Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series. Part II.

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Abstract.

L. Carleson's celebrated theorem of 1965 [1] asserts the pointwise convergence of the partial Fourier sums of square integrable functions. The Fourier transform has a formulation on each of the Euclidean groups R, Z and T. Carleson's original proof worked on T. Fefferman's proof translates very easily to R. M´at´e [2] extended Carleson's proof to Z. Each of the statements of the theorem can be stated in terms of a maximal Fourier multiplier theorem [5]. Inequalities for such operators can be transferred between these three Euclidean groups, and was done P. Auscher and M.J. Carro [3]. But L. Carleson's original proof and another proofs very long and very complicated. We give a very short and very "simple" proof of this fact. Our proof uses PNSA technique only, developed in part I, and does not uses complicated technical formations unavoidable by the using of purely standard approach to the present problems. In contradiction to Carleson's method, which is based on profound properties of trigonometric series, the proposed approach is quite general and allows to research a wide class of analogous problems for the general orthogonal series. Lets suppose that there are general orthogonal series in space $L_2(\Omega), \Omega \subseteq R^d, d = 1,2...$

$$\sum_{n=0}^{\infty} c_n f_n(x), (c_n) \in l_2, f_n \in L_2(\Omega). \tag{1.1}$$

We shall say that sequence $\{f_n(x)\}_{n=1}^{\infty}$ or series (1.1) possesses by LC-property if series (1.1) converges a.e. It is good known that a general orthogonal row does not possesses by LC-property. A problem of possession by LC-property is still open for many orthogonal series, as example for the series by Jakoby's polynomial. In the present work we shall obtain a general sufficient condition guaranteeing the presence of LC-property in series (1.1). We shall say that the general orthogonal series (1.1) in space L_2 is strong orthogonal series, if the following condition is executed:

$$^{\#}\!\left(\int^{\#} f_i(x)^{\#} f_j(x) d^{\#}\mu(x)\right) =^{\bullet} \delta_{ij}. \tag{1.2}$$

Where $\delta_{ij} =^{\bullet} 1$, if $i = j$, and $\delta_{ij} =^{\bullet} 0$, if $i \neq^{\bullet} j$ $i, j \in {}^{\#}N$. In other words it is said here that orthogonality persists in strong (non-paralogical) sense after using of #-mapping. The main result is that strong orthogonality plus condition ${}^{\#}(c_n) \in^{\bullet} {}^{\#}l_2$ pulls the LC-property.

During last time L. Carleson's result [1] was generalized in various directions [4], [5]. For non-orthogonal series a special analogue of L. Carleson's celebrated theorem was obtained in work [4]. A Hilbert space of Dirichlet series is obtained by considering the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ that satisfy. These series converge in the half plane $\operatorname{Re} s > 1/2$ and define a functions that are locally L_2 on the boundary $\operatorname{Re} s = 1/2$. An analog of L.Carleson's celebrated convergence theorem is obtained [2]: each such Dirichlet series converges almost everywhere on the critical line $\operatorname{Re} s = 1/2$. To each Dirichlet series of the above type corresponds a "trigonometric" series $\sum_{n=1}^{\infty} a_n \chi(n)$, where χ is a multiplicative character from the positive integers to the unit circle. The space of characters is naturally identified with the infinite-dimensional torus T^{∞}, where each dimension comes from a prime number. The secondanalog of Carleson's theorem reads: The above "trigonometric" series converges for almost all characters χ. At the same time classical technique was found not enough effective for the study of a problem of LC-property presence in a case of multidimensional Fourier series. Carleson's results are trivially transferred on N-multiple Fourier series, for the case of convergence by cubes, but in the case of arbitral convergence Carleson methods does not works and, in general, the problem for N-multiple Fourier series is still open. Particularly, this problem is open for the case of orbicular amounts $E_M[f(x)]$

$$E_M[f(x)] = (2\pi)^N \sum_{|n|^2 \leq M} f_n \exp(inx), n \in Z^N \tag{1.3}$$

of Fourier series of function $f \in L_2(T^N)$. We shall demonstrate that in the case of orbicular amounts $E_M[f(x)]$ LC-property is right.

In 1971 R. Cooke proved Cantor-Lebesque theorem in two dimentions [6]: if

$$\sum_{|n|^2 = k} c_n \exp(inx) \to 0, k \to \infty \tag{1.4}$$

a.e. on T^2, then

$$\sum_{|n|^2=k} |c_n|^2 \to 0, k \to \infty . \qquad (1.5)$$

As it good known, this Cooke‘s result took the part of the last lacking element in solution of the old problem of representation of two variable quantities function by trigonometric series. Unfortunately, Cooke’s proof is essentially based on specific particular qualities of two-dimensional case and for $N > 2$ it could not be principally adapted. We shall demonstrate that if for $N > 2$ the following condition is executed:

$$\sum_{|n|^2=k} {}^{\#}c_n \exp(inx) \approx^{\bullet} 0, k \in {}^{\#}N \setminus N, n \in {}^{\#}Z^N , \qquad (1.6)$$

#-a.e. on ${}^{\#}\mathrm{T}^N$ then

$$\sum_{|n|^2=k} \left|{}^{\#}c_n\right|^2 \approx 0 ,$$

and by that (1.5) is true.

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