ${}^{\bullet}R$ as a field of classical nonstandard real numbers, or as a field of ***classical*** hyperreal numbers. We also refer to ${}^{\bullet}N$ as the set of ***classical*** nonstandard natural numbers or as the set of ***classical*** hypernatural numbers. Because the ordered field R is Dedecind complete, it follows that extension field ${}^{\bullet}R$ will necessarily have among its new elements both infinitesimal and infinite numbers. Bat it is easily seen that a proper classical extension field ${}^{\bullet}R$ of R ***cannot satisfy literally*** all the properties of R. For example it ***cannot be Dedecind complete***, because the set of finite numbers in ${}^{\bullet}R$ cannot have a least upper bound r, because then $r-1$ would be a smaller upper bound. Thus nonstandard field ${}^{\bullet}R$ may be called ***weak nonstandard*** extension of the field R. **But it is easily seen that a strong nonstandard extension of the field** R, **cannot exist in set theory** ZFC.

Definition IV. Let us call algebraic field Θ in set theory $ZFC^{\#}$, non-classical field, if it may exist in set theory $ZFC^{\#}$, but not in a classical set theory ZFC.

Definition V. Let us call non-classical field Θ- ***paraordered*** field, if ordered relation $(\circ \le \circ)$ - it is a paralogical relation on the $\Theta \times \Theta$.

The starting point of paralogical nonstandard analysis is the construction and use, of an ***paraordered*** field ${}^{\#}R$ which is a proper extension of the classical nonstandard field ${}^{\bullet}R$.

Definition VI. Let us call paraordered field ${}^{\#}R$-**strong nonstandard extension** of the field ${}^{\#}R$, if ${}^{\#}R$ satisfy ***literally*** all the properties of R.

Theorem III. Strong nonstandard extensions of the classical nonstandard field ${}^{\bullet}R$, exist in set theory $ZFC^{\#}$.

Definition VII. Let X be a paralogical set. A paralogical filter on X is a set $\mathbb{F}$ of subsets of X such that :

(1) $X \in F$,

(2) The intersection of any two elements of $\mathbb{F}$ is an element of $\mathbb{F}$,

(3) If $H \in F$ **and** $H \subset G \subset X$ **then** $G \in F$ **,**

(4) $\varnothing \notin F$ **, (5)** $\varnothing \in F$ **.**

Definition VIII. A paralogical filter $\mathbb{F}$ is said to be *fixed* or *principal* if the intersection of all elements of $\mathbb{F}$ is nonempty; otherwise, $\mathbb{F}$ is said to be *free* or *non-principal*.

Definition IX. Let X be a set. A collection U of subsetsof X is an paralogical *ultrafilter* if U is a paralogical filter, and whenever $A \subseteq X$ then either $A \in U$ or $X \setminus A \in U$.

IV.Paralogical nonstandard analysis and paralogical ultraproducts of structures

A basic framework for Paralogical Nonstandard Analysis (PNSA) can be derived in a very natural way from the elementary properties of ultraproducts. Logical formalism provides a convenient way of expressing and using the basic properties of paralogical ultraproducts. Let V be a set of mathematical objects; we want to consider paralogical non-standard extensions of V. Each such extension is based on a set that we will denote by ${}^{\#}V$. To begin we construct these as paralogic ultrapowers of V. Let I be an index set and $U^{\#}$ an paralogical ultrafilter on I. To avoid trivial situations we assume that V and I are infinite and that $U^{\#}$ is countably incomplete. This means that there are sets $(U_n \mid n \in {}^{\#}N)$ in $U^{\#}$ such that $\bigcap_n (U_n \mid n \in {}^{\#}N) = \varnothing$

Note that any nonprincipal paralogical ultrafilter on a countable index set must be countably incomplete.

On the space V^I of all functions $\alpha : I \to V$ define an equivalence relation $\sim_{U^{\#}}$ by

$$\alpha \sim_{U^{\#}} \beta \Leftrightarrow \{i \in I \mid \alpha(i) = \beta(i)\} \in U^{\#} .$$

Then ${}^{\#}V$ is defined to be the set of all equivalence classes of $\sim_{U^{\#}}$ on V^I.

That is,elements of ${}^{\#}V$ are represented by functions $\alpha : I \to V$,and these functions, are identified exactly when they are equal "almost everywhere" in the sense of ${}^{\#}U$. For each function $\alpha : I \to V$ we denote its equivalence class under $\sim_{U^{\#}}$ by $[\alpha]_{\#}$. We regard ${}^{\#}V$ as an extension of V via the diagonal mapping $\#: {}^{\#}V \to {}^{\#}V$

defined for each $a \in V$ by ${}^{\#}a = [\alpha]_{\#}$ where α is the constant function $\alpha : I \to V$ with $\alpha(i) = a$ for all $i \in I$. To obtain a fully equipped paralogical nonstandard extension of V from this construction we define ${}^{\#}A$ for every set $A \subseteq V^m$ and define ${}^{\#}A \subseteq ({}^{\#}V)^m$ for every function $f : A \to B$ where A and B are subsets of (possibly different) cartesian powers of V. First consider $A \subseteq V^m$ and define ${}^{\#}A \subseteq ({}^{\#}V)^m$ by

$$([\alpha_1]_{\#},...,[\alpha_m]_{\#}) \in {}^{\#}A \Leftrightarrow \{i \in I \mid (\alpha_1(i),...,\alpha_m(i)) \in A\} \in U^{\#} .$$

Next consider $f : A \to B$ where $A \subseteq V^m$ and $B \subseteq V^n$ and define ${}^{\#}f : {}^{\#}A \to {}^{\#}B$ by

$${}^{\#}f([\alpha_1]_{\#},...,[\alpha_m]_{\#}) = ([\beta_1]_{\#},...,[\beta_m]_{\#})$$

where $\beta_1,...,\beta_n$ are elements of V^I satisfying

$$\{i \in I \mid (\beta_1(i),...,\beta_n(i)) = f(\alpha_1(i),...,\alpha_m(i))\} \in U^{\#} .$$

Of course one must show that ${}^{\#}A$ and ${}^{\#}f$ are well defined on the equivalence classes: $[\alpha_1]_{\#},...,[\alpha_m]_{\#}$; this is an elementary exercise.

The full system just defined will be referred to as the ${}^{\#}U$-paralogical ultrapower nonstandard extension of V. The most important properties of this ${}^{\#}$mapping are given next. They are all very easy to prove using the elementary roperties of paralogical ultralters. For each $m,n \geq 0$ this ${}^{\#}$mapping satisifes the following conditions:

$(E1)\,\#$ preserves membership and function values: For each $a_1,...,a_m \in V$, if $A \subseteq V^m$, then $[({}^\#a_1,...,{}^\#a_m) \in {}^\#A]^{(n)} \Rightarrow ({}^\#a_1,...,{}^\#a_m) \in {}^\#A \Leftrightarrow (a_1,...,a_m) \in A, n \in N$; moreover, if $f : A \to B$ is a function, with $a_1,...,a_m \in A \subseteq V^m$ and $B \subseteq V^n$, then ${}^\#f({}^\#a_1,...,{}^\#a_m) = {}^\#(f(a_1,...,a_m))$.

$(E2)\,\#$ commutes with Boolean operations: if $A, B \subseteq V^m$, then ${}^\#(A \cap B) = ({}^\#A \cap {}^\#B)$, ${}^\#(A \cup B) = ({}^\#A \cup {}^\#B)$, ${}^\#(V^m \setminus A) = ({}^\#V)^m \setminus {}^\#A$; moreover, ${}^\#(V^m) = ({}^\#V)^m$ and ${}^\#\varnothing = \varnothing$.

$(E3)\,\#$ commutes with Cartesian products: if $A \subseteq V^m$ and $B \subseteq V^n$ then ${}^\#(A \times B) = {}^\#A \times {}^\#B$. We regard $A \times B$ as a subset of V^{m+n}.

$(E4)\,\#$ commutes with coordinate mappings: if $\pi : V^m \to V^n$ is any coordinate mapping on V then ${}^\#\pi : ({}^\#V)^m \to ({}^\#V)^n$ is the corresponding coordinate mapping on ${}^\#V$; moreover ${}^\#(\pi(A)) = ({}^\#\pi)({}^\#A)$ for all $A \subseteq V^m$.

By a "coordinate mapping" in $(E4)$ we mean a map, which omits, identifies and permutes coordinates. Such a map is associated to a sequence $s = s_1,...,s_n$ of elements of $\{1,...,m\}$. The coordinate mapping π_s from V^m to V^n is the one that satisfies $\pi_s(a_1,...,a_m) = (a_{s_1},...,a_{s_n})$ for all $a_1,...,a_m \in V$. Note that this gives a uniform meaning to π_s independent of the set from which the coordinates are taken. In particular, the corresponding coordinate mapping on ${}^\#V$ is given by a similar definition and $(E4)$ includes the assertion that this mapping is equal to ${}^\#\pi_s$.

$(E5)\,\#$ is injective: if $A \subseteq V^m$ is nonempty, then ${}^\#A$ is also nonempty.

Therefore, for any $A, B \subseteq V^m$, $[{}^\#A = {}^\#B]^{(n)} \Rightarrow {}^\#A = {}^\#B \Leftrightarrow A = B$; moreover, if f, g are functions from $A \subseteq V^m$ to $B \subseteq V^n$ then $[{}^\#f = {}^\#g]^{(n)} \Rightarrow {}^\#f = {}^\#g \Leftrightarrow f = g$.

$(E6)\,\#$ preserves finite cardinalities: if $A \subseteq V^m$ is finite, then A and ${}^\#A$ have the same number of elements; if $A \subseteq V^m$ is infinite then ${}^\#A$ is infinite.

$(E7)\,\#$ preserves function graphs: if $f : A \to B$ is a function, with $A \subseteq V^m$ and $B \subseteq V^n$, and if $\Gamma \subseteq V^{m+n}$ is the graph of f, then ${}^\#\Gamma$ is the graph of the function ${}^\#f : {}^\#A \to {}^\#B$.

$(E8)\,\#$ preserves identity functions and commutes with composition of functions: if f is the identity function on $A \subseteq V^m$, then ${}^\#f$ is the identity function on ${}^\#A$; if $f : A \to B$ and $g : B \to C$ are functions, with $A \subseteq V^m$, $B \subseteq V^n$ and $C \subseteq V^k$, then ${}^\#(g \circ f) = ({}^\#g) \circ ({}^\#f)$.

$(E9)\,\#$ is proper: if $A \subseteq V^m$ and A is infinite, then ${}^\#A$ contains elements that are nonstandard; that is ${}^\#A$ contains elements which are not of the form $({}^\#a_1,...,{}^\#a_m)$ with $a_1,...,a_m \in V$.

Definition. (Paralogical Nonstandard Extension of a Set V) Let V be an infinite paralogical set. A paralogical nonstandard extension of V consists of a set ${}^\#V$ together with a mapping $\#$ that embeds V into ${}^\#V$ and that assigns a set ${}^\#A \subseteq ({}^\#V)^m$ to each set $A \subseteq V^m$ and a function ${}^\#f : {}^\#A \to {}^\#B$ to each function $f : A \to B$.

(where $A \subseteq V^m$ and $B \subseteq V^n$) such that conditions $(E1)$ through $(E9)$ are satisfied.

Remain standard intended use of formulas, let x, y be variables ranging over nonempty sets A, B respectively and let $\varphi(x, y)$ and $\psi(x, y)$ denote conditions (formulas) on (x, y) defining subsets Φ and Ψ (respectively) of $A \times B$.

We consider certain logical formulas that can be built up from $\varphi(x, y)$ and $\psi(x, y)$ (on the left below) and the sets that are defined by them (on the right):

$A^* \Leftrightarrow \forall n A^{(n)}$

$\neg^* A \Leftrightarrow \neg(A \wedge \neg A^\bullet)$

$\varphi^\bullet(x, y)$ defines $\varphi(x, y) \wedge \neg^\bullet \varphi(x, y)$

$\neg \varphi(x, y)$ defines the complement Φ in $A \times B$,

$\neg^\bullet \varphi(x, y)$ defines the strong complement Φ in $A \times B$,

$\varphi(x, y) \vee \psi(x, y)$ defines the union $\Phi \bigcup \Psi$,

$\varphi^\bullet(x, y) \vee \psi^\bullet(x, y)$ defines the strong union $\Phi \bigcup \Psi$,

$\varphi(x, y) \wedge \psi(x, y)$ defines the intersection $\Phi \bigcap \Psi$,

$\varphi^\bullet(x, y) \wedge \psi^\bullet(x, y)$ defines the strong intersection $\Phi \bigcap \Psi$,

$\exists x \varphi(x, y)$ defines the projection where is the projection onto the second coordinate,

$\forall y \varphi(x, y)$ defines $\{x \in A \mid \{x\} \times B \subseteq \Phi\}$,

$\forall^\bullet y \varphi(x, y)$ defines $\{x \in A \mid [\{x\} \times B \subseteq \Phi]^\bullet\}$.

Note that the universal quantifier $\forall^\bullet$ can be handled in terms of the existential quantifier, by means of the easy equivalence $\forall y \varphi(x, y) \Leftrightarrow \neg^\bullet \exists y \neg^\bullet \varphi(x, y)$.

In a similar way we use the $*$-implication sign $\to^\bullet$ as in $\varphi(x, y) \to^\bullet \psi(x, y)$, to abbreviate the formula $(\neg^\bullet \varphi(x, y) \vee \psi^\bullet(x, y)$, and we use the $*$-equivalence symbol $\leftrightarrow^\bullet$, as in $\varphi(x, y) \leftrightarrow^\bullet \psi(x, y)$ to abbreviate $[\varphi(x, y) \to^\bullet \psi(x, y)] \wedge [\psi(x, y) \to^\bullet \varphi(x, y)]$.

Now we define the particular logical formulas that will be used in working with V.

Definition. (Formulas Over V) **Let V be a Universal Set.**

(1) Basic formulas over V consist of the formulas $(s_1, \ldots, s_m) \in A$ and $f(s_1, \ldots, s_m) = (t_1, \ldots, t_n)$, where A is any subset of V^m, f is any function from a subset of V^m into V^n, and each element of the sequence $s_1, \ldots, s_m, t_1, \ldots, t_n$ is either a variable ranging over V or an element of V (with repetitions allowed).

(2) Formulas over V are obtained from basic formulas over V by repeatedly applying the connectives $\vee, \wedge, \neg, \neg^{\bullet}, \rightarrow, \rightarrow^{\bullet}, \leftrightarrow, \leftrightarrow^{\bullet}$ and the quantifiers $\exists, \forall, \forall^{\bullet}$ applied to variables which range over V.

Each formula over V has an obvious interpretation in V (equipped with all relations and functions). If $W \subseteq V$ then every formula over W can be transformed into an equivalent formula over V by relativizing all of its quantifiers to W.

Note that functions appear in formulas over V only through their graphs. This allows us to use partially defined functions, and it is less restrictive than it may seem at first. For example suppose f, g and h are nonparalogical functions from V into itself, and we want to express the condition that h is the composition of f and g. This can be done using the following formula $\forall^{\bullet} x \forall^{\bullet} y[h(x) = y \leftrightarrow^{\bullet} \exists z(g(x) = z \wedge f(z) = y)]$ which is a formula over V. Suppose $< \subseteq V^2$ is an ordering relation on V and we want to express the condition that $[f(x) < g(x)]^{\bullet}$ holds for all elements $x \in V$. This can be done using the following formula over V: $\forall^{\bullet} x \forall^{\bullet} \forall^{\bullet} z[(f(x) = y \wedge g(x) = z) \rightarrow^{\bullet} y < z]$. In this way we see how statements involving the composition of functions and the substitution of functions in predicates can be expressed using formulas over V.

Now consider a fixed paralogical nonstandard extension ${}^{\#}V$ of V. Each formula over V can be given a i.e. nonstandard interpretation in ${}^{\#}V$ and this turns out to be the key to proving things about the nonstandard extension.

This interpretation is made precise by defining for each formula over V its $\#$-transform, which is a formula over ${}^{\bullet}V$. The Transfer Principle which we state below gives the exact relation between the meanings of these two formulas. To help keep the two classes of formulas and their interpretations clearly distinct, we use lower case variables such as $x_1,\ldots,x_m$ to range over V and upper case variables such as $X_1,\ldots,X_m$ to range over V.

Definition. ($\#$-Transform of Formulas over V) Let V be an infinite set and let $\varphi(x_1,\ldots,x_m)$ be a formula over V. Consider any nonstandard extension ${}^{\#}V$ of V. The $\#$**- transform of** $\varphi(x_1,\ldots,x_m)$ is a formula over ${}^{\#}V$ which is denoted by ${}^{\#}\varphi(X_1,\ldots,X_m)$ and is constructed as follows:

(1) Find all of the sets $A \subseteq V^m$ that occur in $\varphi(x_1,\ldots,x_m)$ in basic formulas, and replace each such set by its counterpart ${}^{\#}A$ over ${}^{\#}V$; similarly, replace each function $f: A \rightarrow B$ by ${}^{\#}f$ and replace each element a of V by ${}^{\#}a$ and

(2) replace every variable $x_i, i = 1,\ldots,m$ in $\varphi(x_1,\ldots,x_m)$ including the ones that are used with quantifiers, by a corresponding variable $X_i, i = 1,\ldots,m$ which ranges over ${}^{\#}V$.

Theorem. (Paralogical Transfer Principle) Let V be an universal set and consider a fixed nonstandard extension of V.

(1) Let $\varphi(x_1,...,x_m)$ be a formula over V and let ${}^{\#}\varphi(X_1,...,X_m)$ be its $\#$- transform. Suppose $B \subseteq V^m$ is the set defined by $\varphi(x_1,...,x_m)$:

$$B = \{(x_1,...,x_m) \in V^m \mid \varphi(x_1,...,x_m) \text{ is true in } V\}.$$

Then ${}^{\#}B$ is the set defined by ${}^{\#}\varphi(X_1,...,X_m)$:

$${}^{\#}B = \{(X_1,...,X_m) \in ({}^{\#}V)^m \mid {}^{\#}\varphi(X_1,...,X_m) \text{ is true in } {}^{\#}V\}.$$

(2) Let φ be any sentence over V, and let ${}^{\#}\varphi$ be its $\#$- transform. Then

(2a) φ is true in $V \Leftrightarrow {}^{\#}\varphi^{\bullet}$ is true in ${}^{\#}V$.

(2b) φ is true in $V \Rightarrow {}^{\#}\varphi$ is true in ${}^{\#}V$.

The Paralogical Transfer Principle is a flexible and useful result which expresses nearly everything that one needs to know about paralogical nonstandard extensions. In particular it gives precise meaning to the statement "the paralogical nonstandard extension of V possesses all of the properties that V does" It embodies the content of conditions $(E1)$-$(E9)$ in logical notation and makes it possible to derive useful consequences from those conditions with less effort than would otherwise be true. It is proved by a straightforward argument by induction on the construction of formulas over V. In most expositions of Paralogical Nonstandard Analysis the Transfer Principle appears in the definition of the concept of paralogical nonstandard extension which requires the use of logical notation right at the start. We have turned things around so that the Paralogical Transfer Principle is derived from principles that are mathematically more transparent and we view it as a technical tool for working with paralogical nonstandard extensions. When applied to paralogical ultrapower paralogical nonstandard extensions, the Paralogical Transfer Principle is also we call as the Theorem of Los or the Fundamental Theorem of Paralogical Ultrapowers. We will always assume that the basic set V contains the set R of real numbers. The set ${}^{\#}R$ is equipped with binary functions ${}^{\#}+$ and ${}^{\#}\times$ and with a binary paralogical relation ${}^{\#}<$. Equipped with this additional structure, ${}^{\#}R$ is an paraordered field a real closed paralogical field in fact. These properties can be formulated using logical sentences and the Paralogical Transfer Principle is then used to prove that they hold in ${}^{\#}R$ because they are the $\#$-transforms of sentences that hold in R. For ease of notation we will follow the customary practice of dropping the $\#$ and denoting these as $+$, $\times$, and $<$.

Definition. $x <^{\bullet} y \Leftrightarrow (x < y)^{\bullet}; x, y \in {}^{\#}R$.

Moreover we act as though the embedding of ${}^{\bullet}R$ and R into ${}^{\#}R$ given by the $\#$ mapping is an inclusion. With these conventions, ${}^{\#}R$ is an paraodered, **Dedecind**

complete (V-universal set) **field**, extension of R. Moreover the set $^{\#}Z$ which is the set of “paralogical nonstandard integers” is the domain of a subring of $^{\#}R$ which has the property that for every $r \in^{\#} R$ there is a unique $n \in^{\#} Z$ which satisfies $n \leq r \leq n+1$.

A number x in $^{\#}R$ is called **strong infinitesimal** if $-r <^{\bullet} x <^{\bullet} r$ holds for every standard positive real number $r \in R$. A number x in $^{\#}R$ is called **parainfinitesimal** if $-r <^{\bullet} x <^{\bullet} r$ holds for every positive real number $r \in R$,and $\neg \forall r(-r <^{\bullet} x <^{\bullet} r)$. A number x in $^{\#}R$ is called **infinitesimal**, if x strong infinitesimal or parainfinitesimal. If $x, y \in^{\#} R$ we write $x \approx^{\bullet} y$ when $x-y$ is strong infinitesimal, we write $x \approx^{\#} y$ when $x-y$ is parainfinitesimal, and we write $x \approx y$ when $x-y$ infinitesimal. A number $x \in^{\#} R$ is called **strong finite** if there exists a standard positive real number $r \in R$ such that $-r <^{\bullet} x <^{\bullet} r$. A number $x \in^{\#} R$ is called **parafinite** if there exists a standard positive real number $r \in R$ such that $-r < x < r$ and $\neg \exists r_1 (r_1 \in R)[-r_1 <^{\bullet} x <^{\bullet} r_1]$. A number $x \in^{\#} R$ is called **finite** if x strong finite or parafinite. Otherwise $x \in^{\#} R$ is called **strong infinite**, **parainfinite** or **infinite**. Each finite number $x \in^{\#} R$ determines nonempty set $D(x)$ of the Dedekind cuts in R; therefore there is a unique nonepty set $ST(x)$ of the standard real numbers $r \in R$ such $x \approx^{\#} r \in ST(x)$. This set $ST(x)$ is led the parastandard part of x, ad we write $\underline{st}(x) = \inf\{ST(x)\}$- lower standard part of x, and we write $\overline{st}(x) = \sup\{ST(x)\}$- upper standard part of x. If $\overline{st}(x) = \underline{st}(x) = st(x)$ we call $st(x)$- standard part of x. Evidently for all finite numbers $x, y \in^{\#} R$:

(1) $\underline{st}(x) \leq \overline{st}(x)$,

(2) $x \leq^{\bullet} y \Leftrightarrow [(\underline{st}(x) \leq \underline{st}(y)) \wedge (\overline{st}(x) \leq \overline{st}(y))]$,

(3) $\overline{st}(x+y) = \overline{st}(x) + \overline{st}(y)$,

(4) $\underline{st}(x+y) = \underline{st}(x) + \underline{st}(y)$,

(5) if $\overline{st}(x) > 0, \overline{st}(y) > 0$ then $\overline{st}(x \times y) = \overline{st}(x) \times \overline{st}(y)$,

(6) if $\overline{st}(x) > 0, \overline{st}(y) > 0$ then $\underline{st}(x \times y) = \underline{st}(x) \times \underline{st}(y)$.

Remark. The set of finite numbers in $^{\#}R$ is a convex subring of $^{\#}R$ and the mappings $\overline{st}(\circ)$ and $\underline{st}(\circ)$ is a homomorphisms of ordered rings from this subring onto R. The kernel of $\overline{st}(\circ)$ and $\underline{st}(\circ)$ is obviously the set of infinitesimals in $^{\#}R$. Condition $(E9)$ that $^{\#}V$ is proper, implies that there exist non-zero infinitesimals in $^{\#}R$ and that there also exist infinite numbers in $^{\#}R$ as well. In other words $^{\#}R$ is non-Archimedean. Bat when V- universal set, $^{\#}Z$ satisfier literally all the properties of Z. In this sense, we have literally reconstruction the Archimedean properties inside $^{\#}R$.

One of the key ideas in most applications of classical nonstandard methods is the concept of internal set or function on the nonstandard Robinson’s extension $^{\bullet}V$. This concept is the most effectively developed in PNSA when V- universal set, because in this case V has all classical set theoretical structure so that the sets and functions we are interested in are themselves represented by elements of V, and ${}^{\#}V \subseteq V$. In the most general framework for classical NSA this is handled by taking V to be a superstructure; see Section 6 of [ ] for example. Here we will take a more limited point of view. That is, we consider a subset Ξ of V which contains the mathematical objects we want to investigate and assume that V also contains as elements every subset of Ξ^m $(m \geq 0)$ and every function between such subsets. This allows us to define and to conveniently handle internal subsets of $({}^{\#}\Xi)^m$ and internal functions between such subsets, let $E_m \subseteq \Xi^m \times P(\Xi^m)$ denote the membership relation between m- tuples from Ξ and sets of such m- tuples,

$$E_m = \{(a_1,...,a_m, A) \in \Xi^m \times P(\Xi^m) \mid (a_1,...,a_m) \in A\}.$$

(Here $P(\circ)$ is the power set operation, so $P(\Xi^m)$ is the collection of all subsets of Ξ^m.) According to our assumptions, $E_m \subseteq V \subseteq V^{m+1}$ so that ${}^{\#}E_m$ is a well defined subset of $({}^{\#}V)^{m+1}$; in fact This provides the basis for our definition of internal subsets of $({}^{\#}\Xi)^m$:

Definition. (Internal Subsets and Functions on ${}^{\#}\Xi$) A subset B of $({}^{\#}\Xi)^m$ is internal if there exists $b \in {}^{\#}P(\Xi^m)$ which codes B in the sense that

$$B = \{(a_1,...,a_m) \in ({}^{\#}\Xi)^m \mid (a_1,...,a_m, b) \in {}^{\#}E_m\}.$$

A function $f : A \to B$ where $A \subseteq ({}^{\#}\Xi)^m$ and $B \subseteq ({}^{\#}\Xi)^m$ is internal if the graph of f is internal as a subset of $({}^{\#}\Xi)^{m+n}$.

Note that if $A \subseteq \Xi^m$ or if $f : A \to B$ where $A \subseteq \Xi^m$ and $B \subseteq \Xi^m$, then ${}^{\#}A$ is an internal set and ${}^{\#}f$ is an internal function.

Definition. Let call A and f - external if A and f no internal.

Remark. If V - universal set, ${}^{\#}V \subseteq V$ and then ${}^{\#}A$ is an internal set and ${}^{\#}f$ is an internal function automatically.

The collection of internal sets and functions on ${}^{\#}\Xi$ is closed under a wide variety of mathematical operations, as is provable without great effort from the Transfer Principle. For example the collection of internal sets is closed under Boolean operations where defined, and under coordinate mappings the domain and range of an internal function are internal sets the restriction of an internal function to an internal subset of its domain is again an internal function and so forth. The following principle gives the most general result of this kind :

Theorem. (Internal Definition Principle)

Let $\varphi(x_1,...,x_m, y_1,..., y_n)$ be a formula over V. Let $a_1,...,a_n \in {}^{\#}V$ and let B be the subset of $({}^{\#}\Xi)^m$ defined by ${}^{\#}\varphi(X_1,...,X_m, a_1,...,a_n)$:

$$B = \{(X_1,...,X_m) \in ({}^{\#}\Xi)^m |{}^{\#}\varphi^{\bullet}(X_1,...,X_m,a_1,...,a_n) \text{ is true in } {}^{\#}V\}.$$

Then B is internal.

For example in classical NSA for numbers set N we have: ${}^{\bullet}N \setminus N \notin {}^{\bullet}V$,i.e. set ${}^{\bullet}N \setminus N$ - is external set. But in PNSA we have: ${}^{\#}N \setminus N \in V \subseteq {}^{\#}V$,i.e. ${}^{\#}N \setminus N$ - is internal set.

It is interesting to unravel the definition of internal set and function when ${}^{\#}V$ is constructed using paralogical ultraproducts. What we find out is that the internal subsets of $({}^{\#}\Xi)^m$ in this setting correspond exactly to paralogical ultraproducts of subsets of Ξ^m. Let U be an paralogical ultrafilter on the index set I and suppose ${}^{\#}V$ is the U - ultrapower paralogical nonstandard extension of V. Let B be an internal subset of $({}^{\#}\Xi)^m$ and suppose B is coded by $b \in {}^{\#}P({}^{\#}\Xi)^m$ as in Definition. Then b is the equivalence class under $\sim_U$ of some function $F: I \to P(\Xi^m)$. Let $a_1,...,a_m \in {}^{\#}\Xi^m$ for each $j = 1,...,m$ there is a function $\beta_j : I \to \Xi$ for which $a_j = [\alpha_j]_{\#}$. It is easily seen that

$$(a_1,...,a_m) \in B \Leftrightarrow \{i \in I \,|\, (\alpha_1(i),...,\alpha_m(i)) \in U\}.$$

Therefore the paralogical internal set B can be identified with the paralogical U - ultraproduct of the family of sets $(F(i)\,|\, i \in I)$. This analysis can be reversed to show that every such paralogical ultraproduct gives rise to an paralogical internal subset of $({}^{\#}\Xi)^m$.

V. Analysis on the paralogical numerical line ${}^{\#}R$ and general result

Definition 5.1. A function $f(z){:}{}^{\#}R \to {}^{\#}R$ is said to have a limit $\lim_{z \to {}^{\#}a} f(z) = c$ if, for all $\varepsilon > 0$,($\varepsilon \in {}^{\#}R$), there exists a $\delta > 0$ such that $|f(z) - c| < \varepsilon$ whenever $0 < |z - a| < \delta$,($\delta \in {}^{\#}R$).

Definition 5.2. A function $f(z){:}{}^{\#}R \to {}^{\#}R$ is said to have a strong limit

$$s - \lim_{z \to {}^{\bullet}a} f(z) = c$$

if, for all $\varepsilon \overset{\bullet}{>} 0$, there exists a $\delta \overset{\bullet}{>} 0$ such that $|f(z)-c| <^{\bullet} \varepsilon$ whenever $0 <^{\bullet} |z-a| <^{\bullet} \delta$.

Theorem.5.1. Let $f(z): R \to R$-standard function, then:

(1) $\lim_{z \to a} f(z) = c \Rightarrow \lim_{z \to^{\#} a} {}^{\#}f(z) = {}^{\#}c$,
(2) $\lim_{z \to a} f(z) = c \Leftarrow s - \lim_{z \to^{\bullet} a} {}^{\#}f(z) = {}^{\#}c$.

Definition 5.3. A function $f(z): R \to R$ is said to have a $*$-limit $* - \lim_{z \to a} f(z) = c$,if and only if a function $f(z): {}^{\#}R \to {}^{\#}R$ to have a strong limit $\lim_{z \to^{\bullet} a} {}^{\#}f(z) = {}^{\#}c$.

Definition 5.4. (a) the derivative of a function $f(x): {}^{\#}R \to {}^{\#}R$ with respect to the variable *x* is defined as

$$\frac{d^{\#} f(x)}{d^{\#} x} \equiv f^{\#/}(x) \equiv \lim_{h \to^{\#} 0} \frac{f(x+h) - f(x)}{h},$$

(b) the strong derivative of a function $f(x): {}^{\#}R \to {}^{\#}R$ with respect to the variable *x* is defined

as

$$\frac{d^{\#}_{\bullet} f(x)}{d^{\#}_{\bullet} x} \equiv s - f^{\#/}(x) \equiv s - \lim_{h \to^{\bullet} 0} \frac{f(x+h) - f(x)}{h},$$

(c) a function $f(z): R \to R$ is said to have a $*$-derivate $f^{/\bullet}(x)$ if and only if a function $f(z): {}^{\#}R \to {}^{\#}R$ to have a strong derivate s - $f^{\#/}(x) = {}^{\#}[f^{/\bullet}(x)]$.

Definition 5.5. Let the terms in a series be denoted a_i, let the partial sum be given by

$$S_k = \sum_{i=1}^{k} a_i \ , \qquad k \in {}^{\#}N \ ,$$

and let the k- th sequence of partial sums be given by $S_k, k=1,2,...$. If the sequence of partial sums does not converge to a limit (e.g., it oscillates or approaches $\pm^{\#}\infty$), the series is said to #-diverge, or like usual diverge. A series of terms a_n is said to be absolutely convergent if the series formed by taking the absolute values of the a_n, $\sum_n |a_n|, n\in^{\#}N$ converges.

Definition 5.6. (a) a series $\sum_{n=1}^{\#\infty} u_n(x)$ is uniformly convergent to $S(x)$ for a set $E\subseteq^{\#}R$ of values of x if, for each $\varepsilon>0$, an integer $M\in^{\#}N$ can be found such that

$$\left|S_n(x)-S(x)\right|<\varepsilon$$

for $n>M$ and all $x\in E$, (b) a series $\sum_{n=1}^{\#\infty} u_n(x)$ is strong uniformly convergent to $S(x)$ for a set $E\subseteq^{\#}R$ of values of x if, for each $\varepsilon^{\bullet}>0$, an integer $M\in^{\#}N$ can be found such that

$$\left|S_n(x)-S(x)\right|<^{\bullet}\varepsilon.$$

Theorem.5.2. Let: (1) a standard series $u(x)=\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent to $u(x)$ for a set $E\subseteq R$ of values of x, (2) nonstandard series $U(x)=\sum_{n=1}^{\#\infty} {}^{\#}u_n(x)$ is strong uniformly convergent to $U(x)$ for a set ${}^{\#}E\subseteq^{\#}R$ of values of x. Then ${}^{\#}u(x)=u^{\#}(x)$ for all $x\in^{\#}E$.

Definition.5.7. The Riemann integral for nonstandard function $f(x)={}^{\#}g(x)$,

where

$g(x)$- standard function, we defined in that way:

$$\int_a^b f(x)d^{\#}x={}^{\#}\left(\int_a^b g(x)dx\right).$$

In a general case the Riemann integral defined by taking a limit of a hyper finite Riemann sum

$$\int_a^b f(x)d^{\#}x = \lim_{\max mes[\Delta x_k]\to^{\#}0} \sum_{k=1}^{M} mes[\Delta x_k] f(x_k^{/}),$$

where $a \leq^{\bullet} x \leq^{\bullet} b$ and $x_k^{/}$, are arbitrary points in the intervals $\Delta x_k = [x_k, x_{k+1}]$, $x_k <^{\bullet} x_{k+1}$, $mes[\Delta x_k] = x_{k+1} - x_k$, $[a,b] = \bigcup_{k=1}^{M} \Delta x_k$.

Definition.5.8. (1) The Lebesgue integral for nonstandard function $f(x) = {}^{\#}g(x)$, where $g(x)$ - standard function, we defined in that way:

$$\int_a^b f(x)d^{\#}\mu(x) = {}^{\#}\left(\int_a^b g(x)d\mu(x)\right),$$

where $\int_a^b g(x)d\mu(x)$ - the standard Lebesgue integral.

(2) The Lebesgue measure for nonstandard set ${}^{\#}E \subset {}^{\#}R$, $E \subset R$ we defined in that way:

$$\mu^{\#}\left({}^{\#}E\right) = {}^{\#}\mu\left({}^{\#}E\right)$$

where $\mu(\circ)$ - the standard Lebesgue measure.

Theorem.5.3. The hyper finite series may be L-integrated term by term

$$\int_a^b \left(\sum_{n=1}^{M} u_n(x)\right) d^{\#}\mu(x) = \sum_{n=1}^{M} \int_a^b u_n(x) d^{\#}\mu(x)\,, M \in {}^{\#}N \setminus N\,.$$

Definition.5.8. ${}^{\#}\exp(inx) \overset{\Delta}{=} \exp_{\#}(inx)$, ${}^{\#}\pi = \pi_{\#}$.

Theorem.5.4. Let a standard series $\sum_{n=1}^{\infty} a_n^2 = \infty$, $\psi(n): N \to N$, and $\sum_{k=1}^{{}^{\#}\infty} {}^{\#}a_k^2 =^{\bullet} ({}^{\#}\infty)$.

Then hyper finite sum $\sum_{k=1}^{M} {}^{\#}a_k \exp_{\#}(i\pi_{\#}\, {}^{\#}\psi(k))$, $M \in {}^{\#}N \setminus N$ is nonfinite ${}^{\#}\mu$-almost everywhere.

Proof. Let $\Theta_M(x) = \sum_{k=1}^{M} {}^{\#}a_k \exp_{\#}(i\pi_{\#}\, {}^{\#}\psi(k))$ is finite ${}^{\#}\mu$-almost everywhere. Then

$\int_{-\pi_{\#}}^{\pi_{\#}} \Theta_M^2(x) d^{\#}\mu(x) = \sum_{k=1}^{M} {}^{\#}a_k^2$ is finite or parafinite, but no strong infinite for all

$M \in {}^{\#}N \setminus N$. But then $\sum_{n=1}^{\infty} a_n^2 < \infty$, that contradicts to the theorem conditions.

Theorem.5.5. Let $\omega_1(n): N \to N$, $\omega_2(n) \to R$, $\sum_{n=1}^{\infty} \left|\omega_2^{-1}(n)\right| < \infty$, $\sum_{n=1}^{\infty} (\omega_1(n)/\omega_2(n))^2 = \infty$, $\sum_{k=1}^{{}^{\#}\infty} \left({}^{\#}\omega_1(k) / {}^{\#}\omega_2(k)\right)^2 =^{\bullet} ({}^{\#}\infty)$. Then function

$$\Im(x) = \sum_{n=1}^{\infty} \frac{\exp(i \cdot \pi \cdot \omega_1(n) \cdot x)}{\omega_2(n)}\,,$$

does not possess by the almost everywhere finite derivative function.

Proof. Let's suppose that derivative function $\Im^{/\bullet}(x)$ is finite almost everywhere. Then function

$$\Im_l(x)=\sum_{n=l}^{\infty}\frac{\exp(i\cdot\pi\cdot\omega_1(n)\cdot x)}{\omega_2(n)}$$

evidently, possesses by the almost everywhere finite derivative function $\Im_l^{/\bullet}(x)$ for any $l\in N$. Using the paralogical transfer principle, we see that non-standard function

$$^{\#}\left(\Im_L^{/\bullet}(x)\right)=\left({}^{\#}\Im_L(x)\right)^{\#/}$$

will be ${}^{\#}\mu$-almost everywhere finite for all numbers $L\in{}^{\#}N$. According to the definition of function $\Im(x)$, for all hyper-finite $M\in{}^{\#}N\setminus N$ is truly the identity

$${}^{\#}\Im(x)=\sum_{n=1}^{M}\frac{\exp_{\#}(i\cdot\pi_{\#}\cdot{}^{\#}\omega_1(n)\cdot x)}{{}^{\#}\omega_2(n)}+{}^{\#}\Im_{M+1}(x),$$

from which, in a result of differentiation, we shall obtain the following identity

$$\left({}^{\#}\Im(x)\right)^{\#/}=i\pi_{\#}\sum_{n=1}^{M}\frac{{}^{\#}\omega_1(n)}{{}^{\#}\omega_2(n)}\exp(i\cdot\pi_{\#}\cdot{}^{\#}\omega_1(n)\cdot x)+\left({}^{\#}\Im_{M+1}(x)\right)^{\#/}.$$

From this identity, by virtue of theorem (5.4), follows that derivative function $\left({}^{\#}\Im(x)\right)^{\#/}$ cannot be #-almost everywhere finite. Obtained contradiction proves the theorem.

Theorem.5.6. Let $0<\beta\leq 3.5$. Then Gerver's function

$$G_{3,\beta}(x)=\sum_{n=1}^{\infty}n^{-\beta}\exp(ixn^3)$$

does not possess by the almost everywhere finite derivative function.

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