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New exact quasi-classical asymptotic beyond WKB approximation and beyond Maslov formal expansion

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Abstract. New exact quasi-classical asymptotic of solutions to the n-dimensional Schrodinger equation beyond WKB-theory and beyond Maslov's canonical operator theory is presented. Quantum jump nature is considered successfully. We pointed out that an explanation of quantum jumps can be found to result from Colombeau solutions of the Schrödinger equation alone without additional postulates. Such jumps, does not depend on any single measurement of particle position $q(t)$ at instant t and completely obtained only from Schrodinger equation, without reference to any phenomenological master-equation of Lindblad form.

1.Introduction

A number of experiments on trapped single ions or atoms have been performed in recent years [1]-[5]. Monitoring the intensity of scattered laser light off of such systems has shown abrupt changes that have been cited as evidence of "quantum jumps" between states of the scattered ion or atom. The existence of such jumps was required by Bohr in his theory of the atom. Bohr's quantum jumps between atomic states [6] were the first form of quantum dynamics to be postulated. He assumed that an atom remained in an atomic eigenstate until it made an instantaneous jump to another state with the emission or absorption of a photon. Since these jumps do not appear to occur in solutions of the

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Schrodinger equation, something similar to Bohr's idea has been added as an extra postulate in modern quantum mechanics. Stochastic quantum jump equations [7] were introduced as a tool for simulating the dynamics of a dissipative system with a large Hilbert space and their links with quantum measurement. The question arises whether an explanation of these jumps can be found to result from a Colombeau solution $(\Psi_\varepsilon(x,t;\hbar))_\varepsilon$ of the Schrödinger equation [8] alone without additional postulates. We found exact quasi-classical asymptotic of the quantum averages with position variable with well localized initial data. Note that an axiom of quantum measurement is: if the particle is in some state $|\Psi_{t,\varepsilon}\rangle$ that the probability $P(\boldsymbol{x},\delta\boldsymbol{x}), \boldsymbol{x}\in\mathbb{R}^d$ of getting a result $\boldsymbol{x}$ at instant t with an accuracy of $\|\delta\boldsymbol{x}\|\ll 1$ will be given by

$$P_t(\boldsymbol{x},\delta\boldsymbol{x},\boldsymbol{x}_0;\hbar,\varepsilon)=\int_{\|x-x'\|\le\|\delta x\|}\left|\langle\boldsymbol{x}'|\Psi_{t,\varepsilon}\rangle\right|^2 d\boldsymbol{x}'=\int_{\|x-x'\|\le\|\delta x\|}|\Psi_\varepsilon(\boldsymbol{x}',t,\boldsymbol{x}_0;\hbar)|^2 d\boldsymbol{x}'. \quad (1.1)$$

We rewrite now Eq. (1.1) in the form

$$P_t(\boldsymbol{x},\delta\boldsymbol{x},\boldsymbol{x}_0;\hbar,\varepsilon)=\frac{1}{(2\pi)^{d/2}\|\delta x\|^d}\int_{\mathbb{R}^d}|\Psi_\varepsilon(\boldsymbol{x}',t,\boldsymbol{x}_0;\hbar)|^2\exp\left[-\frac{\|x-x'\|^2}{\|\delta x\|^2}\right]d\boldsymbol{x}'. \quad (1.2)$$

Definition 1.1. [8]. We define well localized Colombeau limiting quantum trajectories $\boldsymbol{q}(t)=\boldsymbol{q}(t,\boldsymbol{x}_0)=\{q_1(t,\boldsymbol{x}_0),\dots,q_d(t,\boldsymbol{x}_0)\}\in\mathbb{R}^d$, $\boldsymbol{q}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0, t\in[0,T]$ by $\left(\underline{\lim}_{\|\delta\boldsymbol{q}(t)\|\to 0}\underline{\lim}_{\hbar\to 0}P_t(\boldsymbol{q}(t),\delta\boldsymbol{q}(t),\boldsymbol{x}_0;\hbar,\varepsilon)\right)_\varepsilon=1$ and well localized limiting quantum trajectories $\boldsymbol{q}(t)=\boldsymbol{q}(t,x_0)=\{q_1(t,x_0),\dots,q_d(t,x_0)\}\in\mathbb{R}^d$, $\boldsymbol{q}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0, t\in[0,T]$ by

$$\lim_{\varepsilon\to 0}\lim_{\|\delta\boldsymbol{q}(t)\|\to 0}\lim_{\hbar\to 0}P_t(\boldsymbol{q}(t),\delta\boldsymbol{q}(t),\boldsymbol{x}_0;\hbar,\varepsilon)=1. \quad (1.3)$$

2.Colombeau solutions of the Schrödinger equation and corresponding path integral representation

Let **H** be a complex infinite dimensional separable Hilbert space, with inner product <·,·> and norm |·|.
Let us consider Schrödinger equation:

$$-i\hbar\left(\frac{\partial\Psi(t)}{\partial t}\right)+\widehat{H}(t)\Psi(t)=0, \Psi(0)=\Psi_0(x),\ H(t)=-\left(\frac{\hbar^2}{2m}\right)\Delta+V(x,t). \quad (2.1)$$

Here operator $H(t):\mathbb{R}\times\mathbf{H}\to\mathbf{H}$ is essentially self-adjoint, $\widehat{H}(t)$ is the closure of $H(t)$.

Theorem 2.1. [9]. Assume that: (1) $\Psi_0(x)\in L_2(\mathbb{R}^d)$,(2)$V(x,t)$ is continuous and $\sup_{x\in\mathbb{R}^d,t\in[0,T]}|V(x,t)|)<+\infty$. Then corresponding solution of the Schrödinger equation (2.1) exists and can be represented via formula

$$\Psi(t,x)=\lim_{n\to\infty}\left(\frac{nm}{4\pi it\hbar}\right)^{d(n+1)/2}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}dx_0dx_1\dots dx_n\Psi_0(x_0)\exp\left[\frac{i}{\hbar}S(x_0,x_1,\dots,x_n,x_{n+1};t)\right], \quad (2.2)$$

where we have set $x_{n+1}=x$ and $S(x_0,x_1,\dots,x_n,x_{n+1};t)=\sum_{i=1}^n\left[\frac{m}{4}\frac{|x_{i+1}-x_i|^2}{(t/n)^2}-\mathrm{V}(x_{i+1},t_i)\right]$, where $t_i=\frac{it}{n}$. Let $q_n(t)$ be a trajectory; that is, a function from $[0,t]$ to $\mathbb{R}^d$ with $q_n(0)=x_0$ and set $q_n(t_i)=x_i, i=1,\dots,n+1$.
Trotter and Kato well known classical results give a precise meaning to the Feynman integral when the potential $V(x,t)$ is sufficiently regular [9]. However if potential $V(x,t)$ is a non-regular this is well known problem to represent solution of the Schrödinger equation (2.1)-(2.2) via formula (2.3), see [9]. We avoided this difficulty using contemporary Colombeau framework. Using replacement $x_i\to\frac{x_i}{1+\varepsilon^{2k}|x|^{2k}}, \varepsilon\in(0,1], k\ge 1$ we obtain from potential $V(x,t)$ regularized potential $V_\varepsilon(x,t), \varepsilon\in(0,1]$, such that $V_{\varepsilon=0}(x,t)=V(x,t)$ and
(i) $\left(V_\varepsilon(x,t)\right)_\varepsilon\in G\left(\mathbb{R}^d\right)$, (ii) $\sup_{x\in\mathbb{R}^d,t\in[0,T]}|V_\varepsilon(x,t)|)<+\infty, \varepsilon\in(0,1]$.

Here $G(\mathbb{R}^d)$ is Colombeau algebra of Colombeau generalized functions [8]. Finally we obtain regularized Schrödinger equation of Colombeau form [8]

$$-i\hbar\left(\frac{\partial\Psi_\varepsilon(t)}{\partial t}\right)_\varepsilon + \left(\widehat{H}_\varepsilon(t)\Psi_\varepsilon(t)\right)_\varepsilon = 0, \left(\Psi_\varepsilon(0)\right)_\varepsilon = \Psi_0(x), H_\varepsilon(t) = -\left(\frac{\hbar^2}{2m}\right)\Delta + V_\varepsilon(x,t). \quad (2.3)$$

Using the inequality Theorem 2.1 asserts again that corresponding solution of the Schrödinger equation exists and can be represented via formula [8]:

$$\left(\Psi_\varepsilon(t,x)\right)_\varepsilon = \left(\lim_{n\to\infty}\left(\frac{nm}{4\pi it\hbar}\right)^{d(n+1)/2}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} dx_0 dx_1 \dots dx_n \Psi_0(x_0)\exp\left[\frac{i}{\hbar}S_\varepsilon(x_0,x_1,\dots,x_n,x_{n+1};t)\right]\right)_\varepsilon, \quad (2.4)$$

where we have set $x_{n+1} = x$ and $S_\varepsilon(x_0,x_1,\dots,x_n,x_{n+1};t) = \sum_{i=1}^{n}\left[\frac{m}{4}\frac{|x_{i+1}-x_i|^2}{(t/n)^2} - V_\varepsilon(x_{i+1},t_i)\right]$,

where we have set $t_i = \frac{it}{n}$.

3.Exact quasi-classical asymptotic beyond Maslov canonical operator

Theorem 3.1. Let us consider Cauchy problem with initial data $\Psi_0(x)$ is given via formula

$$\Psi_0(x) = \frac{\eta^{d/4}}{(2\pi)^{d/4}\hbar^{d/4}}\exp\left[-\frac{\eta(x-x_0)^2}{2\hbar}\right], \text{ where } 0 < \hbar \ll \eta \ll 1 \text{ and } x^2 = \langle x,x\rangle.$$

(1) We assume now that: (i) $\left(V_\varepsilon(x,t)\right)_\varepsilon \in G(\mathbb{R}^d)$, (ii) $V_{\varepsilon=0}(x,t) = V(x,t)\colon \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}$ and (iii) $\forall t \in \mathbb{R}_+$ function $V(x,t)$ is a polynomial on variable $x = (x_1,\dots,x_d)$, i.e.

$$V(x,t) = \sum_{\|\alpha\|\le m} g_\alpha(t)x^\alpha, \alpha = (i_1,\dots,i_d), x^\alpha = x_1^{i_1}\times\dots\times x_d^{i_d}, \|\alpha\| = \sum_{r=1}^{d} i_r.$$

(2) Let $u(\tau,t,\lambda,x,y) = \left(u_1(\tau,t,\lambda,x,y),\dots,u_d(\tau,t,\lambda,x,y)\right)$ be the solution of the boundary problem

$$\frac{\partial^2 u^{\mathrm{T}}(\tau,t,\lambda,x,y)}{\partial\tau^2} = \mathrm{Hess}[V(\lambda,\tau)]u^{\mathrm{T}}(\tau,t,\lambda,x,y) + [V'(\lambda,\tau)]^{\mathrm{T}}, u(0,t,\lambda,x,y) = y, u(t,t,\lambda,x,y) = x. \quad (3.1)$$

Here $\lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d, u^{\mathrm{T}}(\tau,t,\lambda,x,y) = \left(u_1(\tau,t,\lambda,x,y),\dots,u_d(\tau,t,\lambda,x,y)\right)^{\mathrm{T}}$,

$$V'(\lambda,\tau) = \left(\left[\frac{\partial V(x,t)}{\partial x_1}\right]_{x=\lambda},\dots,\left[\frac{\partial V(x,t)}{\partial x_d}\right]_{x=\lambda}\right) \text{ and } \mathrm{Hess}[V(\lambda,\tau)] = \left[\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right]_{x=\lambda}.$$

(3) Let $S(t,\lambda,x,y)$ be the master action given by formula

$$S(t,\lambda,x,y) = \int_0^t L(\dot{u}(\tau,t,\lambda,x,y),u(\tau,t,\lambda,x,y),\tau)\,d\tau, \quad (3.2)$$

where master Lagrangian $L(\dot{u},u,\tau)$ are

$$L(\dot{u},u,\tau) = \frac{m}{2}\dot{u}^2(\tau,t,\lambda,x,y) - \hat{V}(u(\tau,t,\lambda,x,y),\tau), \dot{u} = \left(\frac{\partial u_1}{\partial\tau},\dots,\frac{\partial u_d}{\partial\tau}\right), \dot{u}^2 = \langle\dot{u},\dot{u}\rangle,$$

$$\hat{V}\left((\tau,t,\lambda,x,y),\tau\right) = u(\tau,t,\lambda,x,y)\mathrm{Hess}[V(\lambda,\tau)]u^{\mathrm{T}}(\tau,t,\lambda,x,y) + V'(\lambda,\tau)u^{\mathrm{T}}(\tau,t,\lambda,x,y). \quad (3.3)$$

Let $y_{cr} = y_{cr}(t,\lambda,x) \in \mathbb{R}^d$ be the solution of the linear system of algebraic equations

$$\left[\frac{\partial S(t,\lambda,x,y)}{\partial y_i}\right]_{y=y_{cr}} = 0, i = 1,\dots,d. \quad (3.4)$$

(4) Let $\hat{x} = \hat{x}(t,\lambda,x_0) \in \mathbb{R}^d$ be the solution of the linear system of algebraic equations

$$y_{cr}(t,\lambda,\hat{x}) + \lambda - x_0 = 0. \quad (3.5)$$

Assume that for a given values of parameters t,λ,x_0 the point $\hat{x} = \hat{x}(t,\lambda,x_0)$ is not a focal point on a corresponding trajectory is given by corresponding solution of the boundary problem (3.1). Then for the limiting quantum average given via formulae (1.1) the inequality is satisfied:

$$\underline{\lim}_{\substack{\hbar\to0\\ \varepsilon\to0}}|\langle i,t,x_0;\hbar\rangle - \lambda_i| \le 2\left[\left|\det S_{y_{cr}y_{cr}}\left(t,\lambda,\hat{x}(t,\lambda,x_0),y_{cr}\left(t,\lambda,\hat{x}(t,\lambda,x_0)\right)\right)\right|\right]^{-1}|\hat{x}_i(t,\lambda,x_0)|, \quad (3.6)$$

$i = 1, \ldots, d$. Thus one can to calculate the limiting quantum trajectory corresponding to potential $V(x,t)$ by using *transcendental master equation*

$$\hat{x}_i(t,\lambda,x_0) = 0, i = 1,\ldots,d. \quad (3.7)$$

4.Quantum anharmonic oscillator with a cubic potential supplemented by additive sinusoidal driving

In this subsection we calculate exact quasi-classical asymptotic for quantum anharmonic oscillator with a cubic potential supplemented by additive sinusoidal driving. Using theorem 3.1 we obtain corresponding limiting quantum trajectories given by Eq. (1.3). Let us consider quantum anharmonic oscillator with a cubic potential $V(x) = (m\omega^2/2)x^2 - ax^3 + bx, x \in \mathbb{R}, a, b > 0$ supplemented by an additive sinusoidal driving, i.e. $V(x,t) = (m\omega^2/2)x^2 - ax^3 + bx - [A\sin(\Omega t)]x$. The corresponding master Lagrangian is given by (3.3), is

$$L(\dot{u},u,\tau) = (m/2)\dot{u}^2 - m(\omega^2/2 + 3a\lambda/m)u^2 - (m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega t))u.$$

We assume now that $\omega^2/2 + 3a\lambda/m \geq 0$ and rewrite $L(\dot{u},u,\tau)$ in the form $L(\dot{u},u,\tau) = (m/2)\dot{u}^2 - (m\varpi^2\lambda/2)u^2 + g(\lambda,t)u$, where $\varpi(\lambda) = \sqrt{2|\omega^2/2 + 3a\lambda/m|}$ and $g(\lambda,t) = -[m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega \cdot t)]$.

Therefore corresponding master action $S(t,\lambda,x,y)$ given by Eq. (3.2) is

$$S(t,\lambda,x,y) = \frac{m\varpi}{2\sin\varpi t}[(\cos\varpi t)(y^2 + x^2) - 2xy + \frac{2x}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi\tau)\,d\tau +$$

$$+ \frac{2y}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi(t-\tau))\,d\tau - \frac{2}{m^2\varpi^2}\int_0^t\int_0^\tau g(\lambda,\tau)g(\lambda,s)\sin\varpi(t-\tau)\sin(\varpi s)\,ds d\tau]. \quad (4.1)$$

The linear system of algebraic equations (3.4) is

$$\frac{\partial S(t,\lambda,x,y)}{\partial y} = 2y\cos\varpi t - 2x + (2/m\varpi)\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau = 0.$$

Therefore

$$y_{cr}(t,\lambda,x) = \frac{x}{\cos\varpi t} - \frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin\big((\varpi(t-\tau)\big)d\tau. \quad (4.2)$$

The linear system of algebraic equations (3.5) is

$$\frac{x}{\cos\varpi t} - \frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin\big((\varpi(t-\tau)\big)d\tau + \lambda - x_0 = 0. \quad (4.3)$$

Then the solution of the linear system of algebraic equations (3.5) is

$$\hat{x}(t,\lambda,x_0) = \frac{1}{m\varpi}\int_0^t g(\lambda,t)\sin\big((\varpi(t-\tau)\big)d\tau + (\lambda(t) - x_0)\cos\varpi t. \quad (4.4)$$

Transcendental master equation (3.7) is

$$\frac{1}{m\varpi}\int_0^t g(\lambda(t),t)\sin\big((\varpi(t-\tau)\big)d\tau + (\lambda(t) - x_0)\cos\varpi t = 0. \quad (4.5)$$

Finally from Eq. (4.5) one obtains

$$d\big(\lambda(t)\big)\left(\frac{\cos(\varpi t)}{\varpi} - \frac{1}{\varpi}\right) + \frac{A\big(\varpi\sin(\Omega t) - \Omega\sin(\varpi t)\big)}{\varpi^2 - \Omega^2} - (\lambda(t) - x_0)m\varpi\cos(\varpi t) = 0, \quad (4.6)$$

where $d(\lambda) = m\omega^2\lambda + 3a\lambda^2 - b$.

Numerical Example. $x_0 = 0, m = 1, \Omega = 0, \omega = 9, a = 3, b = 10, A = 0$.

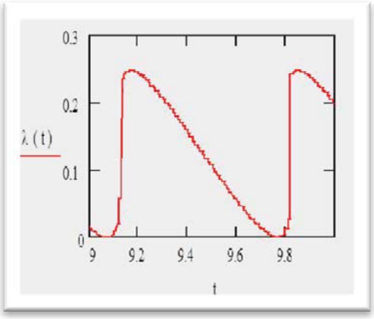

Figure 1. Limiting quantum trajectory $\lambda(t)$ with a jump.

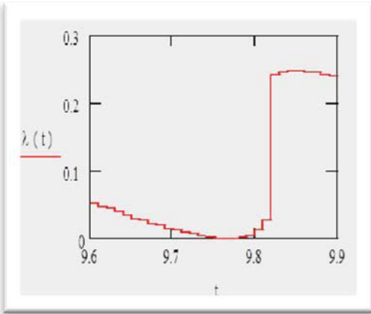

Figure 2. Limiting quantum trajectory $\lambda(t)$ with a jump.

5.Conclusion

We pointed out that there exist limiting quantum trajectories given via Eq. (1.3) with a jump. Such jumps does not depend on any single measurement of particle position $q(t)$ at instant t and obtained without any reference to a phenomenological master-equation of Lindblad form. An axiom of quantum mechanics is that we cannot predict the result of any single measurement of an observable of a quantum mechanical system in a superposition of eigenstates. However we can predict the result of any single measurement of particle position $q(t)$ at instant t with a probability $P(t, q, \delta q) \cong 1$ if the condition is valid: $\lim_{\varepsilon\to 0}\lim_{\|\delta q(t)\|\to 0}\lim_{\hbar\to 0} P_t(q(t), \delta q(t), x_0; \hbar, \varepsilon) = 1$, where $P_t(q(t), \delta q(t), x_0; \hbar, \varepsilon)$ given via Eq. (1.2). Complete explanation is given in our preprint [8].

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