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The Development of Mathematical Thought as Confirmation of Zubiri's Noology

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Introduction

Mathematics, the "Queen of the Sciences", has always had great influence on philosophical thought. From the time of the ancient Greeks, it has been regarded as an archetype of certainty, clarity, and rigor. Yet the nature of mathematics, the ontological status of its objects, and its relationship to reality, have never been fully clarified; and indeed theories about these difficult matters have changed dramatically over the past 2500 years. Zubiri tackled this problem and succeeded in integrating mathematics into his new synthesis by means of his philosophy based upon sentient intelligence. It is fortunate that Zubiri came to maturity in the mid-twentieth century, because it is difficult to see how his rethinking of mathematics could have been carried out prior to developments in mathematical thought which occurred in the last 100 years, specifically, the work of Gödel, Church, and Turing in mathematical logic and computability theory; Lebesgue, Hilbert, Banach and others in measure theory and functional analysis; Cantor and others in set theory; and more recently—perhaps not even known to Zubiri, but a confirmation of his theories—Mandelbrot and others in fractals and chaos theory. Zubiri's philosophy of mathematics has been comprehensively expounded in a recent publication by Guillerma Díaz Muñoz,[1] in which the author shows how Zubiri's philosophy was influenced by developments in the foundations of mathematics during the 20th century.[2] In this essay we will take the opposite approach, and examine how the development of mathematics has confirmed Zubiri's new philosophy of sentient intelligence. We shall begin by reviewing some significant developments in the history of mathematics over the past 2500 years (Part I), then briefly expound the nature of mathematics in Zubiri's philosophy (Part II), and finally (Part III) address three issues which confirm the thesis proposed here:

1. The nature of logic and formal systems.
2. Mathematics and the enlargement of the canon of reality.
3. Why mathematics can be used to describe reality.

I. Some major developments in the history of mathematics

Mathematics was not born in Greece, but it was in Greece that the idea of mathematics as a formal, deductive system had its origin.[3] For the Greeks, as for the rest of the ancient world, mathematics was almost synonymous with geometry—reflecting its origin, "earth measurement" = *geos* + *metros*. The lack of a suitable system of numbers meant that letters had to be pressed into service as numbers, and the resulting absence of a distinction between the two significantly retarded the development of algebra and other forms of analysis for all of the ancient world. For example, Archimedes came within a hair's breadth of inventing calculus, with his development of Eudoxus' Method of Exhaustion; but the notation at his disposal prevented him from taking the crucial step.[4]

Because of its application to earth measurement, geometry was seen as *both* truth about the world *and* the paradigm of mathematical knowledge; by extension, it became the paradigm for all rigorous knowledge. For Aristotle, the same was true of logic: it was about the world, as well as being a deductive system. Whether the Greeks ever clearly recognized the problems of this approach to mathematics is not clear; they seemed to be aware that there are no true circles or lines in the world, but not the full implications of this statement—namely, the need to justify how abstract knowledge such as mathematics can apply to the things of the world, and in particular, how truths deduced from the axioms of Euclidean geometry turn out to be truths about the things of the world. Plato and the Platonists wrestled with this problem, and attempted to solve it with their notion of "participation" of worldly circles and lines in the "ideal" form of circles and lines; but as no independent definition of "participation" could be devised, the basic problem remained. Aristotle and the Aristotelian tradition rejected this approach, and the independent existence of the forms (and universals in general) as substantives. Maziarz and Greenwood summarize the Aristotelian position as follows:

> Aristotle's conception of the status of mathematical objects is in line with his fundamental criticism of Plato's doctrines. Although both place them in an intermediate position, Plato considers them as a distinct class of objects between ideas and particulars, while Aristotle denies them a separate existence. They are intermediate for [Aristotle] insofar as the mind places them between the sensible things out of which they are abstracted and the generic essence of the things, which is reached by a further mental operation. In other words, Aristotle denies that mathematical objects are real substances, but he considers them as substantives in order to incorporate them as subjects in the various propositions of mathematics.[5]

According to these same authors, "...Aristotle combines abstraction and construction in order to give to mathematical objects their being, necessity, coherence, and applicability to natural phenomena."[6] Universals such as circles, squares, and triangles inform actual bodies, and thus determine the bodies' characteristics. Knowledge of these mathematical objects, indeed, only comes by abstraction from actual bodies. But this poses additional, serious problems: what if there is more than one possible mathematical description of the *same* physical object or phenomenon? What if some non-Euclidean geometry better describes the world than Euclidean geometry? In particular, the famous "parallel postulate" gave rise to much consternation, as it was related (erroneously) to the question of whether physical space is infinite.[7] As the potential for most of these difficulties to arise was only dimly perceived by the Greeks, if at all, it is not surprising that they did not come to grips with the problem of mathematics working in the world. And it is unclear that they even recognized the full significance of the problem; that would have to await many further advances in both science and mathematics.

The adoption of Arabic numerals in the late Middle Ages, followed by the development of algebra in the renaissance, and culminating in the invention of analytic geometry by Descartes in the 16th century and that of calculus by Newton and Leibniz in the 17th century, greatly expanded the range of mathematics. In the new picture, geometry played a smaller and smaller role, but remained the paradigm of knowledge, especially mathematical knowledge: rigorous deduction of truths from a small number of axioms. The peculiar nature of Euclid's fifth axiom or postulate, the so-called "parallel postulate", remained as an outstanding issue, though the hope was that it could ultimately be deduced from the other axioms. Despite this great expansion of mathematics, there was no real advance in the understanding of its fundamental relation to reality or the rest of knowledge.

Subsequently mathematics and logic were completely divorced from reality, with equally problematic results. That development occurred with the English empiricists, culminating in Hume's famous distinction between *relations of ideas* and *matters of fact*, with mathematics and logic, of course, falling into the former category. But this left Hume with an insoluble problem: how can mathematics give what appears to be *a priori* knowledge about the world? As a trivial example, if I have two cows in one barn, and three in another, I know that if I bring all of them into the meadow, I will have five cows there—which is a *matter of fact* in Hume's philosophy, but one known with certainty. Kant recognized the fundamental inadequacy of Hume's approach, and attempted to repair the split through his notion of space and time as intuitions, which had structure imposed upon them by the mind itself. Kant was thus the first to explicitly tackle the question of how abstract mathematical knowledge could be a necessary truth about the world, as Euclidean geometry still seemed to be in the late eighteenth century. Kant's well-known solution, that we make it so by synthesizing experience according to the categories and our intuitions of space and time, at least resolved the problem temporarily and allowed the world to go on believing that mathematics as a whole was a necessary truth about the world. This picture was soon shattered forever with the development of non-Euclidean geometry by Georg Riemann (1826-1866) and Nicolai Lobachevsky (1793-1856) in the mid-19th century.[8] Their explicit denial of the parallel postulate, and subsequent construction of consistent geometries based on other assumptions about parallel lines at infinity, revealed two fundamental errors in classical thought: (1) that Euclidean geometry is the only way to do geometry, and therefore is the paradigm for all of mathematics; and (2) that mathematics is necessary truth about the world. Now it became clear that mathematics exists independently of the world of sense experience, and that it must have a foundation that goes deeper than any particular branch of it.

At this point, things began to happen very fast. Indeed, only with the twentieth century, and the remarkable developments that it brought, would the problem start to come into focus. Probability and statistics, which began as the study of games of chance, gradually expanded to become a discipline of mathematics in its own right.[9] Of principal interest is the fact that probability theory, while a branch of mathematics, deals explicitly with *un*certainty rather than certainty. Though the mathematics of probability (proofs and reasoning) is rigorous, and consists of postulates and theorems, e.g., Bayes' theorem and the Law of Large Numbers, the conclusions are not at all like those of more traditional branches of mathematics: characterized by exactitude and certainty. There is a certain probability that x caused y, or A is the most likely source of B, or if one makes enough trials, one will get X percentage of successes. This was rather disconcerting, but rather than look at the abyss and hypothesize that perhaps *certainty* was a special case of *uncertainty*, the world preferred to go on believing that uncertainty was a special case of certainty, one in which we just do not have enough knowledge. If we could somehow get that knowledge, the uncertainty and the probabilities would disappear.

The second half of the 19th century also saw attention paid for the first time to the foundations of mathematics. There was the development of set theory under Georg Cantor (1845-1918), number theory under Guiseppe Peano (1858-1932) and Gottlob Frege (1848-1925). David Hilbert (1862-1943) forged the notion of the unity of mathematics as a rigorous, deductive system, one in which *all* of mathematics (not just a single branch such as geometry) could be deduced from a limited set of axioms. Alfred Norse Whitehead (1861-1947) and Bertrand Russell (1872-1970), sought to realize this goal in their monumental *Principia Mathematica* (1910-1913), wherein they tried to demonstrate that mathematics could be deduced from logic. This program was unexpectedly shattered in 1931 by Kurt Gödel (1906-1978), with his famous paper on the undecidability of any comprehensive formal system of mathematical reasoning such as that of Whitehead and Russell.[10] Gödel's work conclusively demonstrated that Hilbert's attempt to make mathematical truth synonymous with logical deduction from axioms was doomed to failure. The immediate implication is that truth is a broader notion than provability. There are also implications about reality, which Zubiri will draw out a few decades later.

While all of this was happening, work proceeded in another area of mathematics, now known as "measure theory". Henri Lebesgue (1875-1941) and others recognized that the need for mathematical statements to be true in every single case was excessively restrictive, and allowing for a "few" exceptions did not always change what one was trying to calculate. In fact, it permitted many more types of problems to be solved.[11] Not just any exceptions, of course, but those whose cumulative effect was negligible. If these exceptions collectively formed a very small set, one said to be "of measure zero", then calculations could proceed despite them. Thus, mathematical statements are said to be true "presque partout" in French, "almost everywhere" in English, or equivalently, "except on a set of measure zero". The number of exceptions, in fact, can be infinite, provided that their cumulative effect is zero. An easy example is that of calculating the area under Dirichlet's function, which is a function $f(x)$ defined as: $f = 1$ if x is rational, $f = 0$ if x is irrational, and $0 \leq x \leq 1$. This problem cannot be solved using the methods of traditional calculus, because Dirichlet's function is discontinuous at every point, and the usual limiting operations cannot be performed. However, by dismissing the set of points where f is rational, as they form a set of measure zero, the problem becomes trivial. Thus $f = 0$ "almost everywhere", and its integral is trivially equal to zero. This general approach allows mathematics to deal systematically with curves which, by traditional standards, are very badly behaved, that its, are discontinuous or non-differentiable everywhere. As it can be shown that "most" curves fall into this category, measure theory is a significant advance, but it comes at the price of renouncing the kind of certainty found in Euclidean geometry, for example. Significantly, rigorous foundation of probability theory as a branch of mathematics had to await the development of measure theory. Of course, the use of ideas such as "almost everywhere" is a significant departure from classical ideas of mathematics, wherein mathematics is identified with absolute certainty, and exceptions are not permitted.

Finally, another development related to the foregoing occurred over the last four decades or so, the emergence of fractals and chaos theory. Fractals are mathematical objects endowed with somewhat peculiar properties. Some of them are curves which are continuous everywhere, but differentiable nowhere. They can enclose a finite area, yet have infinite perimeter. They exhibit self-similarity on an infinite number of scales, and they have dimensions which are non-integer (e.g., normal lines are of dimension 1, planes of dimension 2, and 3-dimensional space dimension 3). Their dimensions are fractional, e.g., 1.25375...Closely related to fractals is chaos theory. Chaos theory deals with dynamical systems that exhibit chaotic behavior. Chaotic behavior is defined as behavior which is extremely complex and extremely sensitive to initial conditions. Slight changes in initial conditions quickly lead to very divergent states at subsequent times, quite unlike ordinary dynamical systems.

These developments enormously expanded our insight into the nature of mathematics, but at the same time revealed the inadequacy of earlier theories about mathematics and its relationship to the real world. As we shall see, it required Zubiri's new conception of reality to allow mathematics and logic be about reality once again, in a strict and rigorous sense, though with a much different meaning to 'reality'.

II. The Nature of Mathematics in Zubiri's Philosophy

Zubiri had a four-part goal with respect to his philosophy of mathematics. The goal was to:

(1) Make sense of the way mathematicians actually do mathematics (in postulating the existence of mathematical objects and in deducing mathematical truths).

(2) Explain the significance of Gödel's theorem with respect to the reality generated in the postulation process.

(3) Explain the nature of the reality of mathematical objects and our struggle to understand them.

(4) Explain how mathematics can apply to the world.

He could not achieve this ambitious goal without a radical rethinking of the nature of philosophy, and in particular, without the development of his noology, expounded in the three parts of *Sentient Intelligence*.

According to Zubiri, mathematical objects have two moments: one, sensed, as real; the other, freely created.[12] Only by *sensing* the mathematical is it possible to *do* mathematics. This sensing of the mathematical has to do with sensing the transcendental moment of reality itself. We sense the *reality* of mathematical objects just like sensible objects such as chairs; but their *content* is not sensible; rather, it is intelligible, the result of postulation. As Zubiri explains, reality is formality, not a zone of things; objects of mathematics have the same formality as ordinary objects. Thus, when a mathematician (or anyone else) speaks about the number ? or *e*, he is speaking about something which really exists, though neither he nor anyone else grasps the content of these transcendental irrational numbers through ordinary sense perception.

There are five important aspects of Zubiri's philosophy of mathematics:

(1) Mathematical truth, in the sense of proved theorems, is an approximation to mathematical reality, not in the sense of being inaccurate, but in the sense of being incomplete.

(2) All mathematical truth is empirical in the sense that there is no distinction between truths of reason and truths of fact with respect to mathematical objects. 'Empirical', in the case of mathematics, does not mean extra-mental experimentation and testing, as in physical science, nor does it mean that what has been rigorously proved could one day be shown to be false. Rather, it means physical probing of postulated reality. Deduction from axioms is, of course, one way of probing; there are other ways as well, such as use of calculation and computers to learn something about it the mathematical reality of interest. Prior to the recent proof of Fermat's Last Theorem by Andrew Wiles (1994), this area of number theory was probed in large measure by "empirical" means.

(3) Formal logic is not the foundation of mathematics, but is itself founded in the logic of the affirmative intellection of the real.

(4) The mathematical method is not ex-

clusively deduction but a mode of experience: physical corroboration of the real.

(5) The structure of the intelligence is the basis for the application of mathematics to reality (though not in any Kantian sense of synthesis according to categories).

Zubiri was faced with the challenge of making sense of the developments in the history of mathematics, and showing that they support his new philosophical synthesis better than other philosophical systems. To do this, he both explores the consequences of Gödel's theorem and draws out the implications of his own philosophy of the intelligence. In this paper we shall review his efforts and extend them to areas of mathematics which he did not discuss.

III. Mathematical Developments as Confirmation of Zubiri's Philosophy of Sentient Intelligence

A. Nature of Logic and Formal Systems: The Reality of Mathematical Objects and the Priority of Reality Over Truth

The first area of mathematics in which to look for confirmation of Zubiri's new philosophy is that of logic and formal systems. Here, Zubiri tackles head-on the question of the reality of mathematical objects, vis-à-vis his notion of reality. Simply stated, Gödel's theorem tells us that in any formal mathematical system, either there will be true statements which cannot be proved within the system, or else the system will be inconsistent, i.e., both true and false statements can be proved. Since the latter makes the system absolutely useless, it is the former which is of most interest. Historically, Gödel's result has been taken to mean that Hilbert's program (as well as that of Whitehead and Russell) of the complete formalization of mathematics is impossible. Zubiri accepts that, but takes his interpretation much further: the objects of mathematics have a certain reality, one which goes beyond that included in their postulation. In doing mathematics, we postulate mathematical entities, e.g., we say, "let X be a Hilbert space", or "let $\mathbf{P}_n$ be the set of all polynomials of degree n". It is upon this act that Zubiri focuses. We may go on to specify certain characteristics of the object(s) thus postulated, and then explore the consequences by proving theorems and carrying out other forms of investigation. Is this process ever complete, or can it be so? Mathematical thought prior to Gödel believed that it could—that was one motivation for the development of mathematical systems such as Whitehead and Russell's *Principia Mathematica*. It also allowed for nominalistic interpretations of mathematics, that is, considering mathematics as a symbol manipulation process only. There is no reason, under these interpretations, to doubt that all truths about mathematical objects can be known, at least in principle.

For Zubiri, Gödel's result means that the mathematical object, once created, has a reality, and a reality with properties *de suyo*; and this reality is not exhausted by the postulation, indeed, just the opposite. In other words, the reality of these objects goes far beyond the construction used, somewhat analogously to the fact that the reality of a building goes far beyond the architect's blueprints. As this reality includes what can be deduced about the object, the interpretation of Gödel's theorem is that it shows rigorously that the reality of things exceeds what we put into them by postulation. Or in other words, the reality of mathematical objects, like the reality of physical objects, is extremely rich, and cannot be fully captured in any formula (as classical philosophy mistakenly assumed with its notion of essence). Mathematical truth, therefore, is only an approximation to mathematical reality, which cannot be exhausted by formal methods. An additional, non-trivial implication of Zubiri's analysis is that it puts paid to all possible nominalistic interpretations of mathematics—which of course were never satisfactory to mathematicians anyway. Mathematics is about reality, not symbols, just as mathematicians have always believed.

Moreover, the nature of reason and logical inference itself—whose origin is almost never discussed—has its seat in reality as well:

> What is proper to reason or explanation is not evidence nor empirical or logical rigor; rather, it is above all the force of the impression of reality in accordance with which reality in depth is imposed coercively in sentient intellection. The rigor of a reasoning process does not go beyond the noetic expression of the force of reality, of the force with which reality is being impressed upon us, that reality in which we already are by impression.[13]

Ironically, mathematics for Zubiri has returned to being about reality, though in a radically different way than it was for the Greeks. Because reality is formality, and not a "zone of things", mathematical entities are real in the same sense as ordinary physical objects, though they do not exist in the same world as these objects since their content comes not from primordial apprehension, but from postulation. So it makes no sense to look for them in the physical world—how would one look for a Hilbert space there anyway? The vast expansion of the entities investigated by mathematicians, most of which have nothing to do with the world of our day-to-day life, meant that the Greek view had to be abandoned or radically modified.

Indeed, despite the impression often given in mathematics textbooks, that the subject matter came down along with the Ten Commandments with Moses from Mt. Sinai, much mathematical intuition and knowledge of what to try to prove comes from experimentation and the "cut and try" approach. For example, it is unlikely that anyone would have tried to prove Fermat's Last Theorem if they had not first tried to find some integers a, b, and c to fit his famous but simple equation, $a^n+b^n=c^n$, for $n > 2$. The problem of finding the area under the parabolic curve $y = x^2$ was first solved, according to legend, in an empirical fashion by Archimides. The author is currently engaged in research on a famous unsolved problem, the *Random Matrix Eigenvalue problem*, which has implications in many areas. This problem concerns the distribution of eigenvalues for an n^{th} order linear system, and in particular, the probability of obtaining all negative eigenvalues, which implies system stability. This problem is notoriously resistant to theoretical treatment, and all of our understanding of it has been through experimental probing using computers. What Zubiri is suggesting is that in the future, there may be more mathematics done in this fashion, as has been done over the past few decades in chaos theory utilizing computers, because the problems are not tractable theoretically. And this is a reflection of the underlying nature of mathematics as primarily *sensed* and *about reality*, rather than of limitations of the deductive method. The deductive method is wonderful when it can be applied; there are just many cases in mathematics where it cannot be applied, yet knowledge can still be obtained.

Reality is prior to truth: this is the first lesson from the history of mathematics which confirms Zubiri's new philosophy of sentient intellection.

B. Opening of New Frontiers in Mathematics: Expansion of the Canon of Reality

One of Zubiri's persistent criticisms of earlier philosophy is its attempt to force all of reality into a fixed framework, usually expressed in terms of categories. For Zubiri, this is too static to account for the progression of knowledge.

> ...progression is a search not just for new *things* but also for *new forms and new modes of reality*. Upon intellectively knowing the real in the field

> sense, we have not just intellectively known this or that thing, but also just what it is that we call 'real'.[14]

Mathematics illustrates this aspect of knowledge—and the truth of Zubiri's philosophy—quite well. As the brief historical survey indicated, mathematics has come a long way from Euclidean geometry, where clarity and certainty were paramount, indeed, they were the defining characteristics of knowledge in general and mathematics in particular.[15] Indeed, the historical trajectory of mathematics, with its ever-increasing horizons of things encompassed, confirms both Zubiri's notion of the expansible canon of reality, and of reality as fundamentally *open*. Mathematics has gradually expanded both the domain of things that it studies, and the generality with which it studies them, somewhat analogous to the way one's view of the world expands though the window of an airplane: each area which seemed to be everything becomes a small part of a larger whole as the plane gains altitude. So it is with mathematics: Euclidean geometry, which for classical thought, up to Kant, *was* mathematics, is now seen to be only a small piece of it. There have been changes in three directions: (1) we routinely talk about mathematical entities that are totally incomprehensible under the Euclidean paradigm, such as "spaces" of functions, Hilbert and Banach spaces which have infinite dimensions; (2) mathematical entities behave in totally unanticipated ways, such as famous notion of measure of set; and (3) even the type of certainty achievable in mathematics has changed from absolute to less that complete, when we discuss probability theory and prove theorems relying on statements which are only true "almost everywhere".

(1) The sheer number and variety of branches of mathematics, and correlatively, the objects of mathematics, is already staggering and likely to continue to grow. To pick just a few examples, from analysis there are complex numbers; from linear algebra there are matrices and vector spaces; from functional analysis, there are Hilbert and Banach spaces of functions, some of which have infinite dimensions. Likewise from number and set theory there are the transfinite numbers of Cantor. Obviously, each of these developments represents a significant amplification of our canon of mathematical reality.

(2) Similar remarks apply to the notion of *measure* itself within mathematics. A measure is a way of characterizing the "size" of a set, a generalization of the notion of length, if one wishes. Length is an example of what is know as a *set function*: a function which associates a real number to a set. We would like to be able to construct some set function m which assigns to each set E in a collection M of sets of real numbers a nonnegative number (possibly infinite), called the "measure of E" and denote as mE, with four seemingly intuitive properties: (a) it is defined for all sets of real numbers; (b) for a simple interval, it is the length of that interval; (c) the measure of a sequence of disjoint sets is the sum of the measures of the individual sets; and (d) it is translation invariant. Curiously—and counterintuitively—this cannot be done. As Royden explains, "…it is impossible to construct a set function having all four of these properties, and it is not known whether there is a set function satisfying the first three properties".[16] He further notes that "If we assume the continuum hypothesis (that every noncountable set of real numbers can be put in one-to-one correspondence with the set of all real numbers), then such a measure [satisfying the first three properties] is impossible)."[17] This was a totally unexpected result, and an excellent example of how postulated mathematical objects—mathematical reality—behave in unanticipated ways, which go far beyond the intuition of the postulators.

(3) The mathematical reality which is the subject matter of measure theory and probability theory isn't "nice", like that of Euclidean geometry. There is no question

of the mathematical and logical rigor of these disciplines, but of what they tell us though their theorems.

Consider first the theory of probability, which shows that the classical paradigm of strict, deterministic knowledge as the only thing worthy of the name is untenable. Probability theory is especially interesting because it is the systematic attempt to deal with uncertainty in a quantitative manner. This notion is practically an oxymoron for classical thought, and indeed probability theory's development was greatly retarded because of the change in mindset that it required. It required the development of new mathematical objects, specifically, probability distribution functions, whose nature is radically different than that of traditional objects such as lines and circles. For example, I can pick a set of 10 natural numbers at random, and ask, what is the probability that five of them are even? I cannot know, in advance, for any individual case, the exact number that will be even; I can only know the probability that it will be 0 through 10. It is in this sense that there has been a loss of certainty, and correlatively, a new type of reality emerging. As another example, consider differential equation. Deterministic solutions of differential equations are special or limiting cases of more general, probability-based solutions. Once again, this is congruent with Zubiri's philosophy because he does not require strict determinism as a criterion of knowledge; knowledge must be about reality, and we cannot place *a priori* bounds on reality, as both Aristotle and Kant tried to do with the notion of categories. Probability distribution functions are ways of describing reality; but they do not yield the kind of deterministic knowledge, the certainty, that other methods deliver.

Secondly, consider measure theory, which deals with sets. But the nature of these sets is such that much of what we wish to say about them isn't true for all members. In this way, statements such as "...except on a set of measure zero" imply that we are trying somewhat awkwardly to describe some reality which we cannot quite grasp, and which, taken as a whole, is too complex for our understanding. By excluding certain inconvenient cases, we can "tame" the reality, but it remains difficult conceptually. Measure theory shows that postulated reality is *de suyo* in very dramatic ways. The idea of "almost everywhere" true statements, e.g., "$f(x)=0$ almost everywhere"—which would be another oxymoron for classical thought—reveals that reality, even reality by postulation, is rich in unexpected ways, and that we must adapt our thinking to it. Reality is paramount, not preconceived ideas based on limited experience. The reality is *constructed*, and *sensed* as real; we feel that we can "touch" it in a sense; but it resists our efforts to know it, in a way that Euclidean geometry does not. Sets of measure zero are part of our expanded canon of mathematical reality; reason as searching was forced to acknowledge their existence, once certain other mathematical objects were postulated. The mathematical statements that we make and prove are no longer true for all reality, so in this sense the absolute certainty of our statements has disappeared. The canon of mathematical reality now includes both things about which "absolute" statements can be made, and those about which it cannot.

The same can be said of fractals. These types of curves—nowhere smooth and therefore nowhere differentiable—were first discovered by Karl Weierstrauss (1815-1897), but roundly greeted with catcalls because they did not fit the paradigm of "nice" mathematical objects. They present another dramatic example of the richness and sense of reality of mathematical objects, which far exceeds the "content", if one wishes, of their postulation. These objects have become well-known in recent years due to the existence of computers. They illustrate the unsuspected complexity of mathematical objects such as solutions to equations by Newton's method, which were created with no thought whatsoever about fractals. Similarly, chaotic dynamic systems—though completely deterministic and based on Newton's laws—didn't fit the mold of "ordinary" dynamic systems, because they didn't exhibit "nice" behavior: smooth and predictable. For Zubiri, this is just another example of the richness of reality, which goes far beyond our abilities to perceive and capture it. Indeed, it points squarely in the direction of the need for (and the justification of) the equivalent of "empirical" investigation of mathematical objects and mathematical reality.

C. Application of Mathematics to Reality: Reason and the Reality Field

This has been a simmering problem since the time of the Greeks. For them, or at least for Aristotle, mathematics and logic were about reality. However, the reality implicitly assumed by Aristotle was the reality of ordinary sense experience—the only one of which he conceived. This posed several problems. First, how can we have necessary truths about contingent events and things? Why must the things of the world obey, so to speak, our thinking about them? Second, what happens if the mathematics and logic change? As long as mathematics was only Euclidean geometry, Aristotle's view had some plausibility; with the development of branches of mathematics with no obvious relation to the world, such as complex numbers, and even worse, non-Euclidean geometries, it did not. Kant recognized the problem, and attempted to solve it by his theory of knowledge based on synthesizing of experience in conformity with Euclidean geometry. But that approach failed to negotiate the development non-Euclidean geometry as well.

The formalistic approach of Hilbert, Russell and others avoided the problem posed by new mathematical developments such as these new geometries, but at the rather high price of completely disconnecting mathematics and logic from reality, even to the point of making them symbol manipulation schemes. This makes it very difficult to explain how such schemes can ever be useful in the world. The typical approach is to claim that the mathematical or logical system exists by itself, as a purely abstract system (of symbol manipulation, if one likes), and only "applies" to the world when its postulates are interpreted or given meaning. The problem with this approach, for Zubiri, is twofold. First, it evades the question of the reality of mathematical objects, considered without reference to any application in the real world, and in particular, the fact that mathematicians speak about mathematical objects as existing somehow, "Let *x* be a Hilbert space…". And second, it does not explain how logical or mathematical rules, even when interpreted, can explain anything about how things occur in the real world, because we are still in what Zubiri refers to as the "concipient intelligence".

Thus the problem is twofold: what is the reality of mathematical objects, and how can mathematics be about the "real" world, in the sense of giving us apodeictic information about it? The fact that there is the need to come to grips with these problems is a confirmation of Zubiri's philosophy, since it alone is up to the task. The second problem, especially, is typically evaded or overlooked. For Zubiri, what is required to resolve them is the structure of the intelligence, namely its tripartite division into primordial apprehension, logos, and reason. To explain human intelligence, it is necessary that there be some experience already organized (the field, understood in the logos stage), some notes and structure to apply to it (reason), and some unity within reality (the same reality is the object of all three stages). With respect to reason,

> …reason is a structural moment of the intelligence as determined by the nature of the intellection of the real itself. In it reason has its structural origin. And as intellection is formally sentient, it follows that reason itself

is sentient.[18]

The ability to use reason to explain reality perceived at the earlier stages rests on this structure of rational intellection:

> Rational intellection has two moments, viz. the moment of intellection of reality itself as grounding principle, and the moment of intellective knowing of a real determinate content as grounded upon that ground... Reason or explanation, then, is first an intellection of the real ground, and second an intellection of the fact that this ground is of a real thing which one is trying to ground, a ground realized in it. And these two moments taken unitarily in the reality of this thing in the world constitute the free creation of reason.[19]

The key notion is that the two moments of reason—intellection of reality as grounding principle and knowing a real content as grounded upon it—are in fact two moments *of the same thing*, not two separate things.

> But then we see clearly that this intellection has, as I said a bit earlier, a second moment: the attribution of this "reason" or "explanation" freely created to a real thing. And this attribution is free. I can freely intellectively know that in-depth cosmic reality is the classical Hamiltonian ground, or the quantum field ground. And granting this, I intellectively know freely as well that a real field thing has in fact one or the other of those two grounding structures.[20]

This is the key to rational intellection: one freely creates various grounds, and then is free to choose one, which will be the ground of what it is that he seeks to understand in the field. He is not forced to chose one, à la the Kantian synthesis. And this process is at the heart of rational explanation:

> The creation of grounding reason is the actualization of in-depth physical reality in what has been previously intellectively known. And this creation is prolonged in an intellective knowing of a concrete real thing with one or another ground: it is an actualization of the thing in one or another of them. This actualization constitutes the root of realization, the realization of the ground in in-depth reality, and the realization of this ground in the real thing which I want to intellectively know. Reason or explanation, then, is first an intellection of the real ground, and second an intellection of the fact that this ground is of a real thing which one is trying to ground, a ground realized in it.[21]

The application to mathematics is straightforward since it is an enterprise of reason. Reason sketches possibilities to ground what is in the field. These possibilities are verified (or not); this is the truth of reason. The sketch must be adequate to ground what is in the field, and mathematics is one possible way of sketching. A rational intellection can only be scientific if the sketch goes beyond the field, so as to lead to the discovery of new properties of reality itself.[22]

This may be applied to the classic case of Euclidean geometry, which Zubiri believes is widely misunderstood. He points out that it is not the intuitive foundation on which our understanding of spatiality is built, but precisely the opposite: it is something to provide us with in-depth knowledge of that spatial field:

> ...the perceptive spatial field...is not absolute space—that would be absurd—but neither is it a geometric space. Therefore I call it 'pre-geometric space'. It is a space which does not possess strictly conceived characteristics, because when conceiving them it is necessary that this pre-geometric space become a geo-

metric space. Geometric space is therefore an in-depth foundation of pre-geometric space. The diversity of postulates discloses that, above all, both spaces are in fact space, but that the pre-geometric space is different than the geometric space. In particular, it shows us in this way that Euclidean space is not, as has so often be claimed, "intuitive", i.e., it shows us that Euclidean space is a free creation of geometric space.[23]

Conclusion

The development of mathematics since the time of the Ancient Greeks has confirmed the essential elements of Zubiri's philosophy of sentient intelligence. It has done so with respect to logic and the foundations of mathematics, since Gödel's incompleteness theorem illustrates the priority of reality over truth, and the fact that there is more reality in the mathematical creation than that of its postulates. Second, the steady expansion of the nature of mathematical objects, and especially the development of branches of mathematics in which universal certainty is explicitly excluded, demonstrates that the canon of reality is constantly being enlarged, and cannot be confined by categories. Thirdly, the fact that mathematics can be applied to reality shows the need for a structure of reality similar to that of Zubiri, to guarantee this possibility without the need to appeal to complex synthesis methods such as that of Kant—which soon failed.

Notes

1 Guillerma Díaz Múñoz, *Zubiri y la matemática: un nuevo constructivismo*, Washington: Xavier Zubiri Foundation of North America, 1998.

2 See also her article in this issue of the *Review*, pp. 7ff.

3 Altshiller-Court, Nathan, "The Dawn of Demonstrative Geometry", in *From Five Fingers to Infinity*, ed. F. J. Swetz, Chicago: Open Court, 1994, p. 151.

4 Jones, Phillip S., "Irrationals or Incommensurables III: The Greek Solution", in *From Five Fingers to Infinity*, ed. F. J. Swetz, Chicago: Open Court, 1994, p. 178.

5 Edward Maziarz and Thomas Greenwood, *Greek Mathematical Thought*, New York: Barnes & Noble, 1995, p. 165.

6 *Ibid.*, p. 168.

7 M. Kline, *Mathematical Thought from Ancient to Modern Times*, Vol. I, New York: Oxford University Press, 1972, p. 177.

8 See "Nineteenth Century Geometry" in *The Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/geometry-19th/.*

9 Ian Hacking, "Was There a Probabalistic Revolution, 1800-1930?", in *The Probabalistic Revolution*, Volume I: *Ideas in History*, ed. L. Krüger, L. Daston, M. Heidelberger, Cambridge: MIT Press, 1987, p. 45ff.

10 "Über formal unentscheidbare Sätzd der Principia Mathematica und verwandter System I," *Montashefte für Mathematik und Physik 38*, p. 173-198 (1931). English translation: "On formally undecidable propositions of *Principia Mathematica* and related systems I," in *Frege and Gödel*, ed. by Jean van Heijenoort, Cambridge: Harvard University Press, 1970, p. 83ff.

11 Chae, Soo Bong, *Lebesgue Integration*, New York: Marcel Dekker, 1980, p. 42ff.

12 Muñoz, G., "Esbozo de una filosofia zubiriana de la matemática", p. 142.

13 IR, p. 95-96.

14 IR, p. 57.

15 Cf. Aristotle, *Posterior Analytics*, Book 1, Ch. 2, 71b8-15.

16 H. L. Royden, *Real Analysis*, second edition, New York: Macmillan, 1968, p. 52.

17 *Ibid.*

18 IR, p. 137.

19 IR, p. 110-111.

20 IR, p. 112.

21 IR, p. 111-112.

22 cf. Díaz Muñoz, 1998, *op. cit.*, p. 238.

23 IR, p. 131.