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New analytical approach for transition to slow 3-D turbulence

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New analytical approach for transition to slow 3-D turbulence

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Abstract. Analytical non-perturbative study of the three-dimensional nonlinear stochastic partial differential equation with additive thermal noise analogous to that proposed by Nikolaevskii V.N. to describe longitudinal seismic waves is presented. The equation has a threshold of short-wave instability and symmetry providing long wave dynamics. New mechanism of quantum chaos generating in nonlinear dynamical systems with infinite number of degrees of freedom is proposed. The hypothesis says that physical turbulence could be identified with quantum chaos of considered type. It is shown that the additive thermal noise destabilizes dramatically the ground state of the Nikolaevskii system causing it to make a direct transition from a spatially uniform to a turbulent state.

1. Introduction

In the presented work a non-perturbative analytical approach to the studying of problem of quantum chaos in dynamical systems with infinite number of degrees of freedom is proposed. Statistical descriptions of dynamical chaos and investigations of noise effects on chaotic regimes are studied. Proposed approach also allows estimate the influence of additive (thermal) fluctuations on the formation processes of developed turbulence modes in essentially nonlinear processes like electro-convection and others. A principal role of the influence of thermal fluctuations on the dynamics of some types of dissipative systems in the approximate environs of derivation rapid of a short-wave instability was ascertained. Important physical results following from Theorem 2.1 are numerically illustrated by example of 1D stochastic model system:

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$$\frac{\partial u_\eta(x,t,\varepsilon)}{\partial t} + \Delta[\varepsilon - (1+\Delta)^2]u_\eta(x,t,\varepsilon) + \delta \frac{\partial u_\eta(x,t,\varepsilon)}{\partial x_1} u_\eta(x,t,\varepsilon) + f(x,t) - \sqrt{\eta} w(x,t) = 0,$$
$$x \in \mathbb{R}, u_\eta(x,0,\varepsilon) = 0, w(x,t) = \frac{\partial^2 W(x,t)}{\partial x \partial t}, \eta \ll 1, 0 < \delta \quad (1.1)$$

Systematic study of a different type of chaos at onset ''soft-mode turbulence'' based on numerical integration of the simplest 1D Nikolaevskii model has been executed and analyzed by many authors [1]-[3].

2. Main Theoretical Results

We study the stochastic partial differential equation (1.1) in the sense of Colombeau generalized functions [4].

Theorem 2.1. [5]-[6]. (**Strong Large Deviation Principle for Colombeau-Ito's SPDE**)

(**I**) Let $(u_\epsilon(x,t,\varepsilon,\eta,\omega))_\epsilon, \epsilon \in (0,1]$ be a solution of the Colombeau-Ito's SPDE [5]:

$$\frac{\partial (u_\epsilon(x,t,\varepsilon,\eta,\omega))_\epsilon}{\partial t} + \Delta[\varepsilon - (1+\Delta)^2](u_\epsilon(x,t,\varepsilon,\eta,\omega))_\epsilon$$
$$+ \left(\left(F_\epsilon(u_\epsilon(x,t,\varepsilon,\eta,\omega))\right)_\epsilon\right) \sum_{i=1}^{r} \delta_i \left(F_\epsilon\left(\frac{\partial u_\epsilon(x,t,\varepsilon,\eta,\omega)}{\partial x_i}\right)\right)_\epsilon + (f_\epsilon(x,t))_\epsilon$$
$$-\sqrt{\eta}(w_\epsilon(x,t,\omega))_\epsilon = 0, \quad (2.1)$$
$$x \in \mathbb{R}^r, u_\epsilon(x,t,\varepsilon,\eta,\omega) \equiv 0, \delta_j > 0, j = 1, \dots, r. \quad (2.2)$$

Here: (1)$(F_\epsilon(z))_\epsilon \in \mathcal{G}(\mathbb{R}), \mathcal{G}(\mathbb{R})$ is the Colombeau algebra of Colombeau generalized functions and $F_0(z) = z$.

(2) $(f_\epsilon(x,t))_\epsilon \in \mathcal{G}(\mathbb{R}^{r+1})$.

(3)$w_\epsilon(x,t)$ is a smoothed with respect to $\mathbb{R}^r$white noise.

(**II**) Assume that Colombeau-Ito's SDE (2.1)-(2.2) is a strongly dissipative [5].

(**III**) Let $\Re(x,t,\varepsilon,\lambda)$ be the solutions of the linear PDE:

$$\frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial t} + \Delta[\varepsilon - (1+\Delta)^2]\Re(x,t,\varepsilon,\lambda) + \lambda \sum_{i=1}^{r} \delta_i \frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial x_i} - f(x,t) = 0, \lambda \in \mathbb{R}, \quad (2.3)$$
$$\Re(x,0,\varepsilon,\lambda) = -\lambda. \quad (2.4)$$

Then

$$\operatorname{liminf}_{\epsilon \to 0} \mathbf{E}[|u_\epsilon(x,t,\varepsilon,\eta,\omega) - \lambda|^2] \le \Re(x,t,\varepsilon,\lambda).$$

Definition 2.1. (**Differential Master Equation**) The linear PDE [5]-[6]

$$\frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial t} + \Delta[\varepsilon - (1+\Delta)^2]\Re(x,t,\varepsilon,\lambda) + \lambda \sum_{i=1}^{r} \delta_i \frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial x_i} - f(x,t) = 0, \lambda \in \mathbb{R}, \quad (2.5)$$

$$\Re(x,0,\varepsilon,\lambda) = -\lambda \quad (2.6)$$

Definition 2.2. (**Transcendental Master Equation**) The transcendental equation [5]-[6]

$$\Re\big(x,t,\varepsilon,\lambda(x,t,\varepsilon)\big)=0, \tag{2.7}$$

we will call as the transcendental master equation.
We note that concrete structure of the Nikolaevskii chaos is determined by the solution $\lambda(x,t,\varepsilon)$ variety by transcendental master equation (2.7). Master equation (2.7) is only determined by the way of some many-valued function $\lambda(x,t,\varepsilon)$ which is the main constructive object, determining the characteristics of quantum chaos in the corresponding model of Euclidian quantum field theory.

3. Criterion of the existence quantum chaos in Euclidian quantum N-model

Definition 3.1. Let $u_\eta(x,t,\varepsilon,\omega)$ be the solution of the equation (2.1). Assume that for almost all points $(x,t)\in\mathbb{R}^r\times\mathbb{R}_+$ (in the sense of Lebesgue measure on $\mathbb{R}^r\times\mathbb{R}_+$), there exist a function $u(x,t)$ such that $\lim_{\eta\to 0}\mathbf{E}\left[\left(u_\eta(x,t,\varepsilon,\omega)-u(x,t)\right)^2\right]=0$.Then we will say that a function $u(x,t)$ is a quasi-determined solution (QD-solution of the equation (2.1). Assume that there exist a set $\mathfrak{H}\subset\mathbb{R}^r\times\mathbb{R}_+$ that is positive Lebesgue measure, i.e. $\mu_L(\mathfrak{H})>0$ and $\forall(x,t)\{(x,t)\in\mathfrak{H}\to\neg\exists\lim_{\eta\to 0}\mathbf{E}[u_\eta^2(x,t,\varepsilon,\omega)]\}$,i.e., $(x,t)\in\mathfrak{H}$ imply that the limit: $\lim_{\eta\to 0}\mathbf{E}[u_\eta^2(x,t,\varepsilon,\omega)]$ does not exist. Then we will say that Euclidian quantum N-model has the quasi-determined Euclidian quantum chaos (**QD-**quantum chaos). For each point $(x,t)\in\mathbb{R}^r\times\mathbb{R}_+$ we define a set $\{\widetilde{\Re}(x,t,\varepsilon)\}\subset\mathbb{R}$ by the condition:

$$\forall\lambda\left[\lambda\in\{\widetilde{\Re}(x,t,\varepsilon)\}\Leftrightarrow\Re(x,t,\varepsilon,\lambda)=0\right].$$

Definition 3.2. Assume that Euclidian quantum N-model (2.1) has the Euclidian **QD-**quantum chaos. For each point $(x,t)\in\mathbb{R}^r\times\mathbb{R}_+$ we define a set-valued function $\widetilde{\Re}(x,t):\mathbb{R}^r\times\mathbb{R}_+\to 2^{\mathbb{R}}$ by the condition: $\widetilde{\Re}(x,t,\varepsilon)=\{\widetilde{\Re}(x,t,\varepsilon)\}$.We will say that the set-valued function $\widetilde{\Re}(x,t,\varepsilon)$ is a quasi-determined chaotic solution (**QD-**chaotic solution) of the quantum N-model.

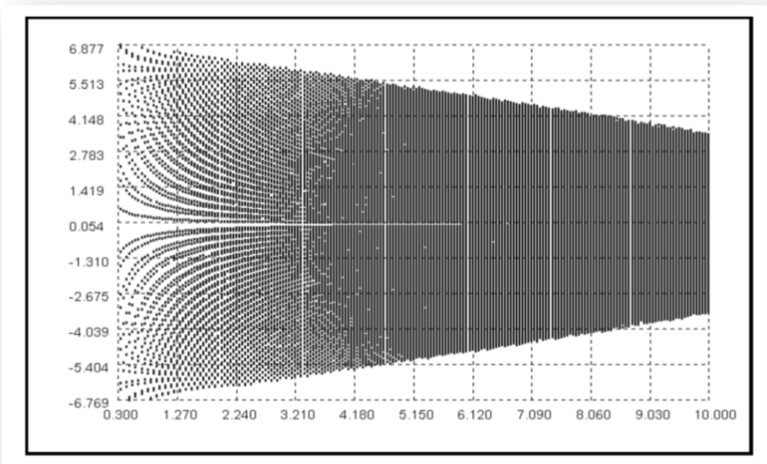

Figure 1. Evolution of **QD-**chaotic solution $\widetilde{\Re}(x,t,\varepsilon)$ in time $t\in[0,10]$ at point $x=3$.
$t\in[0,10],\varepsilon=-10^{-2},\sigma=10^3,p=1.1.$

Theorem 3.1. Assume that $f(x,t) = \sigma \sin(p \cdot x)$. Then for all values of parameters $r, \varepsilon, \sigma, \delta_j, j = 1, \dots, r$ such that $r \in \mathbb{N}, \delta_j \in \mathbb{R}_+, j = 1, \dots, r, \varepsilon \in [-1,1], p \in \mathbb{R}^r, \sigma \neq 0$, quantum N-model (2.1) has the **QD-**chaotic solutions.

Definition 3.3. For each point $(x,t) \in \mathbb{R}^r \times \mathbb{R}_+$ we define the functions such that:
(i)$u_+(x,t,\varepsilon) = \text{limsup}_{\eta \to 0} \mathbf{E}[u_\eta(x,t,\varepsilon,\omega)]$, (ii)$u_-(x,t,\varepsilon) = \text{liminf}_{\eta \to 0} \mathbf{E}[u_\eta(x,t,\varepsilon,\omega)]$,
(iii)$u_w(x,t,\varepsilon) = u_+(x,t,\varepsilon,\omega) - u_-(x,t,\varepsilon,\omega)$. (iv)Function $u_+(x,t,\varepsilon)$ is called upper bound of the **QD-**quantum chaos at point (x,t), (v) function $u_-(x,t,\varepsilon)$ is called lower bound of the **QD-**quantum chaos at point (x,t), (vi) function $u_w(x,t,\varepsilon)$ is called width of the **QD-**quantum chaos at point (x,t).

Definition 3.4. For each point $(x,t) \in \mathbb{R}^r \times \mathbb{R}_+$ we define the functions such that:

$$\text{(i)}\widetilde{\Re}_+(x,t,\varepsilon) = \sup\{\widetilde{\Re}(x,t,\varepsilon)\}, \text{(ii)}\widetilde{\Re}_-(x,t,\varepsilon) = \inf\{\widetilde{\Re}(x,t,\varepsilon)\},$$

$$\text{(iii)}\widetilde{\Re}_w(x,t,\varepsilon) = \widetilde{\Re}_+(x,t,\varepsilon) - \widetilde{\Re}_-(x,t,\varepsilon).$$

Theorem 3.2. (Criterion of QD-quantum chaos in Euclidian quantum N-model) Assume that $\mu_L\{(x,t) | \widetilde{\Re}_w(x,t,\varepsilon) > 0\} > 0$. Then Euclidian quantum N-model has **QD-**quantum chaos.

4. Quasi-determined quantum chaos and physical turbulence nature

The physical nature of quasi-determined chaos is simple and mathematically is associated with discontinuously of the trajectories of the stochastic process $u_\eta(x,t,\varepsilon,\omega)$ on parameter η. In order to obtain the characteristics of this turbulence we have used some appropriate functions [6].

Definition 4.1. The normalized local auto-correlation function is defined by the formula

$$\Phi_{\text{n}}(x,\tau) = \frac{\Phi(x,\tau)}{\Phi(\text{x},0)}. \tag{4.1}$$

Now let us consider 1D Euclidian quantum N-model corresponding to the classical dynamics.
Corresponding differential master equations (2.5)-(2.6) are

$$\frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial t} + \Delta[\varepsilon - (1+\Delta)^2]\Re(x,t,\varepsilon,\lambda) + \lambda\delta \frac{\partial \Re(x,t,\varepsilon,\lambda)}{\partial x} - \sigma \sin(px) = 0, \tag{4.2}$$

$$\Re(x,0,\varepsilon,\lambda) = -\lambda. \tag{4.3}$$

Corresponding transcendental master equations (2.7) are

$$\frac{\{\cos(p \cdot x) - \exp[t \cdot \chi(p)] \cos[p(x - \lambda \cdot \delta \cdot t)]\} \cdot \lambda \cdot \delta \cdot p}{\chi^2(p) + \lambda^2 \cdot \delta^2 \cdot p^2} + \frac{\{\sin(p \cdot x) - \exp[t \cdot \chi(p)] \sin[p(x - \lambda \cdot \delta \cdot t)]\} \cdot \chi(p)}{\chi^2(p) + \lambda^2 \cdot \delta^2 \cdot p^2} + \frac{\lambda}{\sigma} = 0, \tag{4.4}$$

$$\chi(p) = p^2[\varepsilon - (p^2 - 1)^2]. \tag{4.5}$$

We assume now that $\chi(p) = 0$. The result of calculation of the corresponding function $\widetilde{\Re}(x,t,\varepsilon)$ using master equation (4.5) is presented in figure 1. Let us calculate now the corresponding normalized local auto-correlation function $\Phi_{\text{n}}(x,\tau)$. The result of calculation is presented in figure 2.

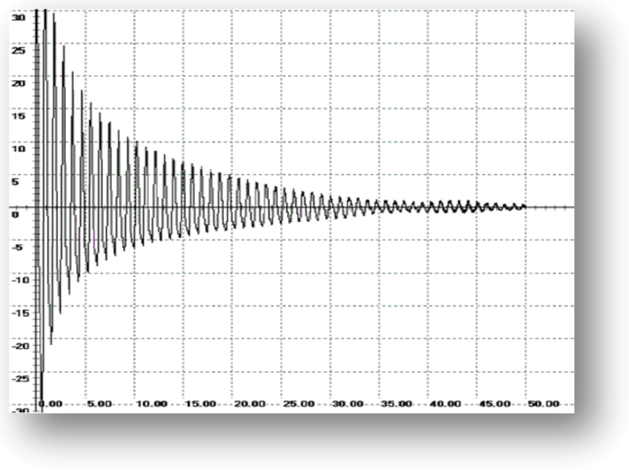

Figure 2. Normalized local auto-correlation function $\Phi_{\mathrm{n}}(1,\tau), \tau \in [0,50], \varepsilon = 10^{-7}, \sigma = 10^{2}, \delta = 1, p = 1.$

5. Conclusion

A non-perturbative analytical approach to the studying of problem of quantum chaos in dynamical systems with infinite number of degrees of freedom is proposed and developed successfully. It is shown that the additive thermal noise dramatically destabilizes the ground state of the system thus causing it to make a direct transition from a spatially uniform to a turbulent state.

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