Paraconsistent First-Order Logic with infinite hierarchy levels of contradiction $\mathbf{LP}^{\#}_{\omega}$. Axiomatical system $\mathbf{HST}^{\#}_{\omega}$, as inconsistent generalization of Hrbacek set theory $\mathbf{HST}$.

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Abstract: In this paper paraconsistent first-order logic $\mathbf{LP}^{\#}_{\omega}$ with infinite hierarchy levels of contradiction is proposed. Corresponding paraconsistent set theory $\mathbf{KSth}^{\#}_{\omega}$ is proposed. Axiomatical system $\mathbf{HST}^{\#}_{\omega}$, as inconsistent generalization of Hrbacek set theory $\mathbf{HST}$ is considered.

Content

List of designations

$\mathbf{V}^{Con}$ - consistent universum
$\mathbf{V}^{Inc}$ - inconsistent universum
$\mathbf{U}$ - complete universum $\mathbf{U} \triangleq \mathbf{V}^{Con} \bigcup \mathbf{V}^{Inc}$
$(\bullet = \bullet)$ - relation of the classical consistent equivalence
$(\bullet =_{\mathbf{s}} \bullet)$ - relation of the strong consistent nonclassical equivalence or $\mathbf{s}$-equivalence
$(\bullet =_{\mathbf{s}} \bullet) \upharpoonright \mathbf{V}^{Con} = (\bullet = \bullet)$
$(\bullet \in \bullet)$ - classical consistent membership relation
$(\bullet \in_{\mathbf{s}} \bullet)$ - strong consistent membership relation or $\mathbf{s}$-membership relation
$(\bullet \in_{\mathbf{s}} \bullet) \upharpoonright \mathbf{V}^{Con} = (\bullet \in \bullet)$
$(\bullet =_{w} \bullet)$ - relation of the weak equivalence or w-equivalence
$(\bullet =_{w} \bullet) \upharpoonright \mathbf{V}^{Con} = (\bullet = \bullet)$
$(\bullet =_{w_1} \bullet)$ - relation of the weak inconsistent equivalence order 1 or w_1-equivalence
$(\bullet =_{w_1} \bullet) \leftrightarrow (\bullet =_{w} \bullet) \wedge (\bullet \neq_{w} \bullet)$
$(\bullet =_{w_n} \bullet)$ - relation of the weak inconsistent equivalence order n or w_n-equivalence
$(\bullet \in_{w} \bullet)$ - weak membership relation or w-membership relation
$(\bullet \in_{w_1} \bullet)$ weak inconsistent membership relation order 1 or w_1-membership relation
$\varnothing_{w}$-weak empty set

I.Introduction.

The real history of non-Aristotelian logic begins on May 18,1910 when N.A. Vasiliev presented to the Kazan University faculty a lecture "On Partial Judgements, the Triangle of Opposition, the Law of Excluded Fourth" [Vasiliev 1910] to satisfy the requirements for obtaining the title of privat-dozent. In this lecture Vasiliev expounded for the first time the key principles of non-Aristotelian, imaginary, logic. In this work he likewise constructed his "imaginary" logic free of the laws of contradiction and excluded middle in the informal, so-to-speak Aristotelian, manner (although imaginary logic is in essense non- Aristotelian).Thus the birthday of new logic was exactly fixed in the annals of history. Vasiliev’s reform of logic was radical, and he did his best to determine whether it was possible for the new logic with new laws and new subject to imply a new logical Universe. Vasiliev began the modern non-classical revolution in logic, but he certainly did not complete it. The founder of paraconsistent logic, N.A. Vasiliev, stated as a characteristic feature of his logic, three kinds of sentence, i.e. "**S** *is* **A**", "**S** *is not* **A**", "**S** *is and is not* **A**". Thus Vasiliev logic rejected the *law of non-contradiction:* $\neg(\mathbf{A} \wedge \neg\mathbf{A})$ and the *law of excluded middle:* $\mathbf{A} \vee \neg\mathbf{A}$. However Vasiliev's logic preserve the *law of excluded fourth:* $\mathbf{A} \vee \neg\mathbf{A} \vee (\mathbf{A} \wedge \neg\mathbf{A})$. Possible formalized versions of Vasiliev's logic with one level of contradiction $\mathbf{LP}_1^{\#}$ was proposed by A.I.Arruda [1]. In this paper we proposed paraconsistent first-order logic $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}$ with infinite hierarchy levels of contradiction. Corresponding paraconsistent set theory $\mathbf{KSth}_{\omega}^{\#}$ is discussed.

The postulates (or their axioms schemata) of Vasiliev-Amida propositional paraconsistent logic VA_1 are the following:

The language $\mathcal{L}_1$ of paraconsistent logic $VA_1 \triangleq VA_1[\mathbf{V}]$ has as

primitive symbols (i) countable set of a clalassical propositional variables, (ii) countable set $\mathbf{V} = \{\mathbf{P}_i\}_{i \in \mathbb{N}}$ of a non clalassical propositional variables,(iii) the connectives $\neg, , \wedge, \vee, \rightarrow$ and (iv) the parentheses (,).

I. Logical postulates:

(1) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A})$,

(2) $(\mathbf{A} \rightarrow \mathbf{B}) \rightarrow ((\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow (\mathbf{A} \rightarrow \mathbf{C}))$,

(3) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B}$,

(6) $\mathbf{A} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(6) $\mathbf{A} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \rightarrow \mathbf{C}) \rightarrow ((\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{A} \vee \mathbf{B} \rightarrow \mathbf{C}))$,

(9) $\mathbf{A} \vee \neg\mathbf{A}$,

(10) $\mathbf{B} \rightarrow (\neg\mathbf{B} \rightarrow \mathbf{A})$ **if** $\mathbf{B} \notin \mathbf{V}$.

II.Rules of a conclusion:

Anrestricted Modus Ponens rule $\mathbf{MP}$: $\mathbf{A}, \mathbf{A} \rightarrow \mathbf{B} \vdash \mathbf{B}$.

Theorem 1.1.[1]. (1) If $\mathbf{B} \notin \mathbf{V}$, then $\mathbf{B}, \neg\mathbf{B} \vdash \mathbf{A}$; (2) $\neg\neg\mathbf{A} \leftrightarrow \mathbf{A}$ **iff** $\mathbf{A} \notin \mathbf{V}$;
(3) $\neg\neg\mathbf{A} \rightarrow \mathbf{A}$.

The postulates (or their axioms schemata) of Vasiliev-Amida propositional paraconsistent logic VA_2 are the following:

The language $\mathcal{L}_2$ of paraconsistent logic $VA_2 \triangleq VA_2[\mathbf{V}]$ has as

primitive symbols (i) countable set of a clalassical propositional variables, (ii) countable set $\mathbf{V} = \{\mathbf{P}_i\}_{i \in \mathbb{N}}$ of a non clalassical propositional variables,(iii) the connectives $\neg, , \wedge, \vee, \rightarrow$ and (iv) the parentheses (,).

I. Logical postulates:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,

(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,

(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9) $\mathbf{A} \vee \neg\mathbf{A}$,

(10) $\mathbf{B} \to (\neg\mathbf{B} \to \mathbf{A})$ **if** $\mathbf{B} \notin \mathbf{V}$,

(11) $\mathbf{P}_i \wedge \neg\mathbf{P}_i$ **iff** $\mathbf{P}_i \in \mathbf{V}, i = 1, 2, \ldots$.

II.Rules of a conclusion:

Anrestricted Modus Ponens rule $\mathbf{MP} : \mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash \mathbf{B}$.

II.Paraconsistent Logic with n levels of contradiction $\mathbf{LP}_n^{\#}$.

Modern formalized versions of Vasiliev's logic with one level of contradiction may be found in Amida [1980], [Puga and Da Costa 1988], Smimov [Smirnov 1987], and [Smimov 1987a, 161-169]. There is also the presentation Smimov given at the International Congress of Logic, Methodology and Philosophy of Science in Uppsala in 1991.

Paraconsistent Logic with one levels of contradiction $\mathbf{LP}_1^{\#}$.

Let us consider now Vasiliev-Amida type paraconsistent logic $\mathbf{LP}_1^{\#} = \mathbf{LP}_1^{\#}[\mathbf{V}, \boldsymbol{\Delta}]$ with one level of contradiction.

The postulates (or their axioms schemata) of propositional paraconsistent logic $\mathbf{LP}_1^{\#}$ are the following:

The language $\mathcal{L}_1^{\#}$ of paraconsistent logic $\mathbf{LP}_1^{\#} \triangleq \mathbf{LP}_1^{\#}[\mathbf{V}, \boldsymbol{\Delta}]$ has as primitive symbols (i) countable set of a clalassical propositional variables, (ii) countable set $\mathbf{V} = \{\mathbf{P}_i\}_{i \in \mathbb{N}}$ of a non clalassical propositional variables, (iii) the connectives $\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \to$ and (iv) the parentheses (,).

Remark.2.1. We distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$.

The definition of formula is the usual. We denote the set of the all formulae of $\mathbf{LP}_1^{\#}[\mathbf{V}_1,\boldsymbol{\Delta}]$ by $\mathcal{F}_1^{\#}$, where $\mathbf{V}_1 = \mathbf{V}^{[0]} \cup \mathbf{V}^{[1]}$ and $\boldsymbol{\Delta}$ is a given subset of $\mathcal{F}_1^{\#}$. Here we used the following definitions: $\mathbf{V}^{[0]} \triangleq \mathbf{V}, \mathbf{V}^{[1]} \triangleq \{\alpha^{[1]} | (\alpha \in \mathbf{V})\}, \alpha^{[1]} \triangleq (\alpha \wedge \neg_w \alpha)$. $\mathbf{A},\mathbf{B},\mathbf{C}$, ... will be used as metalanguage variables which indicate formulas of $\mathbf{LP}_1^{\#}[\mathbf{V},\boldsymbol{\Delta}]$. We assume through that $\mathbf{V}_1 \subset \boldsymbol{\Delta} \subsetneq \mathcal{F}_1^{\#}$.

I. Logical postulates:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,
(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,
(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,
(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,
(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,
(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,
(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,
(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9) $\mathbf{P}_i \wedge \neg_w \mathbf{P}_i$ **iff** $\mathbf{P}_i \in \mathbf{V}, i = 1,2,\ldots$
(10) $\mathbf{A} \vee \neg_w \mathbf{A}$ **iff** $\mathbf{A} \notin \mathbf{V}$,
(11) $\mathbf{B} \to (\neg_w \mathbf{B} \to \mathbf{A})$ **if** $\mathbf{B} \notin \mathbf{V}_1$,

(12) $\mathbf{A} \vee \neg_w \mathbf{A} \vee (\mathbf{A} \wedge \neg_w \mathbf{A})$ **iff** $\mathbf{A} \in \mathbf{V}_1$,
(13) $\mathbf{A} \vee \neg_{\mathbf{s}} \mathbf{A}$ **if** $\mathbf{A} \in \mathcal{F}_{\mathbf{1}}^{\#}$,
(14) $\mathbf{B} \to (\neg_{\mathbf{s}} \mathbf{B} \to \mathbf{A})$ **if** $\mathbf{A},\mathbf{B} \in \mathcal{F}_1^{\#}$.

II. Rules of a conclusion:

Restricted Modus Ponens rule MPR :

$\mathbf{A},\mathbf{A} \to \mathbf{B} \vdash \mathbf{B}$ **iff** $\mathbf{A} \notin \boldsymbol{\Delta}$.

Unrestricted Modus Tollens rules: $\mathbf{P} \to \mathbf{Q}, \neg_w \mathbf{Q} \vdash \neg_w \mathbf{P}; \mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}} \mathbf{Q} \vdash \neg_{\mathbf{s}} \mathbf{P}$.

The rule of a strong contradiction: $\mathbf{A} \wedge \neg_{\mathbf{s}} \mathbf{A} \vdash \mathbf{B}$.

III. Quantification

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_{\mathbf{1}}^{\#}[\mathbf{V},\boldsymbol{\Delta}]$ we construct the corresponding paraconsistent relevant first-order predicate calculus $\overline{\mathbf{LP}}_1^{\#} = \overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}},\tilde{\boldsymbol{\Delta}}\right]$. The language of the paraconsistent predicate calculus $\overline{\mathbf{LP}}_1^{\#}$, denoted by $\overline{\mathcal{L}}_1^{\#}$, is an extension of the language $\mathcal{L}_1^{\#}$ introduced above, by adding, as usually,for every m, denumerable families of m-ary predicate symbols

$R_1^m, R_2^m, \ldots, R_n^m, \ldots,$ and m-ary function symbols $f_1^m, f_2^m, \ldots, f_n^m, \ldots,$ and the universal $\forall$ and existential $\exists$ quantifiers.

We assume throughout that: the language $\overline{\mathcal{L}}_1^{\#}$ contains also

(i) the classical numerals $\bar{0}, \bar{1}, \ldots;$

(ii) countable set Γ of the classical consistent set variables $\Gamma = \{x, y, z, \ldots\};$

(iii) countable set $\breve{\Gamma}$ of the non classical inconsistent set variables $\breve{\Gamma} = \{\breve{x}, \breve{y}, \breve{z}, \ldots\};$

(iv) countable set Θ of the classical non-logical constants $\Theta = \{a, b, c \ldots\};$

(iv) countable set $\breve{\Theta}$ of the non classical non-logical constants $\breve{\Theta} = \left\{\breve{a}, \breve{b}, \breve{c} \ldots\right\};$

Definition 2.1. An $\overline{\mathbf{LP}}_1^{\#}$ wff Φ (well-formed formula Φ) is a $\overline{\mathbf{LP}}_1^{\#}$- sentence iff it hasn't free variables; a wff Ψ is open if it has free variables. We'll use the slang 'k – place open wff' to mean a wff with k distinct free variables.

Definition 2.2. An $\overline{\mathbf{LP}}_1^{\#}$ wff Φ is a classical iff it hasn't non classical variables and non classical constants.

Definition 2.3. An $\overline{\mathbf{LP}}_1^{\#}$ wff Φ is a non classical iff it has an non classical variables or non classical constants.We denote the set of the all formulae of $\overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\boldsymbol{\Delta}}\right]$ by $\overline{\mathcal{F}}_1^{\#}$, where $\tilde{\mathbf{V}} \supset \mathbf{V}_1$ and $\tilde{\boldsymbol{\Delta}} \supset \boldsymbol{\Delta}$ is a given subsets of $\overline{\mathcal{F}}_1^{\#}$.We assume through that $\tilde{\mathbf{V}} \subset \tilde{\boldsymbol{\Delta}} \subsetneq \overline{\mathcal{F}}_1^{\#}$.

The postulates of $\overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\boldsymbol{\Delta}}\right]$ are those of $\mathbf{LP}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\boldsymbol{\Delta}}\right]$ (suitably adapted) plus the following:

(I) $\dfrac{\alpha \to \beta(x)}{\alpha \to \forall x \beta(x)},$

(II) $\forall x \alpha(x) \to \alpha(y),$

(III) $\alpha(x) \to \exists x \alpha(x),$

(IV) $\dfrac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta},$

(V) $\forall x(\alpha(x))^{(1)} \to (\forall x \alpha(x))^{(1)},$

(VI) $\forall x((\alpha(x))^{(1)} \to (\exists x \alpha(x))^{(1)},$

(VII) $\forall x(\alpha(x))^{[1]} \to (\forall x \alpha(x)) \wedge (\forall x \neg_w \alpha(x)),$

(VIII) $\forall x((\alpha(x))^{[1]} \to (\exists x \alpha(x)) \wedge (\exists x \neg_w \alpha(x)),$

where we used the following definitions:

$\alpha^{(0)} \triangleq \alpha, \alpha^{(1)} \triangleq \neg_w(\alpha \wedge \neg_w \alpha)$ and

$\alpha^{[0]} \triangleq \alpha, \alpha^{[1]} \triangleq \alpha \wedge \neg_w \alpha$

and where the variables x and y and the formulas α and β satisfy the usual definition.

From the calculi $\overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\boldsymbol{\Delta}}\right]$,one can construct the following predicate calculus with equality. This is done by adding to their languages the binary predicates symbol of strong equality $(\cdot = \cdot)$ or $(\cdot =_{\mathbf{s}} \cdot)$ and weak equality$(\cdot =_w \cdot)$ with suitable modifications in the concept of formula, and by adding the following postulates:

(IX) $\forall x(x =_{\mathbf{s}} x)$,

(X) $\forall x \forall y\left[(x =_{\mathbf{s}} y)^{[1]} \vdash \mathbf{B}\right]$,

(XI) $\forall x \forall y[x =_{\mathbf{s}} y \to (\alpha(x) \leftrightarrow \alpha(y))]$,

(XII) $\forall x \forall y \forall z[(x =_{\mathbf{s}} y) \wedge (y =_{\mathbf{s}} z) \to x =_{\mathbf{s}} z]$,

(XIII) $\forall y \exists x(y =_w x)$,

(XIV) $\forall y \exists x(y =_w x)^{[1]}$,

(XV) $\forall x \forall y[x =_w y \to (\alpha(x) \leftrightarrow \alpha(y))]$

(XVI) $\forall x \forall y\left[(x =_w y)^{[1]} \leftrightarrow (\alpha^{[1]}(x) \leftrightarrow \alpha^{[1]}(y))\right]$,

(XVII) $\forall x \forall y \forall z[(x =_w y) \wedge (y =_w z) \to x =_w z]$,

(XVIII) $\forall x \forall y \forall z\left[(x =_w y)^{[1]} \wedge (y =_w z)^{[1]} \to (x =_w z)^{[1]}\right]$,

(XIX) $\forall x \forall y \forall z[(x =_w y) \wedge (y =_{\mathbf{s}} z) \to x =_w z]$,

(XX) $\forall x \forall y \forall z\left[(x =_w y)^{[1]} \wedge (y =_{\mathbf{s}} z) \to (x =_w z)^{[1]}\right]$,

(XXI) $\forall x \forall y \forall z[(x =_{\mathbf{s}} y) \wedge (y =_w z) \to x =_w z]$,

(XXII) $\forall x \forall y \forall z\left[(x =_{\mathbf{s}} y) \wedge (y =_w z)^{[1]} \to (x =_w z)^{[1]}\right]$.

II.Rules of a conclusion:

Restricted Modus Ponens rule $\mathbf{MPR}$:

$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash \mathbf{B}$ **iff** $\mathbf{A} \notin \tilde{\mathbf{\Delta}}$.

Unrestricted Modus Tollens rules: $\mathbf{P} \to \mathbf{Q}, \neg_w \mathbf{Q} \vdash \neg_w \mathbf{P}; \mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}} \mathbf{Q} \vdash \neg_{\mathbf{s}} \mathbf{P}$.

The rule of a strong contradiction: $\mathbf{A} \wedge \neg_{\mathbf{s}} \mathbf{A} \vdash \mathbf{B}$.

Definition 2.4. Classical $\mathbf{V}$-object $\mathfrak{I}^{\mathbf{Cl}} = \mathfrak{I}^{\mathbf{Cl}}\left[\tilde{\mathbf{V}}, \tilde{\mathbf{\Delta}}\right]$ is the object such that from any classical formula of the form $P(\mathfrak{I}^{\mathbf{Cl}}) \wedge \neg_w P(\mathfrak{I}^{\mathbf{Cl}})$, where $P(\mathfrak{I}^{\mathbf{NCl}}) \notin \tilde{\mathbf{\Delta}}$ by using principles as in paraconsistent logical calculas $\overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\mathbf{\Delta}}\right]$ using Restricted Modus Ponens rule, one can deduce any formula i.e., classical object $\mathfrak{I}^{\mathbf{Cl}}$ is the object which hasn't any inconsistent property with respect to a weak negation $\neg_w$.

Definition 2.5. Non classical $\mathbf{V}$-object $\mathfrak{I}^{\mathbf{NCl}} = \mathfrak{I}^{\mathbf{NCl}}\left[\tilde{\mathbf{V}}, \tilde{\mathbf{\Delta}}\right]$ of the 1-degree of inconsistency is the object $\mathfrak{I}^{\mathbf{NCl}}$ such that: from any non classical formula of the form $P(\mathfrak{I}^{\mathbf{NCl}}) \wedge \neg_w P(\mathfrak{I}^{\mathbf{NCl}})$, where $P(\mathfrak{I}^{\mathbf{NCl}}) \notin \tilde{\mathbf{\Delta}}$ by using principles as in paraconsistent logical calculas $\overline{\mathbf{LP}}_1^{\#}\left[\tilde{\mathbf{V}}, \tilde{\mathbf{\Delta}}\right]$ using Restricted Modus Ponens rule one can't deduce any formula whatsoever i.e., non classical object of the 1-degree of inconsistency is the object $\mathfrak{I}^{\mathbf{NCl}}$ which has at least one inconsistent property of the 1-degree with respect to a weak negation $\neg_w$.

The simplest example of non classical objects 1-degree inconsistency is inconsistent numbers $\breve{a}$ such that

$$(\breve{a} =_w \bar{1}) \wedge \neg_w(\breve{a} =_w \bar{1}), \tag{2.1}$$

or

$$\left(\breve{b} =_w \bar{1}\right) \wedge \left(\breve{b} =_w \bar{2}\right). \tag{2.2}$$

Remark.2.2. Note that: (i) formula (2.1) meant that $(\breve{a} =_w \bar{1}) \in \tilde{\mathbf{V}}$ and (ii) formula (2.2) meant that $\left(\breve{b} =_w \bar{1}\right) \in \tilde{\mathbf{\Delta}}$ and $\left(\breve{b} =_w \bar{2}\right) \in \tilde{\mathbf{\Delta}}$.

Paraconsistent Logic with n levels of contradiction $\mathbf{LP}_n^{\#}$.

Let us consider now paraconsistent logic $\mathbf{LP}_n^{\#} = \mathbf{LP}_n^{\#}[\mathbf{V},\mathbf{\Delta}]$ with n levels of contradiction.

The postulates (or their axioms schemata) of propositional paraconsistent logic $\mathbf{LP}_n^{\#} = \mathbf{LP}_n^{\#}\left[\hat{\mathbf{V}},\hat{\mathbf{\Delta}}\right]$ are the following:

The language $\mathcal{L}_n^{\#}$ of paraconsistent logic $\mathbf{LP}_n^{\#}$ has as primitive symbols (i) countable set of a clalassical propositional variables, (ii) countable set $\mathbf{V} = \{\mathbf{P}_i\}_{i \in \mathbb{N}}$ of a non clalassical propositional variables, (iii) the connectives $\neg_w, \neg_\mathbf{s}, \wedge, \vee, \rightarrow$ and (iv) the parentheses (,).

Remark 2.3.We distinguish a weak negation $\neg_w$ and a strong negation $\neg_\mathbf{s}$.

The definition of formula is the usual. We denote the set of the all formulae of $\mathbf{LP}_n^{\#}\left[\hat{\mathbf{V}},\hat{\mathbf{\Delta}}\right]$ by $\mathcal{F}_n^{\#}$ where $\hat{\mathbf{V}}$ and $\hat{\mathbf{\Delta}}$ is a given subsets of $\mathcal{F}_n^{\#}$. We assume through that $\hat{\mathbf{V}} \subset \hat{\mathbf{\Delta}} \subsetneq \mathcal{F}_n^{\#}$.

$\mathbf{A}, \mathbf{B}, \mathbf{C}$, ... will be used as metalanguage variables which indicate formulas of $\mathbf{LP}_n^{\#}\left[\hat{\mathbf{V}},\hat{\mathbf{\Delta}}\right]$.

Definition 2.6. (i) $\alpha^{(k)}$ stands for $\alpha^{(k-1)} \wedge (\alpha^{(k-1)})^{(1)}$, where $\alpha^{(0)} \triangleq \alpha$, $\alpha^{(1)} \triangleq \neg_w(\alpha \wedge \neg_w \alpha), 0 \leq k \leq n$.

(ii) the (finite) k-order of the level of a weak consistency (w-consistency) is: $\alpha^{(k)}, 0 \leq k \leq n$.

Definition 2.7. (i) $\alpha^{[k]}$ stands for $\alpha^{[k-1]} \wedge (\alpha^{[k-1]})^{[1]}$, where $\alpha^{[0]} \triangleq \alpha$, $\alpha^{[1]} \triangleq \alpha \wedge \neg_w \alpha, 0 \leq k \leq n$.

(ii) the (finite) k-order of the level of a weak inconsistency (w-inconsistency) is: $\alpha^{[n]}, 1 \leq k \leq n$.

I. Logical postulates:

(1) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A})$,

(2) $(\mathbf{A} \to \mathbf{B}) \to ((\mathbf{A} \to (\mathbf{B} \to \mathbf{C})) \to (\mathbf{A} \to \mathbf{C}))$,
(3) $\mathbf{A} \to (\mathbf{B} \to \mathbf{A} \wedge \mathbf{B})$,
(4) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \to \mathbf{B}$,
(6) $\mathbf{A} \to (\mathbf{A} \vee \mathbf{B})$,
(7) $\mathbf{B} \to (\mathbf{A} \vee \mathbf{B})$,
(8) $(\mathbf{A} \to \mathbf{C}) \to ((\mathbf{B} \to \mathbf{C}) \to (\mathbf{A} \vee \mathbf{B} \to \mathbf{C}))$,

(9) $\mathbf{P} \wedge \neg_w \mathbf{P}$ **iff** $\mathbf{P} \in \mathbf{V}$,
(10) $\mathbf{P}^{[k]}$ **iff** $\mathbf{P} \in \mathbf{V}$,
(11) $\mathbf{A} \vee \neg_w \mathbf{A}$ **if** $\mathbf{A} \notin \hat{\mathbf{V}} = \bigcup_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \mathbf{V}^{[\mathbf{k}]}.$,
(12) $\mathbf{A} \vee \neg_{\mathbf{s}} \mathbf{A}$ **if** $\mathbf{A} \in \mathcal{F}_n^{\#}$,
(13) $\mathbf{B} \to (\neg_{\mathbf{s}} \mathbf{B} \to \mathbf{A})$ **if** $\mathbf{A}, \mathbf{B} \in \mathcal{F}_n^{\#}$,
(14) $(\mathbf{A} \vee \neg_w \mathbf{A}) \vee (\mathbf{A} \wedge \neg_w \mathbf{A}) \vee \mathbf{A}^{[2]} \underbrace{\vee \ldots \vee \mathbf{A}^{[k]} \vee \ldots \vee}_{n-1} \mathbf{A}^{[n]}$ **if** $\mathbf{A} \in \mathcal{F}_n^{\#}$,

(15) $\mathbf{B} \to (\neg_w \mathbf{B} \to \mathbf{A})$ **if** $\mathbf{B} \notin \hat{\mathbf{V}} = \bigcup_{k=0}^{n} \mathbf{V}^{[k]}$.

II.Rules of a conclusion:

Restricted Modus Ponens rule $\mathbf{MPR}$:

$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash \mathbf{B}$ **iff** $\mathbf{A} \notin \hat{\mathbf{V}}$.
Unrestricted Modus Tollens rule: $\mathbf{P} \to \mathbf{Q}, \neg_w \mathbf{Q} \vdash \neg_w \mathbf{P}; \mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}} \mathbf{Q} \vdash \neg_{\mathbf{s}} \mathbf{P}$.

The rule of a strong contradiction: $\mathbf{A} \wedge \neg_{\mathbf{s}} \mathbf{A} \vdash \mathbf{B}$.

III.Quantification

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}_n^{\#}[\hat{\mathbf{V}}]$ we construct the corresponding paraconsistent relevant first-order predicate calculus. These new calculus will be denoted by $\overline{\mathbf{LP}}_n^{\#}[\hat{\mathbf{V}}]$.

The postulates of $\overline{\mathbf{LP}}_n^{\#}[\hat{\mathbf{V}}]$ are those of $\mathbf{LP}_n^{\#}[\hat{\mathbf{V}}]$ (suitably adapted) plus the following:

(I) $\dfrac{\alpha \to \beta(x)}{\alpha \to \forall x \beta(x)}$,

(II) $\forall x \alpha(x) \to \alpha(y)$,

(III) $\alpha(x) \to \exists x \alpha(x)$,

(IV) $\dfrac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta}$,

(V) $\forall x(\alpha(x))^{(k)} \to (\forall x \alpha(x))^{(k)}, k = 1,2,\ldots,n$,

(VI) $\forall x((\alpha(x))^{(k)} \to (\exists x \alpha(x))^{(k)}, k = 1,2\ldots,n$,

(VII) $\forall x(\alpha(x))^{[k]} \to (\forall x \alpha(x))^{[k]}, k = 1,2\ldots,n$.

From the calculus $\overline{\mathbf{LP}}_n^{\#}\left[\hat{\mathbf{V}}\right]$,we can construct the following predicate calculus with equality.This is done by adding to their languages the binary predicates symbol of strong equality $(\bullet = \bullet)$ or $(\bullet =_{\mathbf{s}} \bullet)$ and weak equality$(\bullet =_w \bullet)$ with suitable modifications in the concept of formula, and by adding the following postulates:

(IX) $\forall x(x =_{\mathbf{s}} x)$,

(X) $\forall x\left[(x =_{\mathbf{s}} x)^{[1]} \vdash \mathbf{B} \right]$,

(XI) $\forall x \forall y[x =_{\mathbf{s}} y \to (\alpha(x) \leftrightarrow \alpha(y))]$,

(XII) $\forall x \forall y \forall z[(x =_{\mathbf{s}} y) \wedge (y =_{\mathbf{s}} z) \to x =_{\mathbf{s}} z]$,

(XIII) $\forall y \exists x(x =_w x)^{[k]}, k = 0,1,2,\ldots,n$,

(XIV) $\forall x \forall y\left[(x =_w y)^{[k]} \leftrightarrow \forall \alpha(\circ)(\alpha^{[k]}(x) \leftrightarrow \alpha^{[k]}(y))\right], k = 1,2,\ldots,n$,

(XV) $\forall x \forall y \forall z\left[(x =_w y)^{[k]} \wedge (y =_w z)^{[k]} \to (x =_w z)^{[k]}\right], k = 0,1,2,\ldots,n$,

(XVI $\forall x \forall y \forall z\left[(x =_w y)^{[k]} \wedge (y =_{\mathbf{s}} z) \to (x =_w z)^{[k]}\right], k = 0,1,2,\ldots,n$,

(XVII) $\forall x \forall y \forall z\left[(x =_{\mathbf{s}} y) \wedge (y =_w z)^{[k]} \to (x =_w z)^{[k]}\right], k = 0,1,2,\ldots,n$,

(XVIII) $\forall y \exists x(y =_w x)^{[k]}, k = 0,1,2,\ldots,n$.

III. Paraconsistent Logic with infinite hierarchy levels of contradiction $\mathbf{LP}_{\boldsymbol{\omega}}^{\#}$.

The postulates (or their axioms schemata) of propositional paraconsistent logic

$\mathbf{LP}^{\#}_{\boldsymbol{\omega}} = \mathbf{LP}^{\#}_{\boldsymbol{\omega}}\left[\hat{\mathbf{V}},\hat{\boldsymbol{\Delta}}\right]$ are the following:

The language $\mathcal{L}^{\#}_{\boldsymbol{\omega}}$ of paraconsistent logic $\mathbf{LP}^{\#}_{\boldsymbol{\omega}}$ has as primitive symbols (i) countable set of a clalassical propositional variables, (ii) countable set $\mathbf{V} = \{\mathbf{P}_i\}_{i\in\mathbb{N}}$ of a non clalassical propositional variables, (iii) the connectives $\neg_w, \neg_{\mathbf{s}}, \wedge, \vee, \rightarrow$ and (iv) the parentheses (,).

Remark.3.1.We distinguish a weak negation $\neg_w$ and a strong negation $\neg_{\mathbf{s}}$.

The definition of formula is the usual. We denote the set of the all formulae of $\mathbf{LP}^{\#}_{\boldsymbol{\omega}}\left[\hat{\mathbf{V}},\hat{\boldsymbol{\Delta}}\right]$ by $\mathcal{F}^{\#}_{\boldsymbol{\omega}}$ where $\hat{\mathbf{V}}$ and $\hat{\boldsymbol{\Delta}}$ is a given subsets of $\mathcal{F}^{\#}_{\boldsymbol{\omega}}$. We assume through that $\hat{\mathbf{V}} \subset \hat{\boldsymbol{\Delta}} \subsetneq \mathcal{F}^{\#}_{\boldsymbol{\omega}}$.

$\mathbf{A}, \mathbf{B}, \mathbf{C},$ ... will be used as metalanguage variables which indicate formulas of $\mathbf{LP}^{\#}_{\boldsymbol{\omega}}\left[\hat{\mathbf{V}},\hat{\boldsymbol{\Delta}}\right]$.

Definition 3.1. (i) $\alpha^{(n)}$ stands for $\alpha^{(n-1)} \wedge (\alpha^{(n-1)})^{(1)}$, where $\alpha^{(0)} \triangleq \alpha, \alpha^{(1)} \triangleq \neg_w(\alpha \wedge \neg_w\alpha), 1 \leq n < \boldsymbol{\omega}$.

(ii) $\alpha^{(\boldsymbol{\omega})}$ stands for $\forall n[\alpha^{(n)}]$.

(iii) the finite n-order of the level of a weak consistency (w-consistency) is: $\alpha^{(0)} \triangleq \alpha, \alpha^{(n)}, 1 \leq n < \boldsymbol{\omega}$.

(iv) the ifinite $\boldsymbol{\omega}$-order of level of a weak consistency (w-consistency) is : $\alpha^{(\boldsymbol{\omega})}$.

Definition 3.2. (i) $\alpha^{[n]}$ stands for $\alpha^{[n-1]} \wedge (\alpha^{[n-1]})^{[0]}$, where $\alpha^{[0]} \triangleq \alpha \wedge \neg_w\alpha, 1 \leq n < \boldsymbol{\omega}$.

(ii) $\alpha^{[\boldsymbol{\omega}]}$ stands for $\forall n[\alpha^{[n]}]$.

(iii) the finite n-order of the level of a weak inconsistency (w-inconsistency) is: $\alpha^{[n]}, 1 \leq n < \boldsymbol{\omega}$.

(iv) the ifinite $\boldsymbol{\omega}$-order of the level of a weak inconsistency (w-inconsistency) is: $\alpha^{[\boldsymbol{\omega}]}$.

I. **Logical postulates**:

(1) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A})$,

(2) $(\mathbf{A} \rightarrow \mathbf{B}) \rightarrow ((\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow (\mathbf{A} \rightarrow \mathbf{C}))$,

(3) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B})$,

(4) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{A}$,

(5) $\mathbf{A} \wedge \mathbf{B} \rightarrow \mathbf{B}$,

(6) $\mathbf{A} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(7) $\mathbf{B} \rightarrow (\mathbf{A} \vee \mathbf{B})$,

(8) $(\mathbf{A} \rightarrow \mathbf{C}) \rightarrow ((\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{A} \vee \mathbf{B} \rightarrow \mathbf{C}))$,

(9) $\mathbf{P}_i \wedge \neg_w\mathbf{P}_i$ **iff** $\mathbf{P}_i \in \mathbf{V}, i = 1, 2, \ldots,$

(10) $\mathbf{P}_i^{[n]}$ **iff** $\mathbf{P}_i \in \mathbf{V}, i = 1, 2, \ldots; 1 \leq n < \boldsymbol{\omega}$,

(11) $\mathbf{A} \vee \neg_w\mathbf{A}$ **if** $\mathbf{A} \notin \hat{\mathbf{V}} = \bigcup_{\mathbf{k}\in\mathbb{N}} \mathbf{V}^{[\mathbf{k}]}$,

(12) $\mathbf{A} \vee \neg_{\mathbf{s}}\mathbf{A}$ **if** $\mathbf{A} \in \mathcal{F}^{\#}_{\boldsymbol{\omega}}$,

(14) $\mathbf{B} \rightarrow (\neg_{\mathbf{s}}\mathbf{B} \rightarrow \mathbf{A})$ **if** $\mathbf{A}, \mathbf{B} \in \mathcal{F}^{\#}_{\boldsymbol{\omega}}$,

(15) $\mathbf{A} \vee \neg_w \mathbf{A} \vee \mathbf{A}^{[1]} \vee \mathbf{A}^{[2]} \underbrace{\vee \ldots \vee}_{n} \mathbf{A}^{[n]}$ **if** $\mathbf{A} \in \mathcal{F}^{\#}_{\boldsymbol{\omega}}, 1 \leq n < \boldsymbol{\omega}$,

(16) $\mathbf{B} \to (\neg_w \mathbf{B} \to \mathbf{A})$ **if** $\mathbf{B} \notin \hat{\mathbf{V}} = \bigcup_{\mathbf{k} \in \mathbb{N}} \mathbf{V}^{[\mathbf{k}]}$.

II.Rules of a conclusion:

Restricted Modus Ponens rule $\mathbf{MPR}$:

$\mathbf{A}, \mathbf{A} \to \mathbf{B} \vdash \mathbf{B}$ **iff** $\mathbf{A} \notin \hat{\mathbf{V}}$.

Unrestricted Modus Tollens rule: $\mathbf{P} \to \mathbf{Q}, \neg_w \mathbf{Q} \vdash \neg_w \mathbf{P}; \mathbf{P} \to \mathbf{Q}, \neg_{\mathbf{s}} \mathbf{Q} \vdash \neg_{\mathbf{s}} \mathbf{P}$.

The rule of a strong contradiction: $\mathbf{A} \wedge \neg_{\mathbf{s}} \mathbf{A} \vdash \mathbf{B}$.

III.Quantification

Corresponding to the propositional paraconsistent relevant logic $\mathbf{LP}^{\#}_{\boldsymbol{\omega}}[\hat{\mathbf{V}}]$ we construct the corresponding paraconsistent relevant first-order predicate calculus. These new calculus will be denoted by $\overline{\mathbf{LP}}^{\#}_{\boldsymbol{\omega}}[\hat{\mathbf{V}}]$.

The postulates of $\overline{\mathbf{LP}}^{\#}_{\boldsymbol{\omega}}[\hat{\mathbf{V}}]$ are those of $\mathbf{LP}^{\#}_{\boldsymbol{\omega}}[\hat{\mathbf{V}}]$ (suitably adapted) plus the following:

(I) $\dfrac{\alpha \to \beta(x)}{\alpha \to \forall x \beta(x)}$,

(II) $\forall x \alpha(x) \to \alpha(y)$,

(III) $\alpha(x) \to \exists x \alpha(x)$,

(IV) $\dfrac{\alpha(x) \to \beta}{\exists x \alpha(x) \to \beta}$,

(V) $\forall x(\alpha(x))^{(n)} \to (\forall x \alpha(x))^{(n)}, 1 \leq n < \boldsymbol{\omega}$,

(VI) $\forall x((\alpha(x))^{(n)} \to (\exists x \alpha(x))^{(n)}, 1 \leq n < \boldsymbol{\omega}$,

(VII) $\forall x(\alpha(x))^{[n]} \to (\forall x \alpha(x))^{[n]}, 1 \leq n < \boldsymbol{\omega}$,. .

From the calculus $\overline{\mathbf{LP}}^{\#}_{\omega}[\hat{\mathbf{V}}]$,we can construct the following predicate calculus with equality.This is done by adding to their languages the binary predicates symbol of strong equality $(\bullet = \bullet)$ or $(\bullet =_{\mathbf{s}} \bullet)$ and weak equality$(\bullet =_w \bullet)$ with suitable modifications in the concept of formula, and by adding the following postulates:

(IX) $\forall x(x =_{\mathbf{s}} x)$,

(X) $\forall x[(x =_{\mathbf{s}} x)^{[1]} \vdash \mathbf{B}]$,

(XI) $\forall x \forall y[x =_{\mathbf{s}} y \to (\alpha(x) \leftrightarrow \alpha(y))]$,

(XII) $\forall x \forall y \forall z[(x =_{\mathbf{s}} y) \wedge (y =_{\mathbf{s}} z) \to x =_{\mathbf{s}} z]$,

(XIII) $\forall y \exists x(x =_w x)^{[n]}, 0 \leq n < \boldsymbol{\omega}$,

(XIV) $\forall x \forall y\left[(x =_w y)^{[n]} \leftrightarrow \forall \alpha(\circ)(\alpha^{[n]}(x) \leftrightarrow \alpha^{[n]}(y))\right], 1 \leq n < \boldsymbol{\omega}$,

(XV) $\forall x \forall y \forall z\left[(x =_w y)^{[n]} \wedge (y =_w z)^{[n]} \to (x =_w z)^{[n]}\right], 0 \leq n < \boldsymbol{\omega}$,

(XVI) $\forall x \forall y \forall z\left[(x =_w y)^{[n]} \wedge (y =_{\mathbf{s}} z) \to (x =_w z)^{[n]}\right], 0 \leq n < \boldsymbol{\omega}$,

(XVII) $\forall x \forall y \forall z\left[(x =_{\mathbf{s}} y) \wedge (y =_w z)^{[n]} \to (x =_w z)^{[n]}\right], 0 \leq n < \boldsymbol{\omega}$,

(XVIII) $\forall y \exists x(y =_w x)^{[n]}, 0 \leq n < \boldsymbol{\omega}$.

IV. Paraconsistent set theory $\mathbf{KSth}_{\boldsymbol{\omega}}^{\#}$

Cantor’s "naive" set theory **KSth** was based mainly on two fundamental principles: the postulate of extensionality (if the sets x and y have the same elements, then they are equal), and the postulate of comprehension or separation (every property determines a set, composed of the objects that have this property). The latter postulate, in the standard (first-order) language of set theory, becomes the following schema of formulas:

$$\exists y \forall x(x \in y \leftrightarrow F(x,y)). \tag{4.1}$$

Now, it is enough to replaces the formula $F(x,y)$ in (4.1) by $x \notin x$ to derive Russell’s paradox. That is, the principle of comprehension (4.1) entails an inconsistency. Thus, if one adds (4.1) to classical first-order logic, conceived as the logic of a set-theoretic language, a trivial theory is obtained.

Remark.4.1.We distinguish a weakly inconsisten membership relation $(\circ \in_w \circ)$ and a strongly consisten membership relation$(\circ \in_{\mathbf{s}} \circ)$.

Definition 4.1. (i) the minimal order of the level of a weak consistency (w-consistency) is: $\alpha^{(1)} \triangleq \alpha^{(0)} \wedge \neg_w(\alpha^{(0)} \wedge \neg_w \alpha^{(0)}), \alpha^{(0)} \triangleq \alpha = (x \in_w y)$;

(ii) the minimal order of the level of a weak inconsistency (w -inconsistency) is: $\alpha^{[1]} \triangleq (\alpha^{[0]} \wedge \neg_w \alpha^{[0]}), \alpha^{[0]} \triangleq \alpha = (x \in_w y)$.

Definition 4.2.(i) $x \in_{w,(n)} y$ is to stands for $(x \in_w y)^{(n)}$ and is to mean "x is a weakly consistent member of y of the n-order (of the n-level) of w-consistency ".

(ii) $x \in_{w,[n]} y$ is to stands for $(x \in_w y)^{[n]}$ and is to mean "x is a weakly inconsistent member of y of the n-order (of the n-level) of w-inconsistency ".

Definition 4.1. An $\overline{\mathbf{LP}}_1^{\#}$ wff Φ is a w-wff iff it does not contain the connective: $\neg_w$.
We now replace the formula (4.1) by formulae

$$\exists y \forall x (x \in_{w,(n)} y \leftrightarrow F(x,y)) , \quad n = 0,1,2,\ldots \tag{4.2}$$

and

$$\exists y \forall x (x \in_{w,[n]} y \leftrightarrow F(x,y)), \quad n = 0,1,2,\ldots . \tag{4.3}$$

Theorem 4.1. (1) The collections $\Re_n \triangleq \forall x[(x \in_{w,(n)} \Re_n) \leftrightarrow [\neg_w (x \in_{w,(n)} x)]]$ is contradictory of the $n+1$-order of w-inconsistency.
(2) The collections $\Re_n \triangleq \forall x[(x \in_{w,[n]} \Re_n) \leftrightarrow [\neg_w (x \in_{w,[n]} x)]]$ is contradictory of the $n+1$-order of w-inconsistency.
Theorem 4.2. (1) The collection $\Re_{\boldsymbol{\omega}} \triangleq \forall x \forall n[(x \in_{w,(n)} \Re_{\boldsymbol{\omega}}) \leftrightarrow [\neg_w (x \in_{w,(n)} x)]]$ is contradictory of the $\boldsymbol{\omega}+1$-order of w-inconsistency.
(2) The collection $\Re_{\boldsymbol{\omega}} \triangleq \forall x \forall n[(x \in_{w,[n]} \Re_{\boldsymbol{\omega}}) \leftrightarrow [\neg_w (x \in_{w,[n]} x)]]$ is contradictory of the $\boldsymbol{\omega}+1$-order of w-inconsistency.

The standard non-classical response to these paradoxes is to find fault with the *logical and deduction* principles involved in the deduction. Most standard approaches to the paradoxes take them to be important lessons in the behaviour of a Boolean negation.

However if you wish to define negation non-classically, there are many options available.You can define negation inferentially, taking $\mathbf{A}$ to mean that if $\mathbf{A}$,then something absurd follows,or it can be defined by way of the equivalence between the truth of $\sim\mathbf{A}$ and the falsity of $\mathbf{A}$, and allowing truth and falsity to have rather more independence from one another than is usually taken to be the case: say, allowing statements to be neither true nor false, or both true and false. The former account takes truth as primary, and defines negation in terms of a rejected proposition and implication.

For example, one can to define a strong negation $\sim_{\mathbf{s}} A$ non-classically [16]:

$$\sim_{\mathbf{s}}\mathbf{A} \triangleq \mathbf{A} \to \forall x \forall y[(x \in_w y) \wedge (x =_{\mathbf{s}} y)]. \tag{4.4}$$

Theorem 4.3. The collection $\Re_{\sim_{\mathbf{s}}}$ such that $[x \in_w \Re_{\sim_{\mathbf{s}}} \leftrightarrow [\sim_{\mathbf{s}}(x \in_w x)]]$ i.e.,

$\Re_{\sim_s} \triangleq \hat{x}[\sim_{\mathbf{s}}(x \in_w x)]$ is contradictory.
Proof. Replace $F(x,y)$ in the axiom schema of abstraction (4.2) in the definition of collection by $\sim_{\mathbf{s}}(x \in_w x)$, so that the implicit definition of $\Re_{\sim_s}$ becomes

$$x \in_w \Re_{\sim_s} \leftrightarrow [\sim_{\mathbf{s}}(x \in_w x)]. \quad (4.5)$$

Instantiating in (4.5) x by $\Re_{\sim_s}$ then by unrestricted modus pones $\mathbf{MP}$, we obtain:
(1) $\vdash \Re_{\sim_s} \in_w \Re_{\sim_s} \leftrightarrow \sim_{\mathbf{s}}(\Re_{\sim_s} \in_w \Re_{\sim_s})$.

By unrestricted modus pones $\mathbf{MP}$ one obtain the contradiction

(2) $\vdash \Re_{\sim_s} \in_w \Re_{\sim_s} \wedge \sim_{\mathbf{s}}(\Re_{\sim_s} \in_{\mathbf{w}} \Re_{\sim_s})$.

Thus, if we adds (4.2)-(4.3) to first-order logic $\overline{\mathbf{LP}}^{\#}_{\omega}[\hat{\mathbf{V}}, \hat{\mathbf{\Delta}}]$, conceived as the logic of a set-theoretic language with suitable adapted $\hat{\mathbf{V}}$ and $\hat{\mathbf{\Delta}}$ a nontrivial paraconsistent set theory $\mathbf{KSth}^{\#}_{\boldsymbol{\omega}}$ is obtained.

V. Generalized Incompleteness Theorems.

5.1. Let **Th** be some fixed, but unspecified, paraconsistent, i.e. inconsistent but nontrivial formal theory and in these case we wrote **PTh** or $Pcon(\mathbf{PTh})$ instead **Th**. For later convenience, we assume that the encoding is done in some fixed consistent formal theory **S** and that **PTh** contains **S**. We do not specify **S** — it is usually taken to be a formal system of arithmetic, although a weak set theory is often more convenient. The sense in which **S** is contained in **PTh** is better exemplified than explained: If **S** is a formal system of arithmetic and **PTh** is, say, $\widetilde{\mathbf{ZF}}_n, 1 \leqslant n < \omega$ or $\mathbf{ZFC}^{\#}$ then **PTh** contains **S** in the sense that there is a well-known embedding, or interpretation, of **S** in **PTh**.Since encoding is to take place in **S**, it will have to have a large supply of constants and closed terms to be used as codes. (E.g. in formal arithmetic, one has $\overline{0}, \overline{1}$,... .) **S** will also have certain function symbols to be described shortly.

To each formula Φ, of the language of **PTh** is assigned a closed term, $[\Phi]^c$, called the code of Φ. [N.B. If $\Phi(x)$ is a formula with free variable x, then $[\Phi(x)]^c$ is a closed term encoding the formula $\Phi(x)$ with x viewed as a syntactic object and not as a parameter.] Corresponding to the logical connectives and quantifiers are function symbols, $\mathbf{neg}(\cdot)$, $\mathbf{imp}(\cdot)$, etc., such that, for all formulae Φ, Ψ : **S** $\vdash$

$\mathbf{neg}_{(n)}([\Phi]^c) = [\neg_{(n)}\Phi]^c$, **S** $\vdash \mathbf{imp}([\Phi]^c,[\Psi]^c) = [\Phi \to \Psi]^c$, etc. Of particular importance is the substitution operator sub, represented by the function symbol $\mathbf{sub}(\bullet,\bullet)$. For formulae $\Phi(x)$, terms t with codes $[t]^c$:

$$\mathbf{S} \ \vdash \mathbf{sub}([\Phi(x)]^c,[t]^c) = [\Phi(t)]^c. \tag{5.1}$$

Iteration of the substitution operator sub allows one to define function symbols $\mathbf{sub}_3$, $\mathbf{sub}_4,\ldots,$ such that

$$\mathbf{S} \ \vdash \mathbf{sub}_n([\Phi(x_1,x_2,\ldots,x_n)]^c,[t_1]^c,[t_2]^c,\ldots,[t_n]^c) = [\Phi(t_1,t_2,\ldots,t_n)]^c. \tag{5.2}$$

It well known [17] that one can also encode derivations and have a binary relation $\mathbf{Prov_{Th}}(x,y)$ (read "x proves y " or "x is a proof of y") such that for closed t_1,t_2 :

S $\vdash \mathbf{Prov_{Th}}(t_1,t_2)$ iff t_1 is the code of a derivation in **PTh** of the formula with code t_2.It follows that

$$\mathbf{PTh} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Prov_{PTh}}(t,[\Phi]^c) \tag{5.3}$$

for some closed term t.

Definition 5.1.Thus one can define

$$\mathbf{Pr_{PTh}}(y) \leftrightarrow \exists x \mathbf{Prov_{PTh}}(x,y), \tag{5.4}$$

and therefore one obtain a predicate asserting provability.

Remark 5.1. We note that it is not always the case that :

$$\mathbf{PTh} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Pr_{PTh}}([\Phi]^c). \tag{5.5}$$

It well known [17] that the above encoding can be carried out in such a way that the following important conditions $\mathbf{D}1,\mathbf{D}2$ and $\mathbf{D}3$ are met for all sentences [17]:

$$\mathbf{D}1.\mathbf{PTh} \vdash \Phi \text{ implies } \mathbf{S} \vdash \mathbf{Pr_{PTh}}([\Phi]^c),$$

$$\mathbf{D}2.\mathbf{S} \vdash \mathbf{Pr_{PTh}}([\Phi]^c) \to \mathbf{Pr_{PTh}}([\mathbf{Pr_{PTh}}([\Phi]^c)]^c), \quad (5.6)$$

$$\mathbf{D}3.\mathbf{S} \vdash \mathbf{Pr_{PTh}}([\Phi]^c) \wedge \mathbf{Pr_{PTh}}([\Phi \to \Psi]^c) \to \mathbf{Pr_{PTh}}([\Psi]^c).$$

Generalized Incompleteness Theorems depend on the following.

Theorem 5.1. (Diagonalization Lemma). Let $\Phi(x)$ in the language of $\mathbf{PTh}$ have only the free variable indicated. Then there is a sentence ψ such that

$$\mathbf{S} \vdash \psi \leftrightarrow \Phi([\psi]^c). \quad (5.7)$$

Proof. Given $\Phi(x)$, let $\vartheta(x) \leftrightarrow \Phi(\mathbf{sub}(x,x))$ be the diagonalization of $\Phi(x)$. Let $m = [\vartheta(x)]^c$ and $\psi = \vartheta(m)$. Then we claim that $\mathbf{S} \vdash \psi \leftrightarrow \Phi([\psi]^c)$. For $\Phi(x)$ in **S**, we see that

$$\psi \leftrightarrow \vartheta(m) \leftrightarrow \Phi(\mathbf{sub}(m,m)) \leftrightarrow \Phi(\mathbf{sub}([\vartheta(x)]^c,m)) \leftrightarrow \Phi([\vartheta(m)]^c) \leftrightarrow \Phi([\psi]^c). \quad (5.8)$$

We apply now (5.7) to $\neg_{(n)}\mathbf{Pr_{Th}}(x)$.

Theorem 5.2. (Generalized First Incompleteness Theorem).Let (1) $Pcon_{(n)}(\mathbf{Th})$ and (2) $\mathbf{Th} \vdash \phi \leftrightarrow \neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c)$.

Then (**i**)

$$\mathbf{Th} \nvdash \phi, \quad (5.9)$$

(**ii**) under an additional assumption

$$\mathbf{Th} \nvdash \neg_{(n)}\phi. \quad (5.10)$$

Proof. (i) Observe $\mathbf{Th} \vdash \phi$ implies $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\phi]^c)$ by $\mathbf{D}1$, which implies $\mathbf{Th} \vdash \neg_{(n)}\phi$, contradicting the paraconsistency of $\mathbf{Th}$.

(ii) The additional assumption is a strengthening of the converse to $\mathbf{D}1,$ namely $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\phi]^c)$ implies $\mathbf{Th} \vdash \phi.$ We have $\mathbf{Th} \vdash \neg_{(n)}\phi,$ hence $\mathbf{Th} \vdash \neg_{(n)}\neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c)$ so that $\mathbf{Th} \vdash \mathbf{Pr_{Th}}([\phi]^c)$ and, by the additional assumption, $\mathbf{Th} \vdash \phi,$ again contradicting the paraconsistency of $\mathbf{Th}$.

Theorem 5.3. (Generalized Second Incompleteness Theorem).

Let $Pcon_{(n)}(\mathbf{Th})$ be $\neg_{(n)}\mathbf{Pr_{Th}}([\Lambda_n]^c),$ where $\Lambda_n = \mathbf{A} \wedge \neg_{(n)}\mathbf{A}$ is any convenient n-contradictory statement. Then

$$\mathbf{Th} \nvdash Pcon_{(n)}(\mathbf{Th}). \tag{5.11}$$

Proof. Let ϕ be as in the statement of Theorem 5.2.. We show: $\mathbf{S} \vdash \phi \leftrightarrow Pcon_{(n)}(\mathbf{Th}).$ Observe that $\mathbf{S} \vdash \phi \to \neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c)$ implies $\mathbf{S} \vdash \phi \to \neg_{(n)}\mathbf{Pr_{Th}}([\Lambda_n]^c),$ since $\mathbf{S} \vdash \phi \to \Lambda_n$ implies $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\phi \to \Lambda_n]^c),$ by $\mathbf{D}1,$ which implies $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\Lambda_n]^c) \to \mathbf{Pr_{Th}}([\phi]^c),$ by $\mathbf{D}3.$ But $\phi \to \neg_{(n)}\mathbf{Pr_{Th}}([\Lambda_n]^c)$ is just $\phi \to Pcon_{(n)}(\mathbf{Th})$ and we have proven half of the equivalence. Conversely, by $\mathbf{D}2,$ $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\phi]^c) \to \mathbf{Pr_{Th}}([\mathbf{Pr_{Th}}([\phi]^c)]^c)$, which implies $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\phi]^c) \to\to \mathbf{Pr_{Th}}([\neg_{(n)}\phi]^c),$ by $\mathbf{D}1, \mathbf{D}3,$ since $\phi \to \neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c).$ This yields $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\phi \wedge \neg_{(n)}\phi]^c),$ by $\mathbf{D}1, \mathbf{D}3,$ and logic, which implies $\mathbf{S} \vdash \mathbf{Pr_{Th}}([\phi]^c) \to \mathbf{Pr_{Th}}([\Lambda_n]^c)$ by $\mathbf{D}1, \mathbf{D}3,$ and logic. By contraposition, $\mathbf{S} \vdash \neg_{(n)}\mathbf{Pr_{Th}}([\Lambda_n]^c) \to \neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c),$ which is $\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to \phi,$ by definitions.

Theorem 5.4. $\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to Pcon_{(n)}\left(\mathbf{Th} + \neg_{(n)}Pcon_{(n)}(\mathbf{Th})\right).$

Proof.By the proof of Theorem 5.3, (i) $\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to \neg_{(n)}\mathbf{Pr_{Th}}([\phi]^c),$

(ii) $\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \leftrightarrow \phi.$ Using now $\mathbf{D}2, \mathbf{D}3,$ it follows that

$\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to \neg_{(n)}\mathbf{Pr_{Th}}([Pcon_{(n)}(\mathbf{Th})]^c),$ so that

$$\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to \neg_{(n)}\mathbf{Pr_{Th}}([\neg_{(n)}Pcon_{(n)}(\mathbf{Th}) \to \Lambda_n]^c) \tag{5.12}$$

which gives $\mathbf{S} \vdash Pcon_{(n)}(\mathbf{Th}) \to Pcon_{(n)}\left(\mathbf{Th} + \neg_{(n)}Pcon_{(n)}(\mathbf{Th})\right).$

Definition 5.2. Define: (i)

$$\mathbf{Prov}^{\Re}_{\mathbf{Th}}(x,y) \leftrightarrow \mathbf{Prov}_{\mathbf{Th}}(x,y) \wedge$$
$$\wedge \forall z(w \leq x)[\mathbf{Prov}_{\mathbf{Th}}(z,w) \rightarrow y \neq neg_{(n)}(w) \wedge w \neq neg_{(n)}(y)] \tag{5.13}$$

(ii)

$$\mathbf{Pr}^{\Re}_{\mathbf{Th}}(y) \leftrightarrow \exists x \mathbf{Prov}^{\Re}_{\mathbf{Th}}(x,y) \tag{5.14}$$

and

(iii)

$$Pcon^{\Re}_{(n)}(\mathbf{Th}) \leftrightarrow \mathbf{Pr}^{\Re}_{\mathbf{Th}}([\Lambda_n]^c). \tag{5.15}$$

Theorem 5.5. (Generalized Rossers Theorem).Let (1) $Pcon_{(n)}(\mathbf{Th})$ and
(2) $\mathbf{Th} \vdash \phi \leftrightarrow \neg_{(n)}\mathbf{Pr}^{\Re}_{\mathbf{Th}}([\phi]^c)$.
Then (**i**)

$$\mathbf{Th} \nvdash \phi, \tag{5.16}$$

(**ii**)

$$\mathbf{Th} \nvdash \neg_{(n)}\phi. \tag{5.17}$$

(**iii**)

$$\mathbf{Th} \vdash Pcon^{\Re}_{(n)}(\mathbf{Th}). \tag{5.18}$$

Proof.(i) By the paraconsistency of $\mathbf{Th}$, $\mathbf{Prov}_{\mathbf{Th}}$ and $\mathbf{Prov}^{\Re}_{\mathbf{Th}}$ binumerate the same relation. Hence $\mathbf{D}1^{\mathfrak{R}}$ holds: $\mathbf{Th} \vdash \phi \Rightarrow \mathbf{Th} \vdash \mathbf{Pr}^{\mathfrak{R}}_{\mathbf{Th}}([\phi]^c)$. Thus, the proof of the first part of the First Incompleteness Theorem yields the result.

(ii) This follows from (iii).

(iii) Follows immediately from the remarks that $\mathbf{Th}$ is paraconsistent and $\mathbf{Th} \vdash \neg_{(n)}\Lambda_n$.

Theorem 5.6. (Generalized Löb's Theorem). Let be (1) $Pcon_{(n)}(\mathbf{Th})$ and (2) ϕ be closed. Then

$$\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\phi]^c) \to \phi \text{ iff } \mathbf{Th} \vdash \phi. \tag{5.19}$$

Proof. The one direction is obvious. For the other, assume that $\mathbf{Th} \nvdash \phi$. Then $\mathbf{Th} + \neg_{(n)}\phi$ is consistent and we may appeal to the Generalized Second Incompleteness Theorem to conclude that $\mathbf{Th} + \neg_{(n)}\phi$ does not yield $Pcon_{(n)}(\mathbf{Th} + \neg_{(n)}\phi)$, hence not $\neg_{(n)}\mathbf{Pr}_{\mathbf{Th}}([\phi \to \Lambda_n]^c)$. Thus $\mathbf{Th} + \neg_{(n)}\phi \nvdash \neg_{(n)}\mathbf{Pr}_{\mathbf{Th}}([\phi]^c)$. Contraposition yields $\mathbf{Th} \nvdash \mathbf{Pr}_{\mathbf{Th}}([\phi]^c) \to \phi$.

Let be $Pcon_{(n)}(\mathbf{Th})$. Now we focuses our attention on the following schemata:

(**I**) Generalized Local Reflection Principle $\mathbf{Rfn}(\mathbf{Th})$:

$$\mathbf{Pr}_{\mathbf{Th}}([\phi]^c) \to \phi,\ \phi \text{ closed.} \tag{5.20}$$

(**II**) Generalized First Uniform Reflection Principle $\mathbf{RFN}(\mathbf{Th})$:

$$\forall x \mathbf{Pr}_{\mathbf{Th}}([\phi(x)]^c) \to \forall x \phi(x),\ \phi(x) \text{ has only } x \text{ free.} \tag{5.21}$$

(**III**) Generalized Second Uniform Reflection Principle $\mathbf{RFN}'(\mathbf{Th})$:

$$\forall x[\mathbf{Pr}_{\mathbf{Th}}([\phi(x)]^c) \to x\phi(x)],\ \phi(x) \text{ has only } x \text{ free.} \tag{5.22}$$

Theorem 5.7. (Generalized First Incompleteness Theorem).Let be $Pcon_{(n)}(\mathbf{Th})$. Then for some true, unprovable ϕ

$$\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\phi]^c) \to \phi \qquad (5.23)$$

Theorem 5.8. (Generalized Second Incompleteness Theorem).Let be $Pcon_{(n)}(\mathbf{Th})$. Then for any refutable ϕ

$$\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\phi]^c) \to \phi \qquad (5.24)$$

Theorem 5.6 simply yields

$$\mathbf{Th} \nvdash \mathbf{Pr_{Th}}([\phi]^c) \to \phi \text{ iff } \mathbf{Th} \nvdash \phi, \qquad (5.25)$$

VI. Set theory $\mathbf{HST}_{\boldsymbol{\omega}}^{\#}$.

VI.1.Axiomatical system $\mathbf{HST}_{\boldsymbol{\omega}}^{\#}$, as inconsistent generalization of Hrbacek set theory $\mathbf{HST}$.

In this chapter we introduces $\mathbf{HST}_{\boldsymbol{\omega}}^{\#}$, inconsistent generalization of Hrbacek set theory $\mathbf{HST}$ and describes the basic structure of the $\mathbf{HST}_{\boldsymbol{\omega}}^{\#}$ set universe. Syntactically, $\mathbf{HST}_{\boldsymbol{\omega}}^{\#}$ is a theory in the $\mathbf{st_s}$-$\in_\mathbf{s}$ -$\mathbf{st}_w$-$\in_w$ -language, which contains: (1) a binary consistent predicate of strong or consistent membership $\in_\mathbf{s}$ and consistent unary predicate of strong or consistent standardness $\mathbf{st_s}$ (and strong or consistent equality $=_\mathbf{s}$ of course) as the consistent primary notions and (2) a binary inconsistent predicate of weak or inconsistent membership $\in_w$ and inconsistent unary predicate of weak or inconsistent standardness $\mathbf{st}_w$ (and weak or inconsistent equality $=_w$ of course) as the inconsistent primary notions. Formula $x \in_w y$ reads: x weakly belongs to y, or x is an weak element of y, with the usual set theoretic understanding of inconsistent membership. The formula $\mathbf{st}_w x$ reads: x is a weakly standard, its meaning will be explained below. A $\mathbf{st_s}$-$\in_{\mathbf{s}}$ -$\mathbf{st}_w$-$\in_w$ -formula is a formula of the $\mathbf{st_s}$-$\in_{\mathbf{s}}$ -$\mathbf{st}_w$-$\in_w$ -language. An $\in_w$ -formula is a formula of the $\in_w$ -language having $\in_w$ as the only atomic predicate. Thus an $\in_w$ -formula is a $\mathbf{st}_w$-$\in_w$ -formula in which the standardness predicate does not occur. $\in_w$ -formulas are also called weak internal formulas, in opposition to weak external formulas, i.e., those $\mathbf{st}_w$-$\in_w$ -formulas containing $\mathbf{st}_w$.

VI.2. The universe of $\mathbf{HST}^{\#}_{\omega}$

Inconsistent set theory $\mathbf{HST}^{\#}_{\omega}$ deals with eight major types of sets: (i) strongly external or $\mathbf{s}$-external,(ii) strongly internal or $\mathbf{s}$-internal, (iii) strongly standard or $\mathbf{s}$-standard, (iv) strongly well-founded or $\mathbf{s}$-well-founded,(v) weakly external or w-external,(vi) weakly internal or w-internal, (vii) weakly well-founded or w-well-founded.

First of all, strongly standard sets are those consistent sets x which satisfy $\mathbf{st_s}x$ and weakly standard sets are those inconsistent sets x which satisfy $\mathbf{st}_w x$.Strongly internal sets are those consistent sets y which satisfy $\mathbf{int_s}y,$ where $\mathbf{int_s}y$ is the formula $\exists^{\mathbf{st_s}}x(y \in_{\mathbf{s}} x) \equiv \exists x[\mathbf{st_s}x \wedge (y \in_{\mathbf{s}} x)]$ (saying: y strongly belongs to a strongly standard set), weakly internal sets are those inconsistent sets y which satisfy $\mathbf{int}_w y,$ where $\mathbf{int}_w y$ is the formula $\exists \mathbf{st}_w x(y \in_w x)$ (saying: y weakly belongs to a weakly standard set). Thus,

(**i**) $\mathbf{S_s} = \{x : \mathbf{st_s}x\}_{\mathbf{s}}$ is the class of all consistent standard sets,

(**ii**) $\mathbf{I_s} = \{y : \mathbf{int_s}y\}_{\mathbf{s}} = \{y : \exists^{\mathbf{st_s}}x(y \in_{\mathbf{s}} x)\}_{\mathbf{s}}$ is the class of all consistent internal sets,

(**iii**) $\mathbf{S}_w = \{x : \mathbf{st}_w x\}_w$ is the class of all inconsistent standard sets,

(**iv**) $\mathbf{I}_w = \{y : \mathbf{int}_w y\}_w = \{y : \exists^{\mathbf{st}_w}x(y \in_w x)\}_w$ is the class of all inconsistent internal sets,

(**v**) $\mathbf{S}^{\#} = \mathbf{S_s} \cup_{\mathbf{s}} \mathbf{S}_w = \{x : \mathbf{st_s}x\}_{\mathbf{s}} \cup_{\mathbf{s}} \{x : \mathbf{st}_w x\}_w$ is the class of all consistent and inconsistent standard sets,

(**vi**) $\mathbf{I}^{\#} = \mathbf{I_s} \cup_{\mathbf{s}} \mathbf{I}_w = \{y : \mathbf{int_s}y\}_{\mathbf{s}} \cup_{\mathbf{s}} \{y : \mathbf{int}_w y\}_w$ is the class of all consistent and inconsistent internal sets.

The class $\mathbf{I_s}$ is the source of some typical objects of consistent "nonstandard” mathematics like consistent hyperintegers and consistent hyperreals, the class $\mathbf{I}_w$ is the source of some typical objects of inconsistent "nonstandard” mathematics like inconsistent hyperintegers and inconsistent hyperreals [],

Blanket agreement 1.1. Thus, internal sets are precisely all sets which are elements of consistent or inconsistent standard sets. This understanding of the notion of internality and the associated notions like

$\mathbf{I}^{\#}, \exists^{\mathbf{st_s}}, \exists^{\mathbf{st}_w}, \exists^{\mathbf{st}^{\#}} \equiv \exists^{\mathbf{st_s}} \vee \exists^{\mathbf{st}_w}, \forall^{\mathbf{st_s}}, \forall^{\mathbf{st}^{\#}} \equiv \forall^{\mathbf{st_s}} \wedge \forall^{\mathbf{st}_w}$ is default throughout this paper. All exceptions (e.g., when $\mathbf{IST}^{\#}_{\boldsymbol{\omega}}$ is considered) will be explicitly indicated.

External sets consistent and inconsistent, are simply all sets in the nonstandard universe of $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$. We shall use $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ to denote the class of all consistent and inconsistent external sets. Thus, $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ is the "universe of discourse", the universe of all sets considered by the theory, including the class $\mathbf{WF}^{\#}_{\boldsymbol{\omega}}$ of all well-founded sets. $\mathbf{WF}^{\#}_{\boldsymbol{\omega}}$ will satisfy all axioms of $ZFC^{\#}_{\boldsymbol{\omega}}$. The class $\mathbf{S}^{\#}$ of all standard sets {determined by the predicate st, as above) will be shown to be $\in_{\mathbf{s}}$ -$\in_{w}$ -isomorphic to $\mathbf{WF}^{\#}_{\boldsymbol{\omega}}$. In a sense, $\mathbf{S}^{\#}$ is an "isomorphic expansion" of $\mathbf{WF}^{\#}_{\boldsymbol{\omega}}$ into $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$. Given that $\mathbf{S}^{\#}$ is not transitive, $\mathbf{I}^{\#}$ arises naturally as the class of all elements of sets in $\mathbf{S}^{\#}$. It is viewed as an elementary extension of $\mathbf{S}^{\#}$ {in $\in_{\mathbf{s}}$ -$\in_{w}$ - language), and thereby also of $\mathbf{WF}^{\#}_{\boldsymbol{\omega}}$. Finally, $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ is a comprehensive universe in which all these classes coexist in a reasonable common set theoretic structure, with $\in_{\mathbf{s}}$ -$\in_{w}$ having the natural meaning in all mentioned universes.

VI.3. The axioms of the external inconsistent universe.

This group includes the $ZFC^{\#}_{\boldsymbol{\omega}}$ Extensionality, Pair, Union, Infinity axioms and the schemata of Separation and Collection (therefore also Replacement, which is a consequence of Collection, as usual) for all $\mathbf{st_s}$-$\in_{\mathbf{s}}$ $\mathbf{st}_w$-$\in_w$ -formulas or for all $\mathbf{st}_{\#}$-$\in_{\#}$ -formulas for short.

VI.4. Axioms for standard and internal sets

Notation 4.1. (1).Let quantifiers $\exists^{\mathbf{st_s}}, \forall^{\mathbf{st_s}}, \exists^{\mathbf{st}_w}$ and $\forall^{\mathbf{st}_w}$ be shortcuts meaning: there exists a

strongly standard..., for all strongly standard,there exists a weakly standard..., for all

weakly standard, ..., formally:

(i) $\exists^{\mathbf{st_s}} x\Phi(x)$ means $\exists x[\mathbf{st_s} x \wedge \Phi(x)]$, (ii) $\forall^{\mathbf{st_s}} x\Phi(x)$ means $\forall x[\mathbf{st_s} x \Rightarrow \Phi(x)]$,

(iii) $\exists^{\mathbf{st}_w} x\Phi(x)$ means $\exists x[\mathbf{st}_w x \wedge \Phi(x)]$, (iv) $\forall^{\mathbf{st}_w} x\Phi(x)$ means $\forall x[\mathbf{st}_w x \Rightarrow \Phi(x)]$.

Quantifiers $\exists^{\mathbf{int}}$ and $\forall^{\mathbf{int}}$ (meaning there exists an internal ... , for all internal ...) are introduced similarly. If g, is an E-formula then g,st, the relativization of g> to S, is the for- mula obtained by restriction of all quantifiers in g> to the class S, so that all occurrences of 3 x ... are changed to 3stx ... while all occurrences of V x ... are changed to ystx .... In other words, g,st says that g> is true in S. Rela-tivization g,int, which displays the truth of an e-formula g> in the universe 0, is defined similarly: the quantifiers 3, V change to 3int, yint. The following axioms specify the behaviour of standard and internal sets.

Notation 4.2.For all $\mathbf{st}_{\mathbf{s}}$-$\in_{\mathbf{s}}$ $\mathbf{st}_w$-$\in_w$ -formulas or for all $\mathbf{st}_{\#}$-$\in_{\#}$ - formulas for short.

$ZFC_{\omega}^{\mathbf{st}_{\#}}$: The collection of all formulas of the form g,st, where g> is an e- statement which is an axiom of $ZFC_{\boldsymbol{\omega}}^{\#}$. In other words, it is postulated that the universe $\mathbf{S}^{\#}$ is a $ZFC_{\omega}^{\#}$ universe. (Note that the $ZFC_{\omega}^{\#}$ axioms are assumed to be formulated as certain closed $\in_{\#}$ - formulas in this definition.) This is enough to prove the following statement:

Lemma 4.1. (1) $\mathbf{S}_{\mathbf{s}} \subseteq \mathbf{I}_{\mathbf{s}}$, (2) $\mathbf{S}_w \subseteq \mathbf{I}_w$.

Proof.(1) See [18] Lemma 1.1.3.

(2) Let $x \in_w \mathbf{S}_w$. The formula $\exists y(x \in_w y)$ is a theorem of $ZFC_{\omega}^{\#}$, therefore $[\exists y(x \in_w y)]^{\mathbf{st}_w}$ that is the formula $\exists^{\mathbf{st}_w} y(x \in_w y)$,is true. In other words, x is an element of a standard set, which means $x \in_w \mathbf{I}_w$.

1.Strong or Consistent Transfer (**s-Transfer**): $\Phi^{\mathbf{int}_{\mathbf{s}}} \Leftrightarrow \Phi^{\mathbf{st}_{\mathbf{s}}}$, where Φ is an arbitrary
closed $\in_{\mathbf{s}}$ -formula containing only consistent standard sets as parameters.
To be more exact, Consistent Transfer is the collection of all statements of the form
$\forall^{\mathbf{st}_{\mathbf{s}}} x_1 \ldots \forall^{\mathbf{st}_{\mathbf{s}}} x_n [\Phi^{\mathbf{int}_{\mathbf{s}}}(x_1,\ldots,x_n) \Leftrightarrow \Phi^{\mathbf{st}_{\mathbf{s}}}(x_1,\ldots,x_n)]$

2.Strong Consistent Transitivity of $\mathbf{I}_{\mathbf{s}}$: $\forall^{\mathbf{int}_{\mathbf{s}}} x \forall y(y \in_{\mathbf{s}} x \Rightarrow \mathbf{int}_{\mathbf{s}} y)$.

3.Consistent Regularity over $\mathbf{I}_{\mathbf{s}}$: For any non empty consistent set X there exists $x \in_{\mathbf{s}} X$ such that $x \cap_{\mathbf{s}} X \subseteq_{\mathbf{s}} \mathbf{I}_{\mathbf{s}}$. (The full Regularity of ZFC requires $x \cap_{\mathbf{s}} X = \varnothing_{\mathbf{s}}$.)

4.Consistent Standardization: $\forall X \exists^{\mathbf{st}_w} y(X \cap_{\mathbf{s}} \mathbf{S}_{\mathbf{s}} = \cap_{\mathbf{s}} \mathbf{S}_{\mathbf{s}})$. (Such consistent standard set
Y, unique by Consistent Transfer and Consistent Extensionality, is sometimes denoted by ${}^{\mathbf{S}_{\mathbf{s}}}X$.)

5.Weak Transfer (**w-Transfer**): $\Phi^{\mathbf{int}_w} \Rightarrow \Phi^{\mathbf{st}_w}$, where Φ is an arbitrary closed $\in_{\mathbf{s}}$ -$\in_w$ -formula containing only consistent and inconsistent standard sets as parameters.
To be more exact, Weak Transfer is the collection of all statements of the form
$\forall^{\mathbf{st}_{\mathbf{s}}} x_1 \ldots \forall^{\mathbf{st}_{\mathbf{s}}} x_n \forall^{\mathbf{st}_w} y_1 \ldots \forall^{\mathbf{st}_w} y_m [\Phi^{\mathbf{int}_{\mathbf{s}}}(x_1,\ldots,x_n;y_1,\ldots,y_m) \Rightarrow \Phi^{\mathbf{st}_w}(x_1,\ldots,x_n;y_1,\ldots,y_m)]$

6.Weak Transitivity of $\mathbf{I}_w$: $\forall^{\mathbf{int}_{\mathbf{s}}} x \forall y(y \in_w x \Rightarrow \mathbf{int}_w y)$.

7.Weak Regularity over $\mathbf{I}_w$: For any non empty consistent set X there exists $x \in_w X$ such that $x \cap_w X \subseteq_w \mathbf{I}_w$. (The full Regularity of ZFC requires $x \cap_w X = \varnothing_w$.)

8.Strictly Weak Regularity (**Strictly w-Regularity**): For any non empty inconsistent
set X there exists $x \in_w X$ such that $x \cap_w X =_w \varnothing_w \wedge \neg(x \cap_w X =_w \varnothing_w)$.

9.Weak Standardization (**w-Standardization**): $\forall X \exists^{\mathbf{st}_w} Y(X \cap_w \mathbf{S}_w =_w Y \cap_w \mathbf{S}_w)$.

9.Weak Standardization: $\forall X \exists^{\mathbf{st}_w} y(X \cap_w \mathbf{S}_w = \cap_w \mathbf{S}_w)$. (Such consistent standard set Y, unique by Consistent Transfer and Consistent Extensionality, is sometimes denoted by ${}^{\mathbf{s}_w}X$.)

Such inconsistent standard set Y, w-unique by w-Transfer and weak Extensionality,

is sometimes denoted by ${}^{\mathbf{s}_w}X$.

Remark 4.1. (**i**) w-Transfer can be considered as saying that: $\mathbf{I}_w$, the universe of all inconsistent internal sets, is an elementary extension of $\mathbf{S}_w$ in the $\in_{\mathbf{s}}$ -$\in_w$ -language. It fo llows, by $(ZFC^{\#}_{\omega})^{\mathbf{st}_w}$, that the class $\mathbf{I}_w$ of all inconsistent internal sets satifies $ZFC^{\#}_{\omega}$ (in the $\in_{\mathbf{s}}$ -$\in_w$ -language), in fact, we can replace $(ZFC^{\#}_{\omega})^{\mathbf{st}_w}$ by $(ZFC^{\#}_{\omega})^{\mathbf{int}_w}$, with relativization to $\mathbf{I}_w$, in the list of $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$ axioms. See also Theorem 1.3.9 below.

(**ii**) w-Transitivity of $\mathbf{I}_w$ postulates that: inconsistent internal sets to form the basement of the $\in_{\mathbf{s}}$ -$\in_w$ -structure of the universe $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$. This axiom is very important since it implies that some set operations in $\mathbf{I}_w$ retain their sense in the whole universe $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$.

(**iii**) w-Regularity over $\mathbf{I}_w$ organizes the $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$ set universe $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ in general case as a sort of hierarchy over the internal universe $\mathbf{I}_w$, in the same way as the w-Regularity axiom organizes the universe in the von Neumann w-hierarchy over the w-empty set $\varnothing_w$ in $ZFC^{\#}_{\omega}$.

(**iv**) Strictly w-Regularity organizes the $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$ set universe $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ in the von Neumann w-hierarchy over the w-empty set $\varnothing_w$,but in a strictly inconsistent sense only.

(**v**) w-Standardization postulates that $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$ does not contain collections of standard sets other than those of the form $S \cap_w \mathbf{S}_w$ for inconsistent standard set S.

Remark 4.2. It well known that the ZFC Regularity fails in $\mathbf{H} = \mathbf{H}_{\mathbf{s}}$: the set of all nonstandard $\mathbf{I}_{\mathbf{s}}$-natural numbers does not contain an $\in_{\mathbf{s}}$ -minimal element, (see for example [18], Exercise 1.2. 15(3)). In contrast with a classical case, $ZFC^{\#}_{\omega}$ w-Regularity valid in $\mathbf{H}^{\#}_{\boldsymbol{\omega}}$, but in a strictly inconsistent sense only. For example the set of all nonstandard $\mathbf{I}_w$-natural contain an inconsistent $\in_w$ -minimal element, see [22]-[23].

VII.5. Well-founded inconsistent sets.

Now we can introduce the last principal class: well-founded inconsistent sets. Recall the following notions from general inconsistent set theory.

Definition 5.1. (i) A binary weak relation $\prec_w$ on inconsistent set or inconsistent class X is a strictly well-founded if any nonempty set $Y \subseteq_w X$ contains consistent $\prec_w$ -minimal w-element $x^* \in_w Y$, that is there exists $x \in_w Y$ such that no $y \in_w Y$ satisfies $y \prec_w x$.

(ii) A binary weak relation $\prec_w$ on inconsistent set or inconsistent class X is weakly well- founded (or w-well-founded) if:

(1) $\prec_w$ is not a strictly well-founded and

(2) any nonempty set $Y \subseteq_w X$ contains a $\prec_w$ -minimal w-element $x^* \in_w Y$, that is there exists $y \in_w Y$ satisfies: $y \prec_w x \wedge x \prec_w y$, i.e. $y \prec_w x \wedge \neg(y \prec_w x)$.

(iii) Inconsistent set or inconsistent class X is w-transitive if any $x \in_w X$ satisfies $x \subseteq_w X$, i.e., weak elements of weak elements of X are weak elements of X.

(iv) Inconsistent set or inconsistent class X is w-complete if we have $y \in_w X$ whenever

$y \subseteq_w x \in_w X$, that is a weak subsets of weak elements of X are weak elements of X.

(v) Inconsistent set x is a strictly well-founded if there is a w-transitive set X such that $x \subseteq_w X$ and the restriction $\in_w \restriction X$ is a strictly well-founded weak relation.

(vi) Inconsistent set x is w-well-founded if there is a w-transitive set X such that $x \subseteq_w X$ and the restriction $\in_w \restriction X$ is a w-well-founded weak relation.

Remark 5.1. It is known that all sets are well-founded in ZFC by the Regularity axiom.

This is not the case in $\mathbf{HST}$: the set ${}^*\mathbb{N}$ of all $\mathbf{I_s}$-natural numbers is ill-founded [18].

Remark 5.2. In contrast with a classical case, all inconsistent sets are w-well-founded

in $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$ by the Strictly w-Regularity axiom. For example, the set ${}^{\#}\mathbb{N} = {}^*\mathbb{N}_{\mathbf{inc}}$ of all $\mathbf{I}_w$-natural numbers is w-well-founded by the Strictly Weak Regularity axiom.

Definition 5.2.($\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$). (i) Let $\mathbf{s\text{-}wf}_w x$ mean that x is a strictly well-founded. We put $\mathbf{s\text{-}WF}_w =_w \{x : \mathbf{s\text{-}wf}_w x\}_w$, the class of all strictly well-founded inconsistent sets and

(ii) let $w\text{-}\mathbf{wf}_w x$ mean that x is a w-well-founded. We put $w\text{-}\mathbf{WF}_w =_w \{x : w\text{-}\mathbf{wf}_w x\}_w$, the

class of all w-well-founded inconsistent sets.

Notation 5.1.We introduce quantifiers $\exists^{\mathbf{s\text{-}wf}_w}, \forall^{\mathbf{s\text{-}wf}_w}, \exists^{w\text{-}\mathbf{wf}_w}$ and $\forall^{w\text{-}\mathbf{wf}_w}$ (meaning: there is a well-founded ... , for any well-founded ...) and the relativization (1) $\Phi^{\mathbf{s\text{-}wf}_w}$ to $\mathbf{s\text{-}WF}_w$, (2) $\Phi^{w\text{-}\mathbf{wf}_w}$ to $w\text{-}\mathbf{WF}_w$ similarly to $\exists^{\mathbf{st_s}}, \exists^{\mathbf{st_s}}, \Phi^{\mathbf{st_s}}, \exists^{\mathbf{st}_w}, \exists^{\mathbf{st}_w}, \Phi^{\mathbf{st}_w}$ in §VII.1.3. In other words, $\Phi^{\mathbf{s\text{-}wf}_w}$ says that gj is true in WIF. The main property of the classes $\mathbf{s\text{-}WF}_w$ and $w\text{-}\mathbf{WF}_w$ in $\mathbf{HST}^{\#}_{\boldsymbol{\omega}}$ is that it admits a definable $\in_w$ -isomorphism $\mathbf{w} \mapsto {}^{\#}\mathbf{w}$ onto the class S of all standard sets.

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