Immunity and hyperimmunity for sets of minimal indices*

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December 14, 2011

Abstract

We extend Meyer's 1972 investigation of *sets of minimal indices.* Blum showed that minimal index sets are immune, and we show that they are also immune against high levels of the arithmetic hierarchy. We give optimal immunity results for sets of minimal indices with respect to the arithmetic hierarchy, and we illustrate with an intuitive example that immunity is not simply a refinement of arithmetic complexity. Of particular note here are the fact that there are three minimal index sets located in $\Pi_3 - \Sigma_3$ with distinct levels of immunity and that certain immunity properties depend on the choice of underlying acceptable numbering. We show that minimal index sets are never hyperimmune, however they can be immune against the arithmetic sets. Lastly, we investigate Turing degrees for sets of random strings defined with respect to Bagchi's size-function s.

1 A short introduction to shortest programs

The *set of shortest programs* is

$$\{e : (\forall j < e)\, [\varphi_j \neq \varphi_e]\}. \tag{1.1}$$

In 1967, Blum [4] showed that one can enumerate at most finitely many shortest programs. Five years later, Meyer [13] formally initiated the investigation of minimal index sets with questions on the Turing and truth-table degrees of (1.1).

Meyer's research parallels inquiry from Kolmogorov complexity where one searches for shortest programs generating single numbers or strings. The clearest confluence

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of Kolmogorov randomness and minimal index sets manifests itself in Schaefer's *set of shortest descriptions*, [16]

$$\{e : (\forall j < e)\, [\varphi_j(0) \neq \varphi_e(0)]\}, \tag{1.2}$$

which serves as the set of minimal indices for Kolmogorov complexity. The size-minimal random strings discussed in the last section of this paper are generalizations of both the Kolmogorov numberings and the minimal index set (1.2).

For underlying Kolmogorov numberings φ, the set (1.1) forms a subset of the Kolmogorov random strings. The converse inclusion fails in general since multiple Kolmogorov random indices can represent the same function. Moreover, one can choose a Gödel numbering ψ such that (1.1) lies entirely within the non-random strings, except for a finite set. For example, let $\psi_i = \varphi_j$ whenever $2^j \leq i < 2^{j+1}$. In this case, all minimal indices are of the form 2^i and have a Kolmogorov complexity which is, up to a constant, the same as i.

In contrast to Meyer [13], we shall focus on the set of minimal indices with respect to domains,

$$\mathrm{MIN} = \{e : (\forall j < e)\, [W_j \neq W_e]\},$$

rather than functions. We also consider natural variants of MIN.

Definition 1.1. We call MIN and following sets *sets of minimal indices.* Minimal index sets are based on equivalence relations and each set contains the least representative from each equivalence class:

$$\begin{aligned}
\mathrm{MIN}^* &= \{e : (\forall j < e)\, [W_j \neq^* W_e]\}, \\
\mathrm{MIN}^{\mathrm{m}} &= \{e : (\forall j < e)\, [W_j \not\equiv_{\mathrm{m}} W_e]\}, \\
\mathrm{MIN}^{\mathrm{T}^{(n)}} &= \{e : (\forall j < e)\, [W_j \not\equiv_{\mathrm{T}^{(n)}} W_e]\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{MIN}^{\mathrm{T}^{(\omega)}} &= \bigcap_{n\in\omega} \mathrm{MIN}^{\mathrm{T}^{(n)}} \\
&= \{e : (\forall j < e)(\forall n)\, [(W_j)^{(n)} \not\equiv_{\mathrm{T}} (W_e)^{(n)}]\},
\end{aligned}$$

where $A \equiv_{\mathrm{T}^{(n)}} B$ is shorthand for $A^{(n)} \equiv_{\mathrm{T}} B^{(n)}$. Here $A^{(n)}$ denotes the n^{th} Turing jump of A. If $n = 0$, we omit "(n)" from the notation.

For simplicity, we place ω and $\emptyset$ in the same m-equivalence class as the rest of the recursive sets (for the remainder of this paper). If the particular Gödel numbering is relevant to the discussion, we shall add a subscript, as in MIN_φ.

We recall the following definitions:

Definition 1.2. Let $(D_e)_{e\in\omega}$ be the canonical numbering of the finite sets.

(I) A set is *immune* if it is infinite and contains no infinite r.e. sets.

(II) A set A is *hyperimmune* if it is infinite and there is no recursive function f such that:

   (a) $\left(D_{f(i)}\right)_{i\in\omega}$ is a family of pairwise disjoint sets, and
   (b) $D_{f(i)} \cap A \neq \emptyset$ for all i.

The following is a generalization of Definition 1.2 (I).

Definition 1.3. Let $\mathcal{C}$ be a family of sets. A set is $\mathcal{C}$*-immune* if it is infinite and contains no infinite members of $\mathcal{C}$. If $\mathcal{C}$ is the class of r.e. sets, then we write *immune* in place of $\mathcal{C}$-immune.

Blum showed that MIN is immune [4], and Meyer showed that MIN is not hyperimmune [13]. Sections 2.1–2.2 contain analogous immunity results for the other minimal index sets. In Theorem 2.6, in particular, we use immunity or "thinness" to distinguish among minimal index sets contained in the same level of the arithmetic hierarchy. Section 2.3 provides a counterexample which is useful for intuition: it shows that immunity is not, in fact, a simple refinement of arithmetic complexity. After inspecting the minimal index sets in Definition 1.1, one might suspect that greater immunity implies greater arithmetic complexity, however this is not true in general.

Section 3 shows that the Π_n-immunity of some, but not all, minimal index sets depends on the Gödel numbering. We show that minimal index sets are not hyperimmune (Section 4). Using this fact, we construct a set which neither contains nor is disjoint from any arithmetic set, yet is majorized by a recursive function and contains a minimal index set (Corollary 4.6). Lastly, in Section 5, we show that size-minimal Kolmogorov random strings need not be Turing complete. This contrasts with the more usual random strings, the special case where size is simply length, which are wtt-complete under any Gödel numbering and truth-table complete under any Kolmogorov numbering [7].

For further background on minimal index sets, we refer the reader to [16] and [19]. Notation not mentioned here follows [14] and [18].

2 Immunity and fixed points

2.1 The Π_3-Separation Theorem

Marcus Schaefer [16] made the following observations with regards to minimal functions, but the results translate easily into sets. He attributes the main idea of (i) to Blum [4, Theorem 3] and (ii) to John Case:

Theorem 2.1 (Schaefer [16])**.**

(I) MIN *is immune.*

(II) MIN* *is* Σ_2*-immune.*

Proposition 2.2 and Lemma 2.3 will be needed to prove the Π_3-Separation Theorem.

Proposition 2.2.

(i) $\mathrm{MIN}^* \in \Pi_3$.

(ii) $\mathrm{MIN}^{\mathrm{m}} \in \Pi_3$.

(iii) $\mathrm{MIN}^{\equiv_1} \in \Pi_3$.

Proof. (i). $\{\langle j, e\rangle : W_j =^* W_e\} \in \Sigma_3$ [18]. □

(ii). For any r.e. sets A and B,

$$A \leq_{\mathrm{m}} B \iff (\exists e)(\forall x)\ [\varphi_e(x)\downarrow \quad \wedge \quad (x \in A \iff \varphi_e(x) \in B)]\,,$$

which shows that $A \leq_{\mathrm{m}} B$ is a $\Sigma_2^{\emptyset'}$ relation. It follows that $A \equiv_{\mathrm{m}} B$ is also a $\Sigma_2^{\emptyset'}$ relation. In particular, for

$$C = \{\langle j, e\rangle : W_j \equiv_{\mathrm{m}} W_e\},$$

we have

$$C \in \Sigma_2^{\emptyset'} = \Sigma_3.$$

Hence

$$e \in \mathrm{MIN}^{\mathrm{m}} \iff (\forall j < e)\ [\langle j, e\rangle \notin C]\,,$$

which places $\mathrm{MIN}^{\mathrm{m}} \in \Pi_3$. □

(iii). The same proof idea as for (ii) works because injectivity can be tested with a $\emptyset'$ oracle. □

Lemma 2.3 (i) is an immediate consequence of Schaefer's theorem, $\mathrm{MIN}^* \oplus \emptyset' \equiv_{\mathrm{T}} \emptyset'''$ [16], however we give a more direct proof below.

Lemma 2.3.

(i) $\mathrm{MIN}^* \notin \Sigma_3$.

(ii) $\mathrm{MIN}^{\mathrm{m}} \notin \Sigma_3$.

(iii) $\mathrm{MIN}^{\equiv_1} \notin \Sigma_3$.

Proof. (i). Suppose $\mathrm{MIN}^* \in \Sigma_3$, let a be the *-minimal index for ω and recall that the set of cofinite indices

$$\mathrm{COF} = \{e : W_e =^* \omega\}$$

is Σ_3-complete [18]. Note that

$$W_j \neq^* W_e \quad \iff \quad (\forall y)\,(\exists x > y)\,(\exists s)\,(\forall t > s)\,[W_{j,t}(x) \neq W_{e,t}(x)] \tag{2.1}$$

and

$$\begin{aligned} \mathrm{COF} &= (\mathrm{MIN}^* \cap \mathrm{COF}) \cup (\overline{\mathrm{MIN}^*} \cap \mathrm{COF}) \\ &= \{a\} \cup \{e : (\forall j \leq e)\ [j \in \mathrm{MIN}^* - \{a\} \quad \Longrightarrow \quad W_j \neq^* W_e]\}\,. \end{aligned}$$

Now $\mathrm{COF} \in \Pi_3$, by (2.1) and because $\mathrm{MIN}^* - \{a\} \in \Sigma_3$ by assumption. This contradicts the fact that COF is Σ_3-complete. □

(II). $\{e : W_e \equiv_{\mathrm{m}} C\}$ is Σ_3-complete whenever C is r.e. This set now plays the role of COF from part (I) [20]. □

(III). $\{e : W_e \equiv_1 C\}$ is Σ_3-complete whenever C is r.e., infinite and coinfinite [5]. Since $W_j \equiv_1 W_e$ is decidable in Σ_3, the same argument again applies. □

This completes the proof of the theorem. □

The proofs of Theorem 2.6 and Corollary 2.7 illustrate the connection between immunity for minimal indices and generalized fixed points. In the following theorem, the cases $\equiv_{\mathrm{m}}$ and $\equiv_{\mathrm{T}}$ were first proven by Arslanov [3], [2], and $=^*$ is due to Arslanov, Nadyrov, and Solov'ev [1]. The remaining cases are due to Jockusch, Lerman, Soare and Solovay [6].

Theorem 2.4 (generalized fixed points, Arslanov, Nadyron, Solov'ev, Jockusch, Lerman, Soare, Solovay)**.** *For every $n \leq \omega$,*

(I) $f \leq_{\mathrm{T}} \emptyset' \implies (\exists e)\ [W_e =^* W_{f(e)}]$,

(II) $f \leq_{\mathrm{T}} \emptyset'' \implies (\exists e)\ [W_e \equiv_{\mathrm{m}} W_{f(e)}]$,

(III) $f \leq_{\mathrm{T}} \emptyset^{(n+2)} \implies (\exists e)\ [W_e \equiv_{\mathrm{T}^{(n)}} W_{f(e)}]$.

Furthermore, e can be found effectively from n and an index for f (in an acceptable numbering of a $\emptyset'$-, $\emptyset''$- or $\emptyset^{(n+2)}$-recursive function, respectively).

Definition 2.5. An integer n is an i^{th} *prime power* if $n = p_i^k$ for some $k \geq 1$, where p_i is the i^{th} prime number.

The following theorem shows that immunity can be used to distinguish between certain MIN-sets, even when the arithmetic hierarchy can not.

Theorem 2.6 (Π_3-Separation)**.** $\mathrm{MIN}^{\mathrm{m}}$*,* MIN^{*} *and* $\mathrm{MIN}^{\equiv_1}$ *are all in* $\Pi_3 - \Sigma_3$*, but*

(I) $\mathrm{MIN}^{\mathrm{m}}$ *is Σ_3-immune, whereas*

(II) MIN^{*} *contains an infinite Σ_3 set and*

(III) $\mathrm{MIN}^{\equiv_1}$ *contains an infinite Σ_2 set.*

Proof. We already showed $\mathrm{MIN}^{\mathrm{m}}, \mathrm{MIN}^{*}, \mathrm{MIN}^{\equiv_1} \in \Pi_3 - \Sigma_3$ in Theorem 2.3.

(I). $\mathrm{MIN}^{\mathrm{m}}$ is known to be infinite as there are infinitely many many-one degrees of r.e. sets. If $\mathrm{MIN}^{\mathrm{m}}$ had an infinite Σ_3-subset, then there would be a $\emptyset''$-recursive function f such that $f(e) > e$ and $f(e) \in \mathrm{MIN}^{\mathrm{m}}$ for all e. This would imply

$$(\forall e)\ [W_{f(e)} \not\equiv_{\mathrm{m}} W_e],$$

in contradiction to Theorem 2.4 which says that such a $\emptyset''$-recursive function does not exist. □

(ɪɪ). Recall that

$$\mathrm{INF} = \{e : W_e \text{ is infinite}\}$$

and for every k, let

$$\begin{aligned} P_k &= \{n : n \text{ is a } k^{\text{th}} \text{ prime power}\}, \\ A_k &= \{e : W_e \subseteq^* P_k\} \cap \mathrm{INF}, \\ A &= \{e : (\exists k)\ (\forall j < e)\ [e \in A_k \ \ \wedge \ \ j \notin A_k]\}. \end{aligned}$$

Now $A \subseteq \mathrm{MIN}^*$, as $e \in A$ implies $W_j \neq^* W_e$ for all $j < e$. Since the A_k's are disjoint, any infinite B satisfies $B \subseteq^* A_k$ for at most one k. Moreover, each A_k contributes a distinct element to A, hence A is infinite. Finally,

$$\begin{aligned} W_e \subseteq^* P_k &\iff (\exists y)\ (\forall x \geq y)\ [x \in W_e \implies x \in P_k] \\ &\iff (\exists y)\ (\forall x \geq y)\ [x \notin W_e \ \vee \ x \in P_k] \\ &\iff (\exists y)\ (\forall x \geq y)\ (\forall t)\ [x \notin W_{e,t} \ \vee \ x \in P_k], \end{aligned}$$

which makes $A_k \in \Delta_3$, on account of $\mathrm{INF} \in \Pi_2$. It follows that $A \in \Sigma_3$. □

(ɪɪɪ). Define a sequence of finite sets by

$$A_k = \{x : 0 \leq x \leq k\}.$$

Furthermore, define

$$B_k = \{e : W_e \text{ has } \textit{at least } k \text{ elements}\} \in \Sigma_1,$$

which means that

$$C_k = \{e : W_e \text{ has } \textit{exactly } k \text{ elements}\} = B_k \cap \overline{B_{k+1}} \in \Delta_2.$$

It follows from the Pigeonhole Principle that

$$W_e \equiv_1 A_k \iff e \in C_k,$$

and therefore

$$\{\langle e, k\rangle : W_e \equiv_1 A_k\} \in \Delta_2.$$

Now

$$A = \{e : (\exists k)\ (\forall j < e)\ [W_j \not\equiv_1 A_k \quad \wedge \quad W_e \equiv_1 A_k]\}$$

is a Σ_2 set. Moreover, A is infinite because each A_k represents a distinct $\equiv_1$ class. Since $A \subseteq \mathrm{MIN}^{\equiv_1}$, it follows that $\mathrm{MIN}^{\equiv_1}$ is not Σ_2-immune. □

This completes the proof. □

Remark. It is worth noting that $\mathrm{MIN}^{\equiv_1}$ is immune (simply because it is a subset of MIN).

2.2 Upper minimal index sets

The goal of this section is to determine the immunity of $\mathrm{MIN}^{\mathrm{T}^{(n)}}$.

Corollary 2.7. *For all $n < \omega$, $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ is Σ_{n+3}-immune.*

Proof. We follow the proof of the Π_3-Separation Theorem 2.6(i) and as before, $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ is infinite (this will follow from Corollary 4.5).

Let $n \geq 0$ and let A be an infinite, Σ_{n+3} set. Suppose $A \subseteq \mathrm{MIN}^{\mathrm{T}^{(n)}}$. Since A is infinite and r.e. in $\emptyset^{(n+2)}$, A has a $\emptyset^{(n+2)}$-recursive subset B. Define a $\emptyset^{(n+2)}$-recursive function g by

$$g(e) = (\mu i)\,[i > e \quad \wedge \quad i \in B].$$

Now for all e, $g(e) > e$ and $g(e) \in \mathrm{MIN}^{\mathrm{T}^{(n)}}$. Therefore

$$(\forall e)\,[W_e \not\equiv_{\mathrm{T}^{(n)}} W_{g(e)}],$$

contradicting Theorem 2.4. □

We now show that Corollary 2.7 is optimal. This will follow from a result by Lempp and Lerman:

Theorem 2.8 (Lempp and Lerman [8]). *Any countable partial order P with jump which is consistent with:*

(I) *its order relation,*

(II) *the order-preserving property of the jump operator,*

(III) *the property of the jump operator that the jump of an element is strictly greater than the element, and*

(IV) *the property that a non-jump element lies between $\mathbf{0}$ and $\mathbf{0}'$, a single jump element lies between $\mathbf{0}'$ and $\mathbf{0}''$, etc.*

can be effectively embedded into the r.e. degrees.

The next corollary follows from Theorem 2.8 and will be useful in the proof of Theorem 2.11. In the case of $n = 0$, Corollary 2.9 says that there exists a recursive sequence of low, pairwise minimal r.e. sets.

Corollary 2.9. *For every n, there exists a recursive sequence of r.e. sets $A_0, A_1, \ldots$ such that for all C r.e. in $\emptyset^{(n)}$ and $i \neq j$,*

(I) $\emptyset <_{\mathrm{T}^{(n)}} A_i$.

(II) $(A_i)' \equiv_{\mathrm{T}^{(n)}} \emptyset'$,

(III) $\left[C \leq_{\mathrm{T}} (A_i)^{(n)} \quad \wedge \quad C \leq_{\mathrm{T}} (A_j)^{(n)}\right] \implies C \leq_{\mathrm{T}} \emptyset^{(n)}$.

Definition 2.10. For $n \geq 0$,

(I) $\mathrm{LOW}^n = \{e : W_e \equiv_{\mathrm{T}^{(n)}} \emptyset\}$,

(II) $\mathrm{HIGH}^n = \{e : W_e \equiv_{\mathrm{T}^{(n)}} \emptyset'\}$.

Theorem 2.11. *For all $n \geq 0$, $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ is not Σ_{n+4}-immune.*

Proof. Let $n \geq 0$ and let $A_0, A_1, \ldots$ be the corresponding sequence of sets obtained from Corollary 2.9. Define

$$B_k = \left[\{x : W_x \leq_{\mathrm{T}^{(n)}} A_k\} \ \cap \ \overline{\mathrm{LOW}^n}\right],$$
$$B = \{e : (\exists k)\, (\forall j < e)\, [e \in B_k \quad \wedge \quad j \notin B_k]\}\,.$$

Note that $B \leq_{\mathrm{T}^{(n)}} A$ is a $\Sigma_{n+2}^{B \oplus A'}$ relation. Since, for any x, both $W_x \leq_{\mathrm{T}^{(n)}} \emptyset'$ and $(A_k)' \leq_{\mathrm{T}^{(n)}} \emptyset'$, it follows that

$$\{x : W_x \leq_{\mathrm{T}^{(n)}} A_k\} \in \Sigma_{n+2}^{\emptyset'} = \Sigma_{n+3}.$$

This places $B_k \in \Delta_{n+4}$, on account of $\overline{\mathrm{LOW}^n} \in \Pi_{n+3}$. Therefore $B \in \Sigma_{n+4}$.

It remains to show that B is an infinite subset of $\mathrm{MIN}^{\mathrm{T}^{(n)}}$. Note that $B_i \cap B_j = \emptyset$ for i, j with $i \neq j$. Indeed, if $e \in B_i \cap B_j$, then

$$W_e \leq_{\mathrm{T}^{(n)}} A_i \quad \wedge \quad W_e \leq_{\mathrm{T}^{(n)}} A_j \quad \wedge \quad e \notin \mathrm{LOW}^n,$$

contradicting Property (III) of Corollary 2.9. Now since $B_k \neq \emptyset$ and each B_k contributes exactly one element to B, B must be infinite.

Finally, assume $e \in B$ and let k be such that $e \in B_k$ and $j \notin B_k$ for all $j < e$. Then for $j < e$,

$$W_e \leq_{\mathrm{T}^{(n)}} A_k \quad \wedge \quad W_j \not\leq_{\mathrm{T}^{(n)}} A_k,$$

which implies $W_e \not\equiv_{\mathrm{T}^{(n)}} W_j$. So $e \in \mathrm{MIN}^{\mathrm{T}^{(n)}}$. That is, $B \subseteq \mathrm{MIN}^{\mathrm{T}^{(n)}}$. ☐

Remark. Any set is Δ_n-immune iff it is Σ_n-immune. Therefore our theorems regarding Σ_n-immunity also give the results for Δ_n-immunity.

2.3 Intuition

The immunity results from Sections 2.1–2.2 are summarized in Figure 1. The arithmetic results are optimal by Lemma 2.3 and [19, Theorem 1.3.4]. The set-theoretic inclusions are immediate from the definitions.

Based on this diagram, one might be tempted to believe that minimal index sets which are higher in the arithmetic hierarchy are also more immune. This is not true, and we devote the remainder of this section to a counterexample. Indeed, the set $\mathrm{MIN}^{\text{Thick-}*}$, defined below, is in $\Sigma_4 - \Pi_4$ and only Σ_2-immune, whereas $\mathrm{MIN}^{\mathrm{m}} \in \Pi_3$ is Σ_3-immune. Our omission of $\mathrm{MIN}^{\text{Thick-}*}$ from Figure 1 makes the diagram coherent.

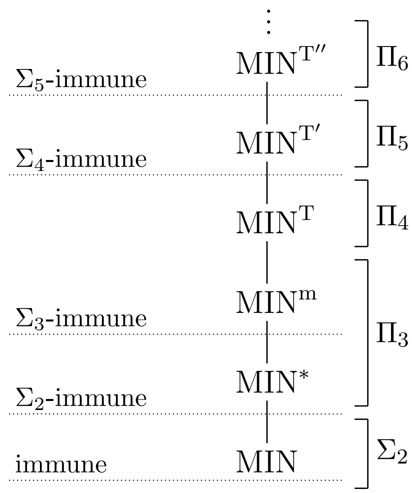

Figure 1: A näive approach to minimal index sets, by reverse inclusion.

Definition 2.12. For $A, B \subseteq \omega$, define the equivalence relation

$$A \equiv_{\text{Thick-}*} B \iff (\forall n)\left[A^{[n]} =^* B^{[n]}\right],$$

where $A^{[n]} = \{x : \langle x, n\rangle \in A\}$.

Theorem 2.13.

(I) $\mathrm{MIN}^{\text{Thick-}*} \in \Sigma_4$.

(II) $\mathrm{MIN}^{\text{Thick-}*} \notin \Pi_4$.

Proof. (I). $\{\langle j, e\rangle : W_j =^* W_e\} \in \Sigma_3$, so $\{\langle j, e\rangle : W_j \equiv_{\text{Thick-}*} W_e\} \in \Pi_4$. □

(II). Let $A \in \Pi_4$. Then there exists a relation $R \in \Sigma_3$ such that

$$x \in A \iff (\forall y)\,[R(x, y)].$$

Since COF is Σ_3-complete [18], there exists a recursive function g such that $R(x, y)$ iff $W_{g(x,y)}$ is cofinite. Therefore

$$x \in A \iff (\forall y)\left[W_{g(x,y)} =^* \omega\right].$$

Define a recursive function f by

$$\varphi^{[y]}_{f(x)} = \varphi_{g(x,y)}.$$

Then

$$\begin{aligned} W_{f(x)} \equiv_{\text{Thick-}*} \omega &\iff (\forall y)\left[W_{g(x,y)} =^* \omega\right] \\ &\iff x \in A, \end{aligned}$$

which makes

$$\text{Thick-COF} = \{e : W_e \equiv_{\text{Thick-}*} \omega\}$$

Π_4-complete.

Suppose towards a contradiction that $\text{MIN}^{\text{Thick-}*} \in \Pi_4$, and let a be the $\equiv_{\text{Thick-}*}$-minimal index for ω. Then

$$\begin{aligned}\text{Thick-COF} &= \{e : W_e \equiv_{\text{Thick-}*} \omega\} \\ &= \{a\} \cup \left\{e : (\forall j < e)\left[j \in \text{MIN}^{\text{Thick-}*} - \{a\} \implies W_j \not\equiv_{\text{Thick-}*} W_e\right]\right\}.\end{aligned}$$

Now Thick-COF $\in \Sigma_4$, since $W_j \equiv_{\text{Thick-}*} W_e$ can be decided in Π_4 and because

$$\text{MIN}^{\text{Thick-}*} - \{a\} \in \Pi_4$$

by assumption. This contradicts the fact that Thick-COF is Π_4-complete. □

Thickness contributes nothing to immunity, as evidenced by Corollary 2.15.

Lemma 2.14 (semi-fixed points)**.** *There exists a recursive function ν such that*

$$(\forall f \leq_{\text{T}} \emptyset')\,(\exists e)\left[W_{\nu(e)} \equiv_{\text{Thick-}*} W_{f(e)}\right].$$

Proof. Using the *s-m-n* Theorem, define a recursive function ν by

$$\varphi_{\nu(x)}(\langle z, n\rangle) = \begin{cases} \varphi_{\varphi_x(n)}(z) & \text{if } \varphi_x(n)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

so that for any $x \in \text{TOT}$,

$$W^{[n]}_{\nu(x)} = W_{\varphi_x(n)}.$$

Let $f \leq_{\text{T}} \emptyset'$ and define, again using the *s-m-n* Theorem, a recursive sequence of $\emptyset'$-recursive functions $\{f_n\}$ by

$$\varphi_{f_n(x)}(z) = \varphi_{f(x)}(\langle z, n\rangle)$$

so that

$$W_{f_n(x)} = W^{[n]}_{f(x)}.$$

By the Generalized Fixed Point Theorem (Theorem 2.4), we can uniformly find a recursive sequence $\{e_n\}$ such that for all n,

$$W_{e_n} =^* W_{f_n(e)}.$$

Let e be an index so that

$$\varphi_e(n) = e_n.$$

Then for all n,

$$W^{[n]}_{\nu(e)} = W_{\varphi_e(n)} = W_{e_n} =^* W_{f_n(e)} = W^{[n]}_{f(e)}.$$

This means that

$$(\exists e)\left[W_{\nu(e)} \equiv_{\text{Thick-}*} W_{f(e)}\right], \tag{2.2}$$

which is what we intended to show. □

Comparing Corollary 2.15 with the results from Section 2.1, we note that the thick operator does not at all affect immunity:

Corollary 2.15. $\mathrm{MIN}^{\text{Thick-}*}$ *is* Σ_2*-immune but not* Σ_3*-immune.*

Proof. $\mathrm{MIN}^{\text{Thick-}*}$ is Σ_2-immune follows immediately from the fact that $\mathrm{MIN}^* \supseteq \mathrm{MIN}^{\text{Thick-}*}$ and Theorem 2.1 (II). We show $\mathrm{MIN}^{\text{Thick-}*}$ is not Σ_3-immune by modifying the proof of Theorem 2.6 (II). All that is needed is to change the definition of A_k so that it only applies to the first row of each r.e. set:

$$A_k = \left\{e : W_e^{[0]} \subseteq^* P_k\right\} \cap \mathrm{INF}.$$

The rest of the proof is the same. □

3 Π_n-immunity

Our discussion from the previous section gives tight bounds with respect to Σ_n-immunity. With the exception of $\mathrm{MIN}^{\mathrm{m}}$ however, in which case Theorem 2.6 gives an optimal immunity result, we are still left with open questions regarding Π_n-immunity. Unlike the other results from Section 2, Π_1-immunity for MIN, Π_2-immunity for MIN^* and Π_{n+3}-immunity for $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ depend on the numbering for the partial-recursive functions.

Theorem 3.1. *There exist Gödel numberings* ψ *and* ν *such that*

(I) MIN_ψ *contains an infinite* Π_1*-set,*

(II) MIN_ν *is* Π_1*-immune.*

Proof. Let φ be a given Gödel numbering from which the numberings ψ and ν are built. W_e denotes $\operatorname{dom} \varphi_e$ throughout this proof.

(I). Define a Gödel numbering ψ such that $\psi_{2^x} = \varphi_x$ and $\operatorname{dom} \psi_y = \{y\}$ when y is not a power of two. Furthermore, define a partial recursive function θ by

$$\theta(x) = \begin{cases} n & \text{if } n \text{ is the first element enumerated into } W_x, \\ \uparrow & \text{otherwise} \end{cases}$$

and a Π_1-set A by

$$A = \{y : (\forall x)\, [y \neq 2^x \quad \wedge \quad [(2^x < y \quad \wedge \quad \theta(x)\downarrow) \implies \theta(x) \neq y]]\}.$$

We now show $A \subseteq \mathrm{MIN}_\psi$. Let $y \in A$, $z < y$ and assume by way of contradiction that $\operatorname{dom} \psi_z = \operatorname{dom} \psi_y$. Now $z = 2^x$ for some x by definition of ψ, since y is not a power of two. It follows that

$$W_x = \operatorname{dom} \psi_{2^x} = \operatorname{dom} \psi_z = \operatorname{dom} \psi_y = \{y\},$$

and so $\theta(x) = y$. On the other hand, $2^x < y$ and $\theta(x)\downarrow$, which means that $\theta(x) \neq y$ by definition of A. This is a contradiction.

It remains to verify that A is infinite. For every $x > 2$, there is a member $y \in A$ between 2^x and 2^{x+1}. This follows from easy cardinality reasons: there are $2^x - 1$ domains, namely $\{\{2^x+1\}, \ldots, \{2^{x+1}-1\}\}$, represented among the ψ-indices between 2^x and 2^{x+1}. The only ψ-indices between 2^x and 2^{x+1} that are not members of A are those which have one of the following domains: $\{\{\theta(0)\}, \ldots, \{\theta(x)\}\}$. It follows that there are at least $(2^x - 1) - (x + 1)$ members of A between 2^x and 2^{x+1}. □

(II). Define the numbering ν such that

$$\nu_0 \text{ is everywhere undefined}$$

and for $x \geq 0$, $j \in \{0, 1, \ldots, 2^x - 1\}$,

$$\nu_{2^x+j} = \begin{cases} \varphi_x & \text{if there are at least } 2^x - j - 1 \text{ indices } n \leq x \\ & \text{such that } \{2^x, 2^x+1, \ldots, 2^x+j\} \subseteq W_n, \\ \nu_0 & \text{otherwise.} \end{cases}$$

Note that $\nu_{2^x+(2^x-1)} = \varphi_x$ for all x, which makes ν a Gödel numbering.

Suppose there were an infinite, Π_1-set $\overline{W_e}$ such that $\overline{W_e} \subseteq \mathrm{MIN}_\nu$. Choose x large so that $x \geq e$ and

$$2^x + j \in \overline{W_e} \subseteq \mathrm{MIN}_\nu. \tag{3.1}$$

Now

$$\{2^x, 2^x+1, \ldots, 2^x+j-1\} \subseteq \overline{\mathrm{MIN}_\nu} \subseteq W_e. \tag{3.2}$$

By the definition of ν and (3.1),

$$\begin{array}{l}\text{There are } 2^x - j - 1 \text{ indices } n \in \{0, 1, \ldots, x\} - \{e\} \\ \text{such that } \{2^x, 2^x+1, \ldots, 2^x+j\} \subseteq W_n.\end{array} \tag{3.3}$$

By (3.2) and (3.3),

$$\begin{array}{l}\text{There are } 2^x - j \text{ indices } n \in \{0, \ldots, x\} \\ \text{such that } \{2^x, 2^x+1, \ldots, 2^x+j-1\} \subseteq W_n.\end{array}$$

Thus $\nu_{2^x+(j-1)} = \varphi_x$, contradicting the fact that $2^x + j \in \mathrm{MIN}_\nu$. This means that MIN_ν is Π_1-immune. □

This completes the proof. □

Theorem 3.2. *There exists Gödel numberings ψ and ν such that*

(I) MIN^*_ψ *contains an infinite* Π_2*-set,*

(II) MIN^*_ν *is* Π_2*-immune.*

Proof. Let φ be a given Gödel numbering from which the numberings ψ and ν are built. W_e denotes $\operatorname{dom}\varphi_e$ throughout this proof.

(I). Let $E_0, E_1, E_2, \dots$ be a recursive partition of the natural numbers into infinitely many infinite sets, e.g.

$$E_n = \{\langle x, n\rangle : x \in \omega\}.$$

Define

$$A = \{n : (\exists k, e)\, [2^e < n \quad \wedge \quad |W_e - E_n| < k \quad \wedge \quad |W_e \cap E_n| > k]\}, \tag{3.4}$$

and let

$$P = \{0, 2^0, 2^1, 2^2, \dots\}.$$

Let $B[e, k, n]$ denote the bracketed clause in (3.4). We verify that $\overline{A} \cap \overline{P}$ is an infinite Π_2-set. Note that for a fixed $\langle k, e\rangle$, $B[e, k, n]$ can be decided with a halting set oracle. It follows that $A \in \Sigma_2$, hence $\overline{A} \cap \overline{P} \in \Pi_2$. Moreover, for each index e, there exists at most one n satisfying $B[e, k, n]$ (whether or not W_n is finite) because the E_n's are pairwise disjoint. It follows that A contains at most $e + 1$ indices below 2^{e+1}. In particular, $\overline{A}$ has a member between 2^e and 2^{e+1} for every $e > 2$, which proves that $\overline{A} \cap \overline{P}$ is infinite.

Define a Gödel numbering ψ so as to satisfy:

- $\psi_{2^n} = \varphi_n$;
- $V_n = E_n$ if $n \in \overline{A} \cap \overline{P}$,

where $V_n = \operatorname{dom} \psi_n$. This can be done as follows. Let $\{A_s\}_{s \in \omega}$ be a recursive Σ_2-approximation of A satisfying

$$n \in A \quad \Longleftrightarrow \quad (\forall^\infty s)\, [n \in A_s].$$

For $n \in \overline{P}$, enumerate $\langle x, n\rangle$ into V_n iff there is a stage $s > x$ such that $n \notin A_s$. Then $V_n = E_n$ if $n \in \overline{A}$, and V_n is finite subset of E_n otherwise.

It remains to show that $\overline{A} \cap \overline{P} \subseteq \mathrm{MIN}^*_\psi$. Assume that $n \in \overline{A} \cap \overline{P}$. By definition of A, for all numbers $2^x \in P$ satisfying $2^x < n$,

$$V_{2^x} = W_x \neq^* E_n = V_n.$$

For the remaining indices $x \notin P$ with $x < n$, we have

$$V_x = E_x \neq^* E_n = V_n,$$

Therefore $n \in \mathrm{MIN}^*_\psi$. □

Remark. The proof above shows even a bit more. Since finite sets are not $=^*$-minimal, we see that there is a recursive set, namely $\overline{P}$, such that $\mathrm{MIN}^*_\psi \cap \overline{P}$ is an infinite Π_2-set.

(II). We use the fact that the Π_2-sets are those which are co-r.e. relative to K. Let $W_0, W_1, W_2, \dots$ be an acceptable numbering of the r.e. sets with corresponding partial recursive functions $\varphi_0, \varphi_1, \varphi_2, \ldots$, let $U_0^K, U_1^K, U_2^K, \ldots$ be an acceptable numbering relative to K and let

$$B = \{2^x + j : 0 \leq j < 2^x \quad \wedge \quad \text{there are at least } 2^x - j - 1 \text{ indices } n \leq x \text{ such that } \{2^x, 2^x+1, \ldots, 2^x+j\} \subseteq U_n^K\}.$$

Since $B \in \Sigma_2$, let $\{B_s\}$ be a recursive approximation to B satisfying

$$z \in B \iff (\exists t)\,(\forall s > t)\,[z \in B_s].$$

We define the numbering $\nu_0, \nu_1, \ldots$ with corresponding domains $V_0, V_1, \ldots$ so that the following three conditions hold:

- $V_0 = \omega$;
- For $j \in \{0, 1, 2, \ldots, 2^x - 1\}$,

$$V_{2^x+j} = W_x \cup \{t : (\exists s > t)\,[2^x + j \notin B_s]\};$$

- $\nu_{2^x+(2^x-1)} = \varphi_x$.

This ordering satisfies

$$V_{2^x+j} =^* \begin{cases} W_x & \text{if } 2^x + j \in B, \\ \omega & \text{otherwise.} \end{cases} \tag{3.5}$$

The third bullet makes ν a Gödel numbering, so it remains only to show that MIN_ν^* does not contain an infinite Π_2-subset. Assume to the contrary, that $\overline{U_e^K} \subseteq \mathrm{MIN}_\nu^*$. As in Theorem 3.1 (ii), choose x large so that $x \geq e$ and

$$2^x + j \in \overline{U_e^K} \subseteq \mathrm{MIN}_\nu^*. \tag{3.6}$$

Note that $j > 0$ because $2^x \notin B$. It now follows from the definition of ν that

$$\{2^x, 2^x+1, \ldots, 2^x+j-1\} \subseteq \overline{\mathrm{MIN}_\nu^*} \subseteq U_e^K. \tag{3.7}$$

From (3.5) and (3.6) we have that $2^x + j \in B$, so by definition of B,

$$\text{There are } 2^x - j - 1 \text{ indices } n \in \{0, 1, \ldots, x\} - \{e\} \text{ such that } \{2^x, 2^x+1, \ldots, 2^x+j\} \subseteq U_n^K. \tag{3.8}$$

Finally by (3.7) and (3.8),

$$\text{There are } 2^x - j \text{ indices } n \in \{0, \ldots, x\} \text{ such that } \{2^x, 2^x+1, \ldots, 2^x+j-1\} \subseteq U_n^K.$$

This means that $2^x + j - 1 \in B$ and therefore $V_{2^x+j-1} =^* W_x$, contradicting that $2^x + j \in \mathrm{MIN}_\nu^*$. □

This completes the proof. □

An analogous result holds for $\mathrm{MIN}^{\mathrm{T}^{(n)}}$, using the following two lemmata.

Theorem 3.3 (Sacks Jump Theorem [15], [18])**.** *Let B be any set and let S be r.e. in B' with $B' \leq_{\mathrm{T}} S$. Then there exists a B-r.e. set A with $A' \equiv_{\mathrm{T}} S$. Furthermore, an index for A can be found uniformly from an index for S.*

Lemma 3.4 (Schwarz [17])**.** *Let B be a Σ_{k+3} set, where $k \geq 0$. Then there exists a recursive function f satisfying*

$$\begin{aligned} x \in B &\implies f(x) \in \mathrm{LOW}^k, \\ x \notin B &\implies f(x) \in \mathrm{HIGH}^k. \end{aligned}$$

Proof. It is known [18, Theorem IV.4.3] that for any $A \in \Sigma_3$, there exists a recursive function f satisfying

$$\begin{aligned} x \in B &\implies f(x) \in \mathrm{COF}, \\ x \notin B &\implies f(x) \in \mathrm{HIGH}^0. \end{aligned}$$

where HIGH^0 is the index set of the Turing complete r.e. sets. This proves the lemma for the case $n = 0$. Relativizing [18, Theorem IV.4.3], we obtain for each $B \in \Sigma_{k+3}$ a recursive g satisfying

$$\begin{aligned} x \in B &\implies W_{g(x)}^{\emptyset^{(k)}} \text{ is cofinite}, \\ x \notin B &\implies W_{g(x)}^{\emptyset^{(k)}} \equiv_{\mathrm{T}} \emptyset^{(k+1)}. \end{aligned}$$

k iterations of the Sacks Jump Theorem 3.3 now yield the result. □

Theorem 3.5. *For every $k \geq 0$, there exist Gödel numberings ψ and ν such that*

(I) $\mathrm{MIN}_{\psi}^{\mathrm{T}^{(k)}}$ *contains an infinite Π_{k+3}-set,*

(II) $\mathrm{MIN}_{\nu}^{\mathrm{T}^{(k)}}$ *is Π_{k+3}-immune.*

Proof. Fix $k \geq 0$. Let φ be any Gödel numbering and let W_e denote $\operatorname{dom} \varphi_e$.

(I). Let $E_0, E_1, \dots$ be a sequence of r.e. sets satisfying

- $(\forall n)\, [(E_n)' \equiv_{\mathrm{T}^{(k)}} \emptyset']$;
- $(\forall i \neq j)\, [E_i \not\equiv_{\mathrm{T}^{(k)}} E_j]$.

For example, we can take $E_0, E_1, \dots$ to be the sets constructed in Corollary 2.9. Let

$$A = \{n : (\forall e)\, [2^e < n \implies W_e \not\leq_{\mathrm{T}^{(k)}} E_n]\}.$$

Since $(E_n)' \equiv_{\mathrm{T}^{(k)}} \emptyset'$ for all k, we have $A \in \Pi_{k+3}$. Let

$$P = \{2^0, 2^1, 2^2, \dots\}.$$

Finally, define the Gödel numbering ψ to satisfy

- $\psi_{2^n} = \varphi_n$;
- $V_n = E_n$ if $n \in \overline{P}$,

where V_n denotes the domain of ψ_n.

Note that $A \cap \overline{P}$ is infinite, as there are at most e non-members below 2^e for every e. As $A \cap \overline{P} \in \Pi_{k+3}$, it remains only to show that $A \cap \overline{P} \subseteq \mathrm{MIN}_\psi^{\mathrm{T}^{(k)}}$. Let $n \in A \cap \overline{P}$. If $2^x < n$, then

$$V_{2^x} = W_x \not\leq_{\mathrm{T}^{(k)}} E_n = V_n.$$

If $x < n$ and $x \notin P$, then

$$V_x = E_x \not\equiv_{\mathrm{T}^{(k)}} E_n = V_n.$$

Hence $n \in \mathrm{MIN}_\psi^{\mathrm{T}^{(k)}}$. □

(II). Let $U_0, U_1, \ldots$ be an acceptable numbering relative to $\emptyset^{(k+2)}$. Define

$$\begin{aligned} B = \{2^x + j : 0 \leq j < 2^x \quad \wedge \quad & \text{there are at least } 2^x - j - 1 \\ & \text{indices } n \leq x \text{ such that } \{2^x, 2^x+1, \ldots, 2^x+j\} \subseteq U_n\}. \end{aligned}$$

Since $B \in \Sigma_{k+3}$, Lemma 3.4 gives off a corresponding recursive function f. Let g be the recursive "jump inversion" from Lemma 3.3 and let

$$g^{(k)} = \underbrace{g \circ g \circ \ldots \circ g}_{k}.$$

We define the Gödel numbering $\nu_0, \nu_1, \ldots$ with corresponding domains $V_0, V_1, \ldots$ by

- $V_0 = K$;
- For $0 \leq j < 2^x - 1$,

$$V_{2^x+j} = g^{(k)}\left((W_x)^{(k)} \oplus \left(W_{f(2^x+j)}\right)^{(k)}\right);$$

- $\nu_{2^x+(2^x-1)} = \varphi_x$.

Now ν satisfies

$$V_{2^x+j} \equiv_{\mathrm{T}^{(k)}} \begin{cases} W_x & \text{if } 2^x + j \in B, \\ K & \text{otherwise.} \end{cases} \tag{3.9}$$

Due to the similarity between (3.9) and (3.5), we can now proceed exactly as in Theorem 3.2 (II). □

This completes the proof. □

Remark. All of the Gödel numberings in this section can be converted into Kolmogorov numberings using a method such as [16, Theorem 2.17].

4 Properties of $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$

We investigate the minimal index set $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$. The main lemma of this section is Corollary 4.1, which follows from Lerman's revision [9] of Theorem 2.8 to account for the join operator. That the jump operator can be included when greatest element is omitted from the language was also mentioned in the discussion following [8, Theorem 7.10].

Corollary 4.1. *There exists a recursive sequence $\{x_k\}$ such that for all n and i,*

$$(W_{x_i})^{(n)} \not\leq_{\mathrm{T}} \bigoplus_{j\neq i} \left(W_{x_j}\right)^{(n)}. \tag{4.1}$$

In particular, $(W_{x_i})^{(n)} \mid_{\mathrm{T}} (W_{x_j})^{(n)}$ whenever $i \neq j$.

A direct proof of Corollary 4.1, without reference to [8] or [9], appears in [19, Theorem 6.1.1].

Remark. According to Lerman's result, it is even possible to replace (4.1) with the stronger relation

$$(W_{x_i})^{(n)} \not\leq_{\mathrm{T}} \left(\bigoplus_{j\neq i} W_{x_j}\right)^{(n)}.$$

Definition 4.2. Let f be a total function and let $A = \{a_0, a_1, \dots\}$ be an infinite set where the a_n are indexed in ascending order: $a_n < a_{n+1}$.

(I) The function $p_A(n) = a_n$ is called the *principal function* of A.

(II) A function f *majorizes* a set A if $(\forall n)\,[f(n) > p_A(n)]$.

Lemma 4.3 (Medvedev [12]). *An infinite set A is hyperimmune iff A is not majorized by a recursive function.*

We obtain the following satisfying result:

Theorem 4.4 (peak hierarchy). $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$

(I) *is infinite,*

(II) *contains no infinite arithmetic subsets, and*

(III) *is not hyperimmune.*

Proof. (I). Corollary 4.1 provides an infinite list of distinct $\equiv_{\mathrm{T}^{(\omega)}}$ classes. □

(II). Follows from Corollary 2.7, because $\mathrm{MIN}^{\mathrm{T}^{(\omega)}} \subseteq \mathrm{MIN}^{\mathrm{T}^{(n)}}$ for every n. □

(III). We verify that $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$ gets majorized. Let $\{x_k\}$ be as in Corollary 4.1. Then for all n and $i \neq j$,

$$W_{x_i} \not\equiv_{\mathrm{T}^{(n)}} W_{x_j}.$$

Without loss of generality, $x_0 < x_1 < \ldots$ since $\{x_k\}$ is recursive. Define the recursive function

$$f(0) = x_1$$
$$f(n+1) = x_{[2f(n)]},$$

and let p be the principal function of $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$. Note that $f(0) > 0 = p(0)$ and assume for the purposes of induction that $f(n) > p(n)$. Note that

$$p(n) \leq x_{p(n)} < x_{f(n)} < x_{f(n)+1} < \ldots < x_{2f(n)} = f(n+1),$$

so at least $f(n)$ x_k's lie strictly between $p(n)$ and $f(n+1)$, namely

$$\{x_{f(n)}, x_{f(n)+1}, \ldots, x_{2f(n)-1}\}.$$

Hence, at least $f(n)$ distinct $\equiv_{\mathrm{T}^{(\omega)}}$-equivalence classes are represented by indices strictly between $p(n)$ and $f(n+1)$. Since less than $f(n)$ classes are represented in indices up to $p(n)$, there necessarily must be a new $\equiv_{\mathrm{T}^{(\omega)}}$-class introduced strictly between $p(n)$ and $f(n+1)$. This forces $p(n+1) < f(n+1)$. Hence f majorizes $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$. The result now follows immediately from Lemma 4.3. □

This completes the proof. □

Consequently, the other minimal index sets in this paper share properties (i) and (iii):

Corollary 4.5. *Every set containing* $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$*, including* MIN^*, $\mathrm{MIN}^{\mathrm{m}}$ *and* $\mathrm{MIN}^{\mathrm{T}}$*, is infinite but not hyperimmune.*

Remark. $\emptyset^{(\omega)}$ is another familiar set which is hyperarithmetic and majorized by a recursive function. However, unlike $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$, $\emptyset^{(\omega)}$ contains a copy of $\emptyset'$. This means that $\emptyset^{(\omega)}$ is not at all immune.

Lusin once constructed a set of reals which neither contains nor is disjoint from any perfect set [10], [11, Theorem 2.25]. By modifying Lusin's construction and gently expanding $\mathrm{MIN}^{\mathrm{T}^{(\omega)}}$, we obtain an analogous construction for the arithmetic hierarchy which contains a familiar subset:

Corollary 4.6. *There exists a set* $X \supseteq \mathrm{MIN}^{\mathrm{T}^{(\omega)}}$ *such that* X*:*

(I) *contains no infinite arithmetic sets,*

(II) *is not disjoint from any infinite arithmetic set, and*

(III) *is majorized by a recursive function.*

5 Size-minimal random strings

We recall a theorem of Arslanov.

Theorem 5.1 (Arslanov Completeness Criterion [2]). *For any r.e. set A,*

$$A \equiv_{\mathrm{T}} \emptyset' \iff (\exists f \leq_{\mathrm{T}} A)\ (\forall x)\ [W_{f(x)} \neq W_x].$$

In this section, s is a recursive function whose name stands for "size." Size-minimal indices and descriptions of smallest size have received attention in [16, Section 3]. Schaefer [16] shows that there exists a recursive size-function s (independent of the Gödel numbering φ) such that

$$\mathrm{MIN}_{\varphi,s} = \{e : (\forall j)\ [s(j) < s(e) \implies \varphi_j \neq \varphi_e]\}$$

is hyperimmune, although this can not happen as long as $s(e) \leq s(e+1)$ for all e. When $\mathrm{MIN}_{\varphi,s}$ is hyperimmune we have $\mathrm{MIN}_{\varphi,s} \not\geq_{\mathrm{wtt}} \emptyset'$ [16] and when s is the identity function we have $\mathrm{MIN}_{\varphi,s} \equiv_{\mathrm{T}} \emptyset''$ [13], however the Turing degree of $\mathrm{MIN}_{\varphi,s}$ remains open in general.

Our investigation of size-minimal indices leads us to a generalization of the Kolmogorov random strings. Recall that the Kolmogorov random strings are defined as

$$R_\varphi = \{x : (\forall j)\ [l(j) < l(x) \implies \varphi_j(0) \neq x]\},$$

where l is the length function for integers encoded in binary. l could be taken to be any recursive function s, however, as in

$$R_{\varphi,s} = \{x : (\forall j)\ [s(j) < s(x) \implies \varphi_j(0) \neq x]\}.$$

Let

$$N = \{x : (\exists j)\ [s(j) < s(x) \quad \wedge \quad \varphi_j(0) = x]\}$$

be the complement of $R_{\varphi,s}$. Clearly N is an r.e. set.

Theorem 5.2. *The Turing degree of N depends on which of the following two cases applies:*

(a) *For all c there is an $x \notin N$ with $s(x) > c$.*

(b) *There is a constant c such that for all $x \notin N$ it holds that $s(x) < c$.*

In the first case, $N \equiv_{\mathrm{T}} K$. In the second case, N can have any many-to-one r.e. degree (other than $\emptyset$ or ω).

Proof. Assume (a). Let t be a recursive function such that $\varphi_{t(e)}(0)$ is the first element enumerated into W_e whenever it exists; so $\varphi_{t(e)}(0)$ is defined iff $W_e \neq \emptyset$. Now define a function f^N such that for every e, $W_{f^N(e)} = \{x\}$, where x is the first number found such that $x \notin N$ and $s(x) > s[t(e)]$. This means $\varphi_{t(e)}(0) \notin W_{f^N(e)}$. It follows that $W_e \neq W_{f^N(e)}$ for all e, and hence the Turing degree of N is fixed-point free. By Arslanov's Completeness Criterion 5.1, $N \equiv_{\mathrm{T}} K$.

Assume (b). In this case, not much can be said about the Turing degree of N. Indeed, the m-degree of N can be chosen to be equivalent to the m-degree of any r.e. B as follows, with $B, \overline{B}$ both not empty.

Given φ and B, one constructs s via a sequence $a_0, a_1, a_2, \ldots$ in stages. For this, let $b_0, b_1, b_2, \ldots$ be a recursive one-one enumeration of the set B. Now $a_0, a_1, a_2, \ldots$ is chosen using the Padding Lemma such that the following holds:

- $a_x \geq a_y + 2$ for all $y < x$;
- $a_x \notin \{2b_0, 2b_0+1\} \cup \{2b_1, 2b_1+1\} \cup \ldots \cup \{2b_x, 2b_x+1\}$;
- $\varphi_{a_x}(0) = \begin{cases} 2b_x & \text{if } s(2b_x) = 1, \\ 2b_x + 1 & \text{if } s(2b_x) = 0; \end{cases}$
- if $x \in \{a_0, a_1, \ldots, a_x\}$ then $s(x) = 0$ else $s(x) = 1$.

In the last condition, s designates $a_0, a_1, \ldots$ to be the "small" indices, all other indices are "large". Note that the first and last condition together imply that $s(x)$ and $s(x+1)$ are never both 0. Thus, according to the third condition, $B \leq_{\mathrm{m}} N$ by $x \in B \Leftrightarrow 2x + 1 - s(2x) \in N$. Furthermore,

$$(N(2x), N(2x+1)) = \begin{cases} (0,0) & \text{if } s(2x) = 0 \text{ and } x \notin B; \\ (0,1) & \text{if } s(2x) = 0 \text{ and } x \in B; \\ (0,0) & \text{if } s(2x) = 1 \text{ and } x \notin B; \\ (1,0) & \text{if } s(2x) = 1 \text{ and } x \in B. \end{cases}$$

This can be used to show that $N \leq_{\mathrm{m}} B$. So N and B are many-one equivalent. □

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