On One-Sided Versus Two-Sided Classification

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Abstract

One-sided classifiers are computable devices which read the characteristic function of a set and output a sequence of guesses which converges to 1 iff the set on the input belongs to the given class. Such a classifier is two-sided if the sequence of its output in addition converges to 0 on sets not belonging to the class. The present work obtains the below mentioned results for one-sided classes (= Σ^0_2 classes) w.r.t. four areas: Turing complexity, 1-reductions, index sets and measure.

There are one-sided classes which are not two-sided. This can have two reasons: (1) the class has only high Turing complexity. Then there are some oracles which allow to construct noncomputable two-sided classifiers. (2) The class is difficult because of some topological constraints and then there are also no nonrecursive two-sided classifiers. For case (1), several results are obtained to localize the Turing complexity of certain types of one-sided sets.

The concepts of 1-reduction, 1-completeness and simple sets is transferred to one-sided classes: There are 1-complete classes and simple classes, but no class is at the same time 1-complete and simple.

The one-sided classes have a natural numbering. Most of the common index sets relative to this numbering have the high complexity Π^1_1: the index sets of the class $\{0,1\}^\infty$, the index set of the equality problem and the index set of all two-sided classes. On the other side the index set of the empty class has complexity Π^0_2; Π^0_2 and Σ^0_2 are the least complexities any non-trivial index set can have.

Any one-sided class is measurable. It is shown that a one-sided class has effective measure 0 if it has measure 0, but that there are one-sided classes having measure 1 without having measure 1 effectively. The measure of a two-sided class can be computed in the limit.

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1 Introduction

Classification means to identify whether an object is contained in a class or not. In the present paper, the objects of classification are sets of natural numbers. Since these objects have no finite description, the classification algorithm makes up its mind from longer and longer approximations of the object and outputs at the same time a sequence of guesses which signal in the limit whether the current object is accepted, i.e., supposed to be in the class, or rejected, i.e., supposed to be outside the class. In particular a symmetric (two-sided) and more general asymmetric (one-sided) version to define acceptance and rejection is given.

The analysis of the task shows, that classification has to cope with two aspects of difficulty: the topological appearance of the class and its computational complexity. The present work is dedicated to look for a certain kind of classes on these topological and computational aspects of classification and the relation between them. Furthermore the process of classification is compared with that to compute or enumerate a set. Before going into the details, a brief summary of the overview on classification w.r.t. neighbouring subjects is given.

One of the first approaches to classification was to design finite automata which decide in the limit whether an infinite string (representing the characteristic function of a language) belongs to a given ω-language or not [3, 14, 15, 26]. The restrictive computational ability of these finite automata led Büchi [3] and his successors to consider non-deterministic automata. The present paper takes the alternate approach of choosing Turing machines as classifiers. In fact already Büchi and Landweber [4, 12] did some first research into this direction.

The next main root of classification is inductive inference. Wiehagen and Smith [27] introduced a model, in which a finite number of classes is given and the classifier has to detect at least one class where the language to be classified belongs to. As in many other definitions in the field of inductive inference [2, 9], the process has only to converge in the limit and so the machine has the right to withdraw hypotheses and to replace them by new ones. Wiehagen and Smith expected the process only to converge on the domain, so they avoided the topological problems which arize when the classifier has also to signal that a language does not belong to any class of the given collection.

Ben-David [1] and Kelly [10] started to investigate these topological aspects of classification: they showed that a (not necessarily computable) device can classify all sets with respect to one class in the limit iff the given class is the union of countably many closed classes and the intersection of countably many open classes at the same time; for this definition they use the Baire topology generated by the subbasis $\mathcal{A}_{x,y} = \{A : A(x) = y\}$ $(x \in \mathbb{N}, y \in \{0,1\})$. Gasarch, Pleszkoch and Velauthapillai [7, 8] extended the result by establishing a close relation between the topological Borel hierarchy and the quantifier-hierarchy of query-languages during classification.

This paper now returns to the computational aspect of classification. Ben-David

and Kelly already showed that a class is classifiable in the limit iff it is a (relativized) Δ^0_2 class; Rogers [21] called the Δ^0_2 classes also $\Delta^{(s)}_2$ classes. An other natural concept from learning theory is reliable inference which is implicit classification: if a function belongs to the inferred class then the learner converges to an index of this function, if a function does not belong to the class then the learner diverges. Such a mechanism is called a one-sided classifier since the learner signals only for the elements in the class the membership by convergence but not for those outside the class. It is equivalent to the notion of Σ^0_2 classes or $\Sigma^{(s)}_2$ classes as Rogers [21] called them. So one can isolate two natural notions of classification which coincide with the quite natural recursion-theoretic notions of Δ^0_2 and Σ^0_2 classes. So a machine H is a classifier for a class $\mathcal{A}$ iff H accepts every set in $\mathcal{A}$ by converging to 1 and rejects every set outside $\mathcal{A}$ by either diverging (one-sided classification) or converging to 0 (two-sided classification); the formal definition follows.

Definition 1.1 H is a one-sided classifier for $\mathcal{A}$ iff
$$(\forall A \in \mathcal{A})\,(\forall^\infty \sigma \preceq A)\,[H(\sigma) = 1]$$
and $$(\forall A \notin \mathcal{A})\,(\exists^\infty \sigma \preceq A)\,[H(\sigma) = 0];$$
M is a two-sided classifier for $\mathcal{A}$ iff
$$(\forall A \in \mathcal{A})\,(\forall^\infty \sigma \preceq A)\,[M(\sigma) = 1]$$
and $$(\forall A \notin \mathcal{A})\,(\forall^\infty \sigma \preceq A)\,[M(\sigma) = 0].$$

This paper now relates these two natural concepts of classification to each other. For each n, the concepts of Σ^0_n sets relate to that of the Δ^0_n almost in the same way as that of the enumerable ($= \Sigma^0_1$) sets to to that of the computable ($= \Delta^0_1$) sets. This work will show that on one hand for one-sided versus two-sided classes this analogy basically also holds but that on the other hand the similarities are much more restricted than the parallel definitions suggest at the first glance.

Example 1.2 [21] The notions of effective topology follow the definitions as stated by Ko [11, p. 72, p. 165] and Rogers [21, Chapter 15]: A class $\mathcal{A}$ is recursively open iff there is a recursive sequence $\sigma_0, \sigma_1, \ldots$ of strings such that $A \in \mathcal{A} \Leftrightarrow (\exists n)\,[\sigma_n \preceq A]$. A class $\mathcal{A}$ is a recursively G_δ class iff there is a recursive array $\sigma_{m,n}$ of strings such that $A \in \mathcal{A} \Leftrightarrow (\exists m)\,(\forall n)\,[\sigma_{m,n} \not\preceq A]$.

Now any recursively open class is two-sided and any recursively G_δ class is one-sided. There are recursively G_δ classes which are one-sided but not two-sided, e.g., the class of all finite sets.

One important tool in recursion theory is that there is an acceptable numbering of all enumerable sets. Similarly one can obtain an acceptable numbering of all one-sided classes given by total one-sided classifiers.

Theorem 1.3 *There is an effective list $H_0, H_1, \ldots$ of one-sided classifiers such that every one-sided class is generated by such a classifier and every machine H_e is total. The so defined numbering of the one-sided classes is acceptable: every further effective numbering $\mathcal{G}_0, \mathcal{G}_1, \ldots$ can be represented via a computable function: $\mathcal{H}_{s(e)} = \mathcal{G}_e$.*

This effective list of classifiers H_e is generated from an acceptable numbering φ_e of all partial computable functions which of course contains all classifiers. The formal definition is

$$H_e(\sigma) = \begin{cases} \varphi_e(\tau) & \text{for the longest } \tau \preceq \sigma \text{ such that} \\ & \varphi_e(\tau) \text{ outputs 0 or 1 within } |\sigma| \text{ steps;} \\ 0 & \text{if there is no such } \tau. \end{cases}$$

It is easy to verify that whenever φ_e is a one-sided classifier for $\mathcal{A}$, then so is H_e; and whenever φ_e is a two-sided classifier for $\mathcal{A}$, then so is H_e.

At many places this list H_e of one-sided classifiers will be quite useful; in particular it is much more handy to use the H_e in diagonalizations than the φ_e since the H_e are always total and $\{0,1\}$-valued. $\mathcal{H}_e$ denotes the one-sided class generated by H_e.

Both, one-sided and two-sided classes form a lattice. Furthermore a class is two-sided iff the class and its complement are one-sided [21, Chapter 15]. The next theorem shows, that a further well-known result from the hierarchy of sets, namely that every infinite Σ^0_n set has an infinite Δ^0_n subset, does not hold for classes and needs "uncountable" instead of "infinite".

Theorem 1.4 *Every uncountable one-sided class has a two-sided subclass of same cardinality. There is a one-sided infinite class which has no two-sided infinite subclass.*

Proof $\mathcal{A}_k = \{A \in \mathcal{A} : H \text{ outputs on input } A \text{ exactly } k \text{ 0s}\}$. Every class $\mathcal{A}_k$ is two-sided via the following algorithm M_k: $M_k(\sigma) = 1$ iff H outputs on exactly k inputs $\tau \preceq \sigma$ a 0 and $M_k(\sigma) = 0$ otherwise; so M makes on each set $A \in \mathcal{A}_k$ at most two mind changes. Since the cardinality of $\mathcal{A}$ is not countable, one of the countably many two-sided subclasses $\mathcal{A}_k$ has the same cardinality as $\mathcal{A}$. So the first statement of the theorem holds.

The second statement is proven by constructing an infinite one-sided class $\mathcal{A} = \{A_0, A_1, \ldots\}$ without any infinite two-sided subclass. For each machine H_k and each finite set D, $F(D,k,n)$ denotes the length of the shortest string $\sigma \preceq D$ such that either $H_k(\tau) = 1$ for at least n strings $\tau \preceq \sigma$ or $H_k(\tau) = 0$ for all strings τ with $\sigma \preceq \tau \preceq D$. The function F is computable relative to K. The class $\mathcal{A}$ is given via the following inductive definition:

$$\begin{aligned} A_0 &= \emptyset; \\ G(n) &= n + F(A_n, 0, n) + F(A_n, 1, n) + \ldots + F(A_n, n, n); \\ A_{n+1} &= A_n \cup \{G(n)\} = \{G(0), G(1), \ldots, G(n)\}. \end{aligned}$$

G is computable relative to K and has therefore a computable approximation G_s. Now the following machine H is a one-sided classifier for $\mathcal{A}$:

$$H(\sigma) = \begin{cases} 1 & \text{if there is a } \tau \text{ such that } \sigma = \tau 0 \text{ and} \\ & range(\tau) = \{G_{|\sigma|}(0), G_{|\sigma|}(1), \ldots, G_{|\sigma|}(ones(\tau))\}; \\ 0 & \text{otherwise;} \end{cases}$$

where $ones(\tau)$ is the number of all x with $\tau(x)\downarrow = 1$. Assume that A has the finite cardinality n. H converges to 1 if $A = A_n$ and H converges to 0 if $A \neq A_n$. Otherwise A is infinite and infinitely many $\sigma \preceq A$ have the form $\tau 1$. So H outputs for these the value 0 and H rejects every infinite set.

It remains to show that every two-sided subclass of $\mathcal{A}$ is finite. Let the machine $M = H_e$ be any two-sided classifier for some subclass of $\mathcal{A}$ and consider the set $A = \{G(0), G(1), \ldots\} = range(G)$. If M outputs during the classification of some set A_k with $k \geq e$ at least k 1s, then M outputs at least k of them on strings $\tau \preceq A_k$ which are shorter than $G(k)$. Thus these τ are also prefixes of A and M outputs on A at least k 1s. Since M converges on A to 0, M outputs on A only finitely often a 1 and there are only finitely many A_k with $k \geq e$ which M classifies with 1 in the limit. It follows that M is a classifier for a finite subclass of $\mathcal{A}$ and so also the third statement holds. ∎

2 Classification and Turing Complexity

Post [19] studies the enumerable sets w.r.t. their Turing degrees. His main question was whether for the decision procedures of all noncomputable enumerable sets have the same complexity. The work to solve this and similar problems initiated a large study of the Turing degrees of enumerable sets which Soare [25] gives a comprehensive overview.

The analogous question for one-sided classes is to determine the amount of complexity which is necessary to compute a two-sided classifier for them. The straight forward implementation of this idea would be to identify each class with the easiest two-sided classifier for it – but somehow such two-sided classifier sometimes do not exist and if they exist, there may be no one of least complexity.

Therefore classes are (in general) not related to a single Turing degree but to a collection of Turing degrees. This collection is called the Turing complexity of a class and consists of all oracles which allow to compute a two-sided classifier for the given class.

The one-sided classes are ordered in terms of their Turing complexity: So $\mathcal{A}$ has Turing complexity below that of $\mathcal{B}$ iff the class of all oracles relative to which $\mathcal{A}$ is two-sided is a superset of of the corresponding class for $\mathcal{B}$. W.r.t. this ordering, there are one-sided classes of least and greatest Turing complexity. Furthermore among those of intermediate Turing complexity, there are some classes whose Turing complexity can be identified with a single Turing degree: Such a class has Turing degree $\mathbf{a}$ iff the Turing-degrees of the two-sided classifiers just form a cone above $\mathbf{a}$:

$$\{deg_T(M) : M \text{ is a two-sided classifier for } \mathcal{A}\} = \{\mathbf{b} : \mathbf{a} \leq \mathbf{b}\}.$$

So there are four types of one-sided classes with respect to their Turing complexity.

- A two-sided class has the least possible Turing complexity since it is two-sided relative to every oracle. $\emptyset$ and $\{0,1\}^\infty$ are examples of two-sided classes. Every two-sided class has a Turing degree, namely the degree $\mathbf{0}$ of the computable sets.

- There are one-sided classes which are not two-sided relative to any oracle. So they have the greatest possible Turing complexity: Example 2.6 gives three such one-sided classes.

- There is a one-sided class which has a non-recursive Turing degree.

- There is a one-sided class which is two-sided relative to some oracles but which does not have a Turing degree.

The next results deal with the classes of intermediate Turing complexity according to the third and fourth case. Sacks [22, Section II.4] called a set A a Π^0_2 singleton iff A is the only set B which satisfies $(\forall x)(\exists y)[R(x,y,B)]$ for some recursive predicate R. Such Π^0_2 singletons allow to construct a one-sided classes $\mathcal{A}$ which has a nonrecursive Turing degree.

Theorem 2.1 *The cosingle class $\mathcal{A} = \{B : B \neq A\}$ has a Turing degree which is exactly that of A. Furthermore $\mathcal{A}$ is one-sided iff A is a Π^0_2 singleton. So every hyperarithmetic set is below the Turing degree of some one-sided class.*

Proof Assume that M classifies two-sided $\mathcal{A}$. Then there is a finite string $\sigma \preceq A$ such that $M(\tau) = 0$ for all τ with $\sigma \preceq \tau \preceq A$. The binary tree

$$T = \{\tau : (\forall \eta \preceq \tau)\,[\eta \preceq \sigma \vee M(\eta) = 0]\}$$

is recursive in M and A is its only recursive branch. Therefore $M \geq_T A$. On the other hand, an A-oracle is obviously sufficient for two-sided classification since this means only to compare with A.

To show the second statement, assume that $\mathcal{A}$ is one-sided via H. Then let

$$R(x,y,B) = \begin{cases} 1 & \text{if } H(B(0)B(1)\ldots B(z)) = 0 \text{ for at least } x \text{ numbers } z \leq y; \\ 0 & \text{otherwise.} \end{cases}$$

R is recursive. It is easy to see that $(\forall x)\,(\exists y)\,[R(x,y,B)]$ iff H outputs on B infinitely many 0s iff $B = A$. So A is a Π^0_2 singleton.

For the other way round let A be a Π^0_2 singleton, i.e., there is a recursive predicate R such that A is the only set B for which $(\forall x)\,(\exists y)\,[R(x,y,B)]$ holds. W.l.o.g. $R(x,y,B)$ is computed only by making queries at B below y, to obtain this one could replace y by a pair $\langle y,z\rangle$, R by R' and say that $R'(x,\langle y,z\rangle,B)$ is satisfied iff $R(x,y,B)$ holds and z is the place of the largest query to B during the computation of $R(x,y,B)$. Now let

$$h(B,n) = \max\{m \leq n : (\forall x \leq m)\,(\exists y \leq n)\,[R(x,y,B)]\}$$

and

$$H(\sigma a) = \begin{cases} 1 & \text{if } h(\sigma a, |\sigma a|) = h(\sigma, |\sigma|); \\ 0 & \text{otherwise, i.e., } h(\sigma a, |\sigma a|) > h(\sigma, |\sigma|); \end{cases}$$

where the expression $h(\sigma, n)$ abbreviates $\max\{h(B, n) : \sigma \preceq B\}$. The statement $h(\sigma, |\sigma|)$ can in fact be computed without looking at the values of any B beyond $|\sigma|$ and so depends only on σ. Having this in mind one can see that H outputs on input B infinitely often a 0 iff $\lim_{n\to\infty} h(B, n) = \infty$ iff for every x there is an y such that $R(x, y, B)$ holds. So H outputs infinitely many 0s on input A and only finitely many on any input $B \neq A$; thus H is a one-sided classifier for $\mathcal{A}$.

The third statment follows from the fact that the hyperarithmetic sets are just the Turing closure downward of the Π^0_2 singletons [22, Section II.4]. ∎

The single one-sided classes $\{A\}$ are less complex than the cosingle ones: they are already two-sided without oracle and so have least Turing complexity. Some of the cocountable classes, which are a natural generalization of the cosingle classes, do not have a Turing degree, but they all are two-sided relative to some hyperarithmetic oracle. Theorem 2.2 gives an example of a class which does not have a Turing degree and whose Turing complexity is just the collection of all high Turing degrees. There are also countable and cocountable examples with the same property, but in order to have a better readable proof the easiest example is chosen.

Theorem 2.2 *There is a one-sided class $\mathcal{A}$ whose Turing complexity is the collection of all high Turing degrees. $\mathcal{A}$ does not have a Turing degree.*

Proof Let $\mathcal{A}$ contain all non-empty sets A such that W_e is finite for $e = \min(A)$. A one-sided classifier H for $\mathcal{A}$ is given via

$$H(\sigma) = \begin{cases} 1 & \text{if } \sigma \succeq 0^e 1 \text{ and } W_{e,|\sigma|+1} = W_{e,|\sigma|}; \\ 0 & \text{otherwise.} \end{cases}$$

If $A \succeq 0^e 1$ and W_e is finite, then $W_{e,s+1} = W_{e,s}$ for almost all s. Thus H converges to 1 on these A. Otherwise $A = \emptyset$ and H always outputs 0 or $A \succeq 0^e 1$ and W_e is infinite. In this latter case, $W_{e,s+1} \neq W_{e,s}$ for infinitely many s and $H(\sigma) = 0$ for infinitely many $\sigma \preceq A$. So H is a one-sided classifier for $\mathcal{A}$.

On the other hand it is easy to see that a two-sided classifier M converges on $0^e 1^\infty$ to 1 iff W_e is finite. Since this problem is m-equivalent to K', the set K' can be computed in the limit relative to M and this is possible iff M has high Turing degree.

The second statement of the theorem follows from the fact that the high Turing degrees do not form a cone. ∎

The next three theorems establish further results for classes of intermediate Turing complexity.

Theorem 2.3 *If a one-sided class is two-sided relative to some oracle, then it is two-sided relative to an oracle in Δ^1_2.*

Proof Let H be a computable one-sided classifier for $\mathcal{C}$. Assume furthermore that $\mathcal{C}$ is two-sided relative to some oracle. Now any, not necessarily recursive, two-sided classifier M satisfies the following Π^1_1 equation:

$$\begin{aligned}(\forall A)\,[\,(\,(\exists^\infty \sigma \preceq A)\,[H(\sigma)=0]\;\Rightarrow\;(\forall^\infty \sigma \preceq A)\,[M(\sigma)=0]\,)\;\wedge\\ (\,(\forall^\infty \sigma \preceq A)\,[H(\sigma)=1]\;\Rightarrow\;(\forall^\infty \sigma \preceq A)\,[M(\sigma)=1]\,)\,]\end{aligned}$$

It states that for all A, whenever H accepts or rejects A one-sidedly, so does M two-sidedly. It follows that M is a solution to the predicate iff M is a two-sided classifier for $\mathcal{C}$. So M is specified via a Π^1_1 predicate. Provided that this predicate has at least one solution, Addison and Kondo [22, Corollary 9.4] showed that there is a further Π^1_1 predicate $(\forall A)[P(M,A)]$ which has exactly one solution and whose solution N is also a solution to the original predicate. The sets $\{\sigma : N(\sigma)=c\}$ are in Σ^1_2 for $c=0,1$:

$$N(\sigma)=c \;\;\Leftrightarrow\;\; (\exists M)\,(\forall A)\,[M(\sigma)=c \wedge P(M,A)].$$

Since one set is the complement of the other, it follows that the machine N is in Δ^1_2. ∎

Theorem 2.4 *Let H be a one-sided classifier for $\mathcal{A}$ and for any $A \notin \mathcal{A}$ let $f_A(m)$ denote the first $k > m$ such that $H(A(0)A(1)\ldots A(k))=0$. Then $\mathcal{A}$ has a two-sided classifier of degree* **a** *iff there is a function g of degree* **a** *which dominates the functions f_A for all $A \notin \mathcal{A}$.*

Proof Assume that g dominates the functions f_A for all $A \notin \mathcal{A}$. The following $M \leq_T g$ is a two-sided classifier for $\mathcal{A}$:

$$M(A(0)A(1)\ldots A(n)) = \begin{cases} 1 & \text{if } g(m) < n \text{ for all } m \leq n \\ & \text{with } H(A(0)A(1)\ldots A(m))=0; \\ 0 & \text{otherwise.}\end{cases}$$

If $A \in \mathcal{A}$ then there are only finitely many m with $H(A(0)A(1)\ldots A(m))=0$. Almost all n are greater than $g(m)$ for each such m and thus $M(A(0)A(1)\ldots A(n))=1$ for almost all n. Otherwise $A \notin \mathcal{A}$ and g dominates f_A. So for almost all m there is a k with $m < k \leq g(m)$ and $H(A(0)A(1)\ldots A(k))=0$. So for almost all n, the greatest $m \leq n$ with $H(A(0)A(1)\ldots A(m))=0$ satisfies this condition and since by the choice of m there is no such k between m and n, the relation $n \leq g(m)$ holds. It follows that $M(A(0)A(1)\ldots A(n))=0$ for almost all n. So M is a two-sided classifier for $\mathcal{A}$.

For the other way round let $M \leq_T E$ be a two-sided classifier for $\mathcal{A}$. The following function g is computable relative to E:

$$\begin{aligned} g(n) &= 2+\max\{|\sigma| : \sigma \in T_n\} \text{ where} \\ T_n &= \{\sigma : (\forall \tau \preceq \sigma)\,[\,|\tau| \leq n \vee (M(\tau)=0 \wedge H(\tau)=1)\,]\,\}. \end{aligned}$$

Assume that g would be undefined for some n. Then T_n is infinite and by König's Lemma the tree T_n has an infinite branch A. M converges on A to 0 while H outputs on input A only finitely many 0s, i.e., M and H classify A differently in contradiction to the choice of M and H. So T_n is finite and g is total. Since T_n is computable relative to E, its maximal string can be found using the oracle E and $g \leq_T E$.

Let $A \notin \mathcal{A}$. There is an n such that $M(A(0)A(1)\ldots A(m)) = 0$ for all $m \geq n$. Assume now by way of contradiction that $f_A(m) > g(m)$ for some $m \geq n$. Then $M(\sigma) = 0$ and $H(\sigma) = 1$ for all $\sigma \preceq A$ with $m < |\sigma| \leq g(m)$ and the string $A(0)A(1)\ldots A(g(m))$ is in T_m in contradiction to $g(m)$ being greater than the length of all strings in T_m. Thus such an m does not exist and $f_A(m) \leq g(m)$ for almost all m. ∎

A consequence of this is that every one-sided class, which is two-sided relative to a hyperimmune-free oracle, is already two-sided via a classifier without any access to oracles.

Theorem 2.5 *If $\mathcal{A}$ has a Turing degree then $\mathcal{A}$ has a hyperarithmetic Turing degree.*

Proof Assume that $\mathcal{A}$ has Turing degree $\mathbf{a}$ and let M be a two-sided classifier for $\mathcal{A}$ of degree $\mathbf{a}$. Theorem 2.4 implies that $\mathbf{a} \leq \mathbf{b}$ whenever every $\mathbf{a}$-recursive function is dominated by a $\mathbf{b}$-recursive function. There is a function f_0 dominating every function computable relative to $\mathbf{a}$ such that whenever g is a majorant of f_0 then $M \leq_T g$. First it has to be shown that this can be done via a single index, i.e.,

$$(\exists e, f)\,(\forall \text{ majorants } g \text{ of } f)\,[M = \{e\}^g]$$

and in a second step it is deduced that M is a hyperarithmetic. The existence of such e and f is shown via an algorithm which either provides the information to find e and f or which constructs a majorant of g of f_0 such that $M \not\leq_T g$. In this proof $\sigma_0, \sigma_1, \ldots$ denote strings of numbers and not of bits. σ_0 is the empty string.

> Given σ_n and f_n check whether there is a majorant f_{n+1} of f_n and an extension $\sigma_{n+1} \succeq \sigma_n$ such that
>
> - $\sigma_n \prec \sigma_{n+1} \preceq f_{n+1}$ and
> - there is some $\eta \in \{0,1\}^*$ such that either $\{n\}^{\sigma_{n+1}}(\eta)\downarrow \neq M(\eta)$ or $\{n\}^g(\eta)\uparrow$ for all majorants g of f_{n+1} with $g \succeq \sigma_{n+1}$.
>
> If there are such σ_{n+1}, f_{n+1} the algorithm proceeds with them in the next step otherwise it terminates.

If the algorithm goes through all steps, then $g = \lim_n \sigma_n$ exists and is a majorant of f_0. By construction, for all e there is some $\eta \in \{0,1\}^*$ such that either $\{e\}^g(\eta)\downarrow \neq M(\eta)$ (since $\{e\}^{\sigma_{e+1}}(\eta)\downarrow \neq M(\eta)$ and $\sigma_{e+1} \preceq g$) or $\{e\}^g(\eta)\uparrow$ (since g is a majorant of f_{e+1}).

So M is not computed relative to g via any e in contradiction to the choice of f_0.

Thus the algorithm terminates in some stage n. Now for each g the following set is not empty:

$$F(g,\eta) = \{\tau \succeq \sigma_n : \{n\}^\tau(\eta)\downarrow \wedge (\forall m \in dom(\tau) - dom(\sigma_n))\,[\tau(m) \geq g(m)]\}.$$

Furthermore whenever g is a majorant of f_n and $\tau \in F(g,\eta)$ then $\{n\}^\tau(\eta)\downarrow = M(\eta)$. Using these two facts it is possible to construct e:

> If within $|\eta|$ steps no triple (θ,τ_1,τ_2) has been enumerated witnessing inconsistency in the way that $\tau_1,\tau_2 \in F(g,\theta)$ and $\{n\}^{\tau_1}(\theta) \neq \{n\}^{\tau_2}(\theta)$
> then $\{e\}^g(\eta) = \{n\}^\tau(\eta)$ for the first τ found in $F(g,\tau)$
> else $\{e\}^g(\eta)\uparrow$.

So $M = \{e\}^g$ for all majorants g of f_n and $\{e\}^g$ is partial if $M \neq \{e\}^g$. The second step is now easy. The sets $M_c = \{\eta : M(\eta) = c\}$ are Π^1_1 according to the following definition:

$$M(\eta) = c \;\Leftrightarrow\; (\forall g)\,[\{e\}^g \text{ is total } \Rightarrow \{e\}^g(\eta) = c].$$

Since M_0 is the complement of M_1, both sets are in Δ^1_1 and M is hyperarithmetic. ∎

Example 2.6 *The following classes are one-sided but not relatively two-sided:*
(a) $\mathcal{A} = \{A : A \text{ is cofinite}\}$ [7, Corollary 3.3].
(b) $\mathcal{B} = \{B : B \text{ is primitive recursive}\}$.
(c) $\mathcal{C} = \{C \oplus D : D \neq C'\}$.

Proof
(a) The one-sided classifier H outputs a on input σa. It takes almost always the value 1 iff the input is a cofinite set. On the other hand this class is not relatively two-sided since for any function g there is a set $A \notin \mathcal{A}$ such that g does not dominate f_A. Namely for given g, this set is $A = \mathbb{N} - \{x_0, x_1, \ldots\}$ where $x_0 = g(0) + 1$ and $x_{n+1} = g(x_n) + x_n + 1$. From the definition of H it follows that $f_A(x_n) = x_{n+1} = g(x_n) + x_n + 1 > g(x_n)$ and g does not dominate f_A.

(b) Let $A_0, A_1, \ldots$ be a uniform enumeration of all primitive recursive sets. Now on input σa, the one-sided machine looks for the first k with $\sigma \preceq A_k$. If then also $\sigma a \preceq A_k$ it outputs 1 otherwise it outputs 0. It is easy to see that the machine outputs 1 on almost all inputs $\sigma \preceq A$ iff $A = A_k$ for some k. The other part of the proof uses that $\mathcal{B}$ is not the intersection of countably many open sets and thus not a relativized Π^0_2 class [21, Theorem IX(b)].

(c) For each set C there is a uniform approximation of C' via strings γ^C_n of C such that each γ^C_n queries C only at the places $0, 1, \ldots, n$, $|\gamma^C_n| \leq n$ and $\gamma^C_n \preceq C'$ infinitely often. Now the one-sided machine H outputs on strings of odd length an 1 and processes strings of even length as follows:

$$H(C(0)D(0)C(1)D(1)\ldots C(n)D(n)) = \begin{cases} 0 & \text{if } \gamma^C_n \preceq D(0)D(1)\ldots D(n); \\ 1 & \text{otherwise.} \end{cases}$$

As mentioned, γ_n^C can be computed using only the $C(m)$ with $m \leq n$, thus the whole procedure needs no oracle but retrieves the answers from the input. If $D \neq C'$ then $\gamma_n^C \not\preceq D$ for almost all n; therefore H accepts all sets in $\mathcal{C}$. If $D = C'$ then there are infinitely many n with $\gamma_n^C \preceq D(0)D(1)\ldots D(n)$. H takes 0 at these n and thus rejects all sets outside $\mathcal{C}$. So H is a one-sided classifier for $\mathcal{C}$.

Assume now by the way of contradiction that $\mathcal{C}$ is two-sided via M. Now M can also be viewed as a set and so one can consider the class $\{C \oplus D : C \neq M \vee D \neq M'\}$. A machine to identify this set can be derived from M as follows:

$$N(\sigma) = \begin{cases} M(\sigma) & \text{if } \sigma(x) = M(\frac{x}{2}) \text{ for all even } x \in dom(\sigma); \\ 1 & \text{otherwise;} \end{cases}$$

N is obviously recursive in M. On the other hand, the new class is cosingle and contains all sets except $M \oplus M'$. Then $M \oplus M'$ has to be recursive in M, a contradiction. ∎

Singleton one-sided classes are always two-sided and therefore less complex than cosingleton ones which can have a Turing degree above every given hyperarithmetic Turing degree. Somehow the relation between countable and cocountable one-sided classes is the other way round: While some countable one-sided classes can have the highest possible Turing complexity this is not true for cocountable classes.

Theorem 2.7 *Any cocountable one-sided class $\mathcal{A} = \{B : (\forall n)\,[B \neq A_n]\}$ is two-sided relative to some hyperarithmetic set.*

Proof Let H be a one-sided classifier for $\mathcal{A}$. Sacks [22, Theorem III.6.2] showed that every Σ_1^1 class either contains a perfect subclass (and is just uncountable) or has only members below some hyperarithmetic set. Since each one-sided class is defined without quantification over sets or functions, its complement is a Σ_1^1 class (indeed it is even a Δ_1^1 class). So there is a hyperarithmetic set C such that $A_n \leq_T C$ for all n. Now for each n the function f_{A_n} as defined in Theorem 2.4 is recursive in A_n. Some function $g \leq_T C'$ dominates all functions computable relative to C, in particular g dominates each function f_{A_n}. By Theorem 2.4 the class $\mathcal{A}$ is two-sided relative to C' which has hyperarithmetic Turing degree since the hyperarithmetic Turing degrees are closed under the jump. ∎

3 Complete Classes

There are some other reducibilities between sets besides Turing reduction. Post [19] introduced the concept of 1-reduction: A set A is 1-reducible to B iff there is a one-one computable function f such that $x \in A \Leftrightarrow f(x) \in B$. The set K is complete within the enumerable sets, i.e., every enumerable set can be 1-reduced to K.

It is possible to transfer the notion of 1-reduction to the world of classification.

Here a 1-reduction from a class $\mathcal{A}$ to a class $\mathcal{B}$ is a one-one computable and continuous operator Γ translating every set A into a set $\Gamma(A)$ such that $A \in \mathcal{A} \Leftrightarrow \Gamma(A) \in \mathcal{B}$.

Definition 3.1 A computable operator Γ is called a 1-reduction from $\mathcal{A}$ to $\mathcal{B}$ if

- Γ is strictly monotone (w.r.t. $\preceq$), i.e., $\Gamma(\sigma) \preceq \Gamma(\tau)$ iff $\sigma \preceq \tau$ for all $\sigma, \tau \in \{0,1\}^*$.
- $A \in \mathcal{A}$ iff $\Gamma(A) = \lim_{\sigma \preceq A} \Gamma(\sigma) \in \mathcal{B}$ for all sets A.

A class $\mathcal{A}$ is called 1-complete iff every one-sided class is 1-reducible to it and $\mathcal{A}$ itself is one-sided.

It is easy to see that if $\mathcal{A} \leq_1 \mathcal{B}$ and $\mathcal{B}$ is two-sided via a (nonrecursive) machine M then $\mathcal{A}$ is also two-sided via a classifier computable relative to M. Since there are classes which are not relatively two-sided, the following 1-complete class is not relatively two-sided and does not have a Turing degree.

Theorem 3.2 *The class* $\mathcal{K} = \{A : (\forall^\infty \text{even } x)[x \in A]\}$ *is 1-complete.*

Proof The classifier

$$H(a_0 a_1 \ldots a_n) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ a_n & \text{if } n \text{ is even;} \end{cases}$$

witnesses that $\mathcal{K}$ is one-sided. Assume now that L is a computable one-sided classifier for a further class $\mathcal{A}$. A 1-reduction Γ from $\mathcal{A}$ to $\mathcal{K}$ is defined as follows:

$$\begin{aligned} \Gamma(\lambda) &= L(\lambda); \\ \Gamma(\sigma a) &= \Gamma(\sigma) a L(\sigma a). \end{aligned}$$

From this equation it follows that $H(\Gamma(\sigma)) = L(\sigma)$ and that whenever $\eta \preceq \Gamma(A)$ and $H(\eta) = 0$ then $\eta = \Gamma(\sigma)$ for some σ. Therefore the following two statements are equivalent:

$L(\sigma) = 0$ for infinitely many $\sigma \preceq A$,
$H(\eta) = 0$ for infinitely many $\eta \preceq \Gamma(A)$.

It follows that $A \in \mathcal{A}$ iff $\Gamma(A) \in \mathcal{K}$ and $\mathcal{A}$ is 1-reducible to $\mathcal{K}$. So $\mathcal{K}$ is 1-complete. ∎

Theorem 3.3 *The class* $\mathcal{A}$ *of all cofinite sets has greatest Turing complexity but is not 1-complete.*

Proof By Example 2.6 (a), the class $\mathcal{A}$ of all cofinite sets has greatest Turing complexity. But $\mathcal{A}$ is not 1-complete: Consider the full class $\{0,1\}^\infty$ of all sets. If $\{0,1\}^\infty$ is 1-reducible to $\mathcal{A}$ via Γ then every infinite branch of the tree $\Gamma(\{0,1\}^*)$ would be contained in $\mathcal{A}$. This contradicts the fact that $\mathcal{A}$ contains only countably many sets. ∎

Post [19] showed that simple sets are not complete under various constructions. Indeed it is possible to define something analogue to simple set and to show that it is

not 1-complete: A one-sided class is called simple iff it intersects every other infinite one-sided class.

Theorem 3.4 *No 1-complete class is simple.*

Proof Let $\mathcal{A}$ be a 1-complete class. Then the class $\mathcal{C}$ of all cofinite sets is 1-reducible to $\mathcal{A}$ via some reduction Γ. Now it is shown that $\mathcal{A}$ is not simple via showing that the class $\{\Gamma(A_0), \Gamma(A_1), \ldots\}$ is a two-sided infinite class disjoint to $\mathcal{A}$ where $A_x = \{x\}$. No set $A_x = \{x\}$ is contained in $\mathcal{C}$. Thus also no set $\Gamma(A_x)$ is in $\mathcal{A}$ and $\mathcal{B} = \{\Gamma(A_0), \Gamma(A_1), \ldots\}$ is an infinite class disjoint to $\mathcal{A}$. It remains to be shown that $\mathcal{B}$ is one-sided; indeed it will be shown that the following machine M is a two-sided classifier for $\mathcal{B}$.

$$M(\sigma) = \begin{cases} 1 & \text{if there is an } x \text{ with } \Gamma(0^x1) \preceq \sigma \preceq \Gamma(0^x10^\infty); \\ 0 & \text{otherwise.} \end{cases}$$

The check whether such an x exists, is computable: Only the $x \leq |\sigma|$ have to be considered since the string $\Gamma(0^x1)$ is longer than σ for $x > |\sigma|$. Furthermore the sets $\Gamma(A_x)$ are uniformly recursive, so the whole check and thus M is a computable procedure.

During the classification of any set A, M makes only two mind changes: from the initial guess 0 to 1 if it turns out that $\Gamma(0^x1) \preceq A$ for some x. A further mind change back to 0 if A turns out to be different from $\Gamma(A_x)$. Since no string $\Gamma(0^y1)$ with $y \neq x$ extends $\Gamma(0^x1)$, there is no danger of a third mind change from 0 to 1 because of such a $\Gamma(0^y1)$ being a prefix of A.

It is now easy to verify is that M converges on the sets $\Gamma(A_x)$ to 1 and on all other sets to 0; so M is a classifier for $\mathcal{B}$.

In particular $\mathcal{B}$ is a one-sided infinite class which is disjoint to $\mathcal{A}$ and thus witnesses that $\mathcal{A}$ is not simple. ∎

4 Index Sets of One-Sided Classes

Let $\mathcal{G}$ be a collection of classes and $\mathcal{H}_e$ denote the class generated by the e-th one-sided classifier H_e. Then the set $E = \{e : \mathcal{H}_e \in \mathcal{G}\}$ is called the index set of $\mathcal{G}$ and every such set E belonging to such a $\mathcal{G}$ is called an index set. So this section tries to look at the analogon of the index sets of classes of enumerable sets; while those are mostly situated in the arithmetical hierarchy these index sets of one-sided classes have often the complexity Π^1_1. The first example of such an index set is the equality problem $\{\langle e, e'\rangle : \mathcal{H}_e = \mathcal{H}_{e'}\}$.

Theorem 4.1 *The set* $\{\langle e, e'\rangle : \mathcal{H}_e = \mathcal{H}_{e'}\}$ *is* Π^1_1 *complete.*

Proof The formula

$$(\forall A)\,[\,(\exists^\infty \sigma \preceq A)\,[H_e(\sigma) = 0] \;\Leftrightarrow\; (\exists^\infty \sigma \preceq A)\,[H_{e'}(\sigma) = 0]\,]$$

witnesses that equality is in Π^1_1. Fixing e' to be an index of $\{0,1\}^\infty$ the next Theorem 4.2 witnesses that the set is also Π^1_1 complete. ∎

Theorem 4.2 *The sets $I = \{e : \mathcal{H}_e$ is two-sided$\}$ and $J = \{e : \mathcal{H}_e = \{0,1\}^\infty\}$ are Π^1_1 complete.*

Proof The set $J = \{e : (\forall A)\,(\forall^\infty \sigma \preceq A)\,[H_e(\sigma) = 1]\,\}$ in in Π^1_1. Furthermore $e \in I$ iff there is an index e' such that $\mathcal{H}_e = \mathcal{H}_{e'}$ and $H_{e'}$ makes on any A only finitely many mind changes, i.e.,

$$e \in I \;\Leftrightarrow\; (\exists e')\;\; [\,(\forall A)\,[(\exists^\infty \sigma \preceq A)\,[H_e(\sigma) = 0] \;\Leftrightarrow\; (\exists^\infty \sigma \preceq A)\,[H_{e'}(\sigma) = 0]\,] \;\wedge\; (\forall A)\,(\exists c)\,(\forall^\infty \sigma \preceq A)\,[H_{e'}(\sigma) = c]\,]$$

So it follows that also I is in Π^1_1.

Now it is shown that both sets are complete via the same m-reduction s. Let $T_0, T_1, \ldots \subseteq \mathbb{N}^*$ be a computable enumeration of all primitive recursive trees. The set

$$E = \{e : T_e \text{ is well-founded}\}$$

is Π^1_1 complete [22] where a tree is well-founded iff it does not contain an infinite branch. For any string σ say that σ codes a finite branch $a_0 a_1 \ldots a_n$ iff $\sigma = 1^{\langle a_0,b_0\rangle}01^{\langle a_1,b_1\rangle}0\ldots 1^{\langle a_n,b_n\rangle}0$. E is m-reducible to both index sets via the following reduction:

$$H_{s(e)}(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ codes a finite branch of } T_e, \\ & \text{i.e., if } \sigma = 1^{\langle a_0,b_0\rangle}01^{\langle a_1,b_1\rangle}0\ldots 1^{\langle a_n,b_n\rangle}0 \text{ and } a_0a_1\ldots a_n \in T_e; \\ 1 & \text{otherwise.} \end{cases}$$

If $H_{s(e)}$ outputs infinitely many 0 on some set A then A codes an infinite branch of T_e. So if T_e is well-founded then $H_{s(e)}$ outputs on each A only finitely many 0s. Therefore $\mathcal{H}_{s(e)} = \{0,1\}^\infty$ and is a two-sided class; in particular $s(e) \in I$ and $s(e) \in J$.

Otherwise T_e has an infinite branch $a_0a_1\ldots$ and for any sequence $b_0b_1\ldots$ the set A with the characteristic function $1^{\langle a_0,b_0\rangle}01^{\langle a_1,b_1\rangle}0\ldots$ is not in $\mathcal{H}_{s(e)}$. It follows that $\mathcal{H}_{s(e)} \neq \{0,1\}^\infty$ and $s(e) \notin J$. Furthermore for each function g the sequence $b_0b_1\ldots$ can be chosen such that $\langle a_{n+1}, b_{n+1}\rangle \geq b_{n+1} > g(c_n)$ where $c_n = n + \langle a_0,b_0\rangle + \langle a_1,b_1\rangle + \ldots + \langle a_n,b_n\rangle$. It follows that $f_A(c_n) > g(c_n)$ for each n and that g does not dominate f_A. So there is no function g dominating the functions f_A for all $A \notin \mathcal{H}_{s(e)}$. By Theorem 2.4, $\mathcal{H}_{s(e)}$ is not two-sided – even not relative to any oracle – and $s(e) \notin I$. ∎

Theorem 4.2 has an immediate application: it shows that there is nothing equivalent to a Friedberg numbering. If all one-sided classes would have a Friedberg numbering, then there would be also a numbering where one class, namely $\{0,1\}^\infty$, is omitted. But such a numbering does not exist.

Theorem 4.3 *No numbering contains all one-sided classes except $\{0,1\}^\infty$.*

Proof Assume by the way of contradiction that there is a computable function s such that the numbering $\mathcal{H}_{s(0)}, \mathcal{H}_{s(1)}, \ldots$ covers all one-sided classes except $\{0,1\}^\infty$. Then the set $\{e : (\exists e')[\mathcal{H}_e = \mathcal{H}_{s(e')}]\}$ is in Π^1_1 since Π^1_1 is closed under quantification on numbers as e' and since the equality problem is in Π^1_1. For any given e such an e' exists iff $\mathcal{H}_e \neq \{0,1\}^\infty$. So the complement of this Π^1_1 set is the Π^1_1 complete index set of $\{0,1\}^\infty$ and the function s can not exist, at least s can not be recursive. ∎

Theorem 4.4 *The sets* $\{e : \mathcal{H}_e \leq_1 \mathcal{A}\}$ *and* $\{e : \mathcal{H}_e \equiv_1 \mathcal{A}\}$ *are in* Π^1_1 *for every one-sided class* $\mathcal{A}$. *In particular* $\{e : \mathcal{H}_e$ *is 1-complete*$\}$ *is in* Π^1_1.

Proof The proofs are very similar. There is an enumeration of all operators Γ_i such that whenever Γ_i is total then it is strictly monotone. Furthermore there is a one-sided classifier H for $\mathcal{A}$. Now

$$\begin{aligned}
\mathcal{H}_e \leq_1 \mathcal{A} \quad \Leftrightarrow \quad & (\exists i)\,(\forall A)\,[\,\Gamma_i \text{ is total } \wedge \\
& ((\forall^\infty n)\,[H_e(A(0)A(1)\ldots A(n)) = 1] \Leftrightarrow \\
& (\forall^\infty n)\,[H(\Gamma_i(A(0)A(1)\ldots A(n))) = 1])\,] \\
\mathcal{H}_e \equiv_1 \mathcal{A} \quad \Leftrightarrow \quad & (\exists i,j)\,(\forall A)\,[\,\Gamma_i \text{ and } \Gamma_j \text{ are total } \wedge \\
& ((\forall^\infty n)\,[H_e(A(0)A(1)\ldots A(n)) = 1] \Leftrightarrow \\
& (\forall^\infty n)\,[H(\Gamma_i(A(0)A(1)\ldots A(n))) = 1]) \ \wedge \\
& ((\forall^\infty n)\,[H(A(0)A(1)\ldots A(n)) = 1] \Leftrightarrow \\
& (\forall^\infty n)\,[H_e(\Gamma_j(A(0)A(1)\ldots A(n))) = 1])\,]
\end{aligned}$$

Since the existential quantifier ranges over numbers, these expressions are Π^1_1. They characterize the two index sets. ∎

These classes are not Π^1_1 complete for every $\mathcal{A}$. In particular if $\mathcal{A} = \emptyset$ then they are in Π^0_2: $\mathcal{H}_e \equiv_1 \emptyset$ iff $\mathcal{H}_e = \emptyset$ iff for each n there is an m such that every string $\sigma \in \{0,1\}^m$ has at least n prefixes $\tau \preceq \sigma$ with $H_e(\tau) = 0$. The difference in the complexity of the question whether $\mathcal{H}_e$ is empty or equals $\{0,1\}^\infty$ is the mirror image of the fact that the question whether $W_e = \emptyset$ is Π^0_1 complete while the question whether $W_e = \mathbb{N}$ is Π^0_2 complete.

Furthermore it can be shown that the index set $\{e : \mathcal{H}_e = \emptyset\}$ has the least complexity of an index set of classes. Rice [20] showed for the world of enumerable sets that every non-trivial index set is Π^0_1 hard or Σ^0_1 hard. In the world of one-sided classes it can be shown that every nontrivial index set E is Π^0_2 hard or Σ^0_2 hard. In particular it is shown that the Σ^0_2 complete set Fin is m-reducible to E or $\overline{E}$.

Theorem 4.5 *Let* E *be a non-trivial index set of some collection* $\mathcal{G}$ *of classes. Then the set* $Fin = \{e : W_e$ *is finite*$\}$ *is m-reducible either to* E *or to* $\overline{E}$.

Proof First consider the case $\{0,1\}^\infty \in \mathcal{G}$. In this case it is shown that $Fin \leq_m E$ via a m-reduction s. This s then witnesses that E has at least complexity Π^0_2. Since E is not trivial there is some one-sided class $\mathcal{A} \notin \mathcal{G}$ with some computable one-sided

classifier H. Now s is defined implicitly via giving an informal description for the classifier $H_{s(e)}$:

> $H_{s(e)}$ outputs on A at least n 0s iff $|W_e| \geq n$ and H outputs on A at least n 0s.

If W_e is finite, $H_{s(e)}$ outputs on every A only finitely often a 0 and thus accepts every set; so $\mathcal{H}_{s(e)} = \{0,1\}^\infty$ and $s(e) \in E$ for every $e \in Fin$. If W_e is infinite then $H_{s(e)}$ accepts a set A iff H does; so $\mathcal{H}_{s(e)} = \mathcal{A}$ and $s(e) \notin E$ for every $e \notin Fin$. It follows that J is m-reducible to E via s.

The other case that $\mathcal{G}$ does not contain the class $\{0,1\}^\infty$ just gives an m-reduction from Fin to $\overline{E}$ using the above proof with $\overline{E}$ in place of E and $\{\mathcal{A} : \mathcal{A} \notin \mathcal{G}\}$ in place of $\mathcal{G}$. ∎

5 Classification and Measure

The measure ν given by $\nu(\emptyset) = 0$, $\nu(\{0\}) = \nu(\{1\}) = 0.5$, $\nu(\{0,1\}) = 1$ has an infinite product μ on the space $\{0,1\}^\infty$. This can be extended in such a way that every subclass of a class with measure 0 is again measurable and has measure 0.

For the classes of measure 0 – and thus indirectly by considering the complements also for the classes of measure 1 – there is a characterization via martingales [6, 13, 23]. A martingale is a function m which associates to every $\sigma \in \{0,1\}^*$ a rational number such that:

- $m(\sigma 0) + m(\sigma 1) = 2m(\sigma)$;
- $m(\sigma) > 0$ and $m(\lambda) = 1$.

A martingale witnesses that *a class $\mathcal{A}$ has measure* 0 iff for each $A \in \mathcal{A}$ and for each k there is an n such that $m(A(0)A(1)\ldots A(n)) \geq k$. It witnesses that a *class has measure* 1 iff it witnesses that the complement of this class has measure 0. A class has *recursive measure* 0 or 1 iff some recursive martingale witnesses that is has measure 0 or 1, respectively.

It is well-known that every class, which is Borel, is also measurable. Since every one-sided class is Borel [1], every one-sided class is measurable. For two-sided classes it is even possible to compute the measure in the limit from the classifier.

Theorem 5.1 *The measure $\mu(\mathcal{A})$ of a two-sided class can be computed from any index e of a two-sided classifier H_e for $\mathcal{A}$.*

Proof For each set A there is a unique n such that H_e converges at $A(0)A(1)\ldots A(n)$ either to 1 or to 0. Now let $\mathcal{A}_e = \{A : H_e \text{ converges to 1 at } n\}$ and $\mathcal{B}_e = \{A : H_e \text{ converges to 0 at } n\}$. Since H_e converges on every set A either to 1 or to 0,

these classes $\mathcal{A}_0, \mathcal{A}_1, \ldots$ and $\mathcal{B}_0, \mathcal{B}_1, \ldots$ form a partition of $\{0,1\}^\infty$. In particular $\mu(\mathcal{A}_0) + \mu(\mathcal{B}_0) + \mu(\mathcal{A}_1) + \mu(\mathcal{B}_1) + \ldots = 1$. Now consider the computable sequence

$$q_n = 2^{-1-n} \cdot \sum_{a_0,a_1,\ldots,a_n \in \{0,1\}} H_e(a_0 a_1 \ldots a_n)$$

of rational numbers. This sequence converges to $\mu(\mathcal{A})$ since $\mu(\mathcal{A}_0) + \mu(\mathcal{A}_1) + \ldots + \mu(\mathcal{A}_n) \leq q_n \leq 1 - (\mu(\mathcal{B}_0) - \mu(\mathcal{B}_1) - \ldots - \mu(\mathcal{B}_n))$ and therefore $|\mu(\mathcal{A}) - q_n| \leq \epsilon_n$ where $\epsilon_n = \mu(\mathcal{A}_{n+1}) + \mu(\mathcal{B}_{n+1}) + \mu(\mathcal{A}_{n+2}) + \mu(\mathcal{B}_{n+2}) + \ldots$; the ϵ_n converge to 0 since the sum $\mu(\mathcal{A}_0) + \mu(\mathcal{B}_0) + \mu(\mathcal{A}_1) + \mu(\mathcal{B}_1) + \ldots + \mu(\mathcal{A}_n) + \mu(\mathcal{B}_n)$ approaches monotonly to 1. Thus the sequence of the q_n converges to $\mu(\mathcal{A})$ and so the measure of $\mathcal{A}$ can be computed in the limit from any two-sided classifier for $\mathcal{A}$. ∎

This is not longer true for one-sided classes. The class class of all sets A which are lexicographic before K' has the measure $2^{-1}K'(0) + 2^{-2}K'(1) + \ldots$ which can not be computed in the limit since otherwise K' would be computable in the limit. So one might ask whether the measure is at least in those cases computable where the measure of the class is a recursive real. Easier than computing such a measure is to verify that a class has measure 0 or 1 via presenting a recursive martingale which either succeeds on the class or on its complement. But – as the next example shows – also this fails for certain one-sided classes with measure 1: there is just no such martingale.

Example 5.2 *There is a cosingle one-sided class $\mathcal{A}$ which does not have recursive measure* 1.

Proof Every set $A \leq_T K$ is a π^0_2 singleton. Therefore it follows that for every $A \leq_T K$ the class $\mathcal{A} = \{B : B \neq A\}$ is one-sided and it remains to be shown that there is some $A \not\leq_T K$ such that no recursive martingale succeeds on A. This is just the well-known fact that there is a random-set $A \leq_T K$. ∎

Somehow it is easier to determine that a one-sided class is small than that it is large. As already seen the index set of the empty class is much easier than that of the full class $\{0,1\}^\infty$. This result has a parallel w.r.t. measure. While some cosingle one-sided classes do not have measure 1, every one-sided class of measure 0 does also have recursive measure 0.

Theorem 5.3 *If a one-sided class $\mathcal{A}$ has measure* 0 *then $\mathcal{A}$ has recursive measure* 0.

Proof Let $\mathcal{A}$ be one-sided via a machine H and have measure 0. Now a recursive martingale m is constructed in order to witness that $\mathcal{A}$ has recursive measure 0. The inductive definition starts with $m(\lambda) = 0$.

Now in each step chose (one of) the shortest σ such that $m(\sigma)$ is defined but not $m(\sigma 0)$ and $m(\sigma 1)$. Let L_n be the set of all $\tau \in \{0,1\}^n$ such that $H(\sigma\eta) = 1$ for all non-empty $\eta \preceq \tau$. If for each n the cardinality of L_n would be larger than 2^{n-2} then H would output on a "quarter" of all sets $A \succeq \sigma$ never a 0 after processing σ which implies $\mu(\mathcal{A}) \geq 2^{-|\sigma|-2}$ in contradiction to the choice of $\mathcal{A}$. Thus there is an $n > 0$

such that L_n has at most 2^{n-2} elements, w.l.o.g. n is the smallest such number.

Now let $m(\sigma\tau) = 1.5 \cdot m(\sigma)$ for every $\tau \in L_n$ and let $m(\sigma\tau) = \frac{2^n - 1.5 \cdot |L_n|}{2^n - |L_n|} \cdot m(\sigma)$ for the other strings τ of length n. All values are above 0 and there sum is $2^n \cdot m(\sigma)$. Furthermore define $m(\sigma\eta)$ for strings η of length $n-1, n-2, \ldots, 0$ according to the formula $m(\eta) = \frac{1}{2} \cdot (m(\eta 0) + m(\eta 1))$. This finishes the extension step.

Each such step finishes and since each step takes the shortest σ with $m(\sigma 0), m(\sigma 1)$ being undefined, m becomes a total function. Furthermore all values of m are positive rational numbers and it can be verified that the equation $m(\sigma 0) + m(\sigma 1) = 2 \cdot m(\sigma)$ holds for all σ. Thus m is a recursive martingale.

Let now $A \in \mathcal{A}$. Let $\sigma_0 \preceq \sigma_1 \preceq \ldots \preceq A$ being that sequence of strings such that σ_{n+1} is always a string of the form $\sigma_n \tau$ when m is extended at σ_n. By the construction the following holds:

$$\begin{aligned} m(\sigma_{n+1}) < m(\sigma_n) \quad &\Leftrightarrow \quad H(\eta) = 0 \text{ for some } \eta \text{ with } \sigma_n \prec \eta \preceq \sigma_{n+1} \\ m(\sigma_{n+1}) = 1.5 \cdot m(\sigma_n) \quad &\Leftrightarrow \quad H(\eta) = 1 \text{ for all } \eta \text{ with } \sigma_n \prec \eta \preceq \sigma_{n+1} \end{aligned}$$

Since H outputs on A almost always 1s, the second case holds for almost all n and it follows that m takes on A arbitrary large values. So $\mathcal{A}$ has recursive measure 0 witnessed by the recursive martingale m. ∎

Example 5.2 showed already that one-sided classes of measure 1 do not need to have recursive measure 1. So it is natural to look for the help of oracles and the next result states, that a K-oracle is sufficient to do the job: If a class has measure 1 then it has already K-recursive measure 1, i.e., a K-recursive martingale witnesses that the class has measure 1.

Theorem 5.4 *If a one-sided class $\mathcal{A}$ has measure 1 then $\mathcal{A}$ has K-recursive measure 1.*

Proof For given one-sided class $\mathcal{A}$ with measure 1 a K-recursive martingale m is constructed which succeeds on every $A \notin \mathcal{A}$ and so witnesses that $\mathcal{A}$ has K-recursive measure 1. Let H be a one-sided classifier for $\mathcal{A}$. Let μ denote the standard measure on $\{0,1\}^\infty$ and for any computable tree T let

$$\nu(T, \eta) = \mu(\{A \succeq \eta : A \text{ is infinite branch on } T\}).$$

Starting with $m(\lambda) = 1$, the inductive definition of m runs as follows:

(1) Choose the shortest σ such that $m(\sigma 0)\uparrow$ and $m(\sigma 1)\uparrow$.
Indeed the domain of m will be a tree at each stage and by extending the domain on some shortest leaf, it is guaranteed that m is total at the end. This σ can be found using the oracle K.

(2) Let $T_n = \{\tau : |\{m \in dom(\tau) : H(\tau(0)\tau(1)\ldots\tau(m)) = 0\}| \leq n\}$. Find using the oracle K a suitable n such that $\nu(T_n, \sigma) > 2^{-|\sigma|-1}$.

Such an n exists since the union of all T_n contains almost all branches through σ and so $\nu(\sigma, T_n)$ must approach to $2^{-|\sigma|}$ by the continuity of μ. Furthermore

$$\nu(\sigma, T_n) = 2^{-|\sigma|} - \sum\nolimits_{\tau \text{ is leaf of } T_n} 2^{-|\tau|}$$

is computable via K-oracle and thus a suitable n can be found.

(3) Define m on all nodes of T_n above σ such that $m(\tau) \geq 1.5 \cdot m(\sigma)$ for every leaf τ of T_n above σ.
The definition $m'(\tau) = m(\sigma) \cdot (0.95 \cdot 2^{|\tau|-|\sigma|} \cdot \nu(\tau, T_n)/\nu(\sigma, T_n) + 0.05)$ satisfies all requirements but has the disadvantage of not giving rational numbers. But m' can be approximated by an extension of m onto T_n above σ such that m has on this extended domain the same computational complexity as m', takes only rational values and satisfies $0.9 \cdot m'(\eta) \leq m(\eta) \leq 1.1 \cdot m'(\eta)$ for all η. Since even $m'(\tau) \geq 1.9 \cdot m(\sigma)$ for all leaves of T_n, it follows that $m(\tau) \geq 0.9 \cdot 1.9 \cdot m(\sigma) \geq 1.5 \cdot m(\sigma)$ for these τ.

At each stage of the definition, a set A stays on the corresponding tree T_n only if H outputs a finite number of 0s on input A. Thus starting with $\sigma_0 = \lambda$, a given $A \notin \mathcal{A}$ leaves this tree through a leaf σ_1 and when m is extended on a tree above σ_1 then A leaves this tree through a leaf σ_2 and so on. So A goes through an infinite sequence $\sigma_0, \sigma_1, \ldots$ of nodes such that $m(\sigma_{k+1}) \geq 1.5 \cdot m(\sigma_k)$ for all these nodes. It follows that m takes on A arbitrary high values and so m witnesses that $\overline{\mathcal{A}}$ has K-recursive measure 0. ∎

Let I be an interval of real numbers. It follows from the definition of the Lebesgue measure, that every measurable set $E \subseteq I$ is approximable via a F_σ set F in the sense that the symmetric difference of E and F has measure 0. Lusin [18, Satz 8.2] showed a function $f : I \to I$ is measurable iff for each $\epsilon > 0$ there is a set D of measure less than ϵ such that the restriction of f to the domain $I - D$ is continuous. These results motivate to look at the question to which extend from the view-point of measure theory, a one-sided class can be approximated by a two-sided one.

Theorem 5.5 *For every one-sided class $\mathcal{A}$ and every $\epsilon > 0$ there is a two-sided class $\mathcal{B}$ such that the symmetric difference of both classes has a measure less than ϵ. But there is also a one-sided class $\mathcal{A}$ such that every two-sided class differs from $\mathcal{A}$ on a set of positive measure.*

Proof First note that every one-sided class is Borel [1] and thus measurable. For given one-sided class $\mathcal{A}$ and $\epsilon > 0$ consider the classes $\mathcal{A}_k$ containing all sets A on which H outputs at most k times a 0. These classes are two-sided and they approximate $\mathcal{A}$ from below. Since the measure is continuous, $\mu(\mathcal{A})$ is the upper limit of the $\mu(A_k)$ and so $\mu(\mathcal{A}) - \epsilon < \mu(\mathcal{A}_k) \leq \mu(\mathcal{A})$ for some k. Since $\mathcal{A}_k \subseteq \mathcal{A}$, the symmetric difference has the measure $\mu(\mathcal{A}) - \mu(\mathcal{A}_k)$ which is less than ϵ.

For the second result let G be 1-generic and below K. Consider the class $\mathcal{A} =$

$\{A : A <_{lex} G\}$. This class is one-sided via outputting 1 if $\sigma <_{lex} G_{|\sigma|}$ and 0 otherwise where G_s is a recursive approximation to G. Now let $\mathcal{B}$ be any two-sided class with classifier M. M converges on G to some value a, with only finitely many changes one can obtain that $M(\sigma) = a$ for all $\sigma \preceq M$. Thus G avoids the computable set $\{\tau : M(\tau) \neq a\}$ and so there is some prefix σ such that $M(\tau) = a$ for all $\tau \preceq \sigma$. On the other hand there are $\eta_0, \eta_1 \succeq \tau$ such that all sets extending η_0 belong to $\mathcal{A}$ and all sets extending η_1 belong to $\overline{\mathcal{A}}$, in particular all sets $A \succeq \eta_{1-a}$ are in the symmetric difference of $\mathcal{A}$ and $\mathcal{B}$ so that this symmetric difference has a measure larger than $2^{|\eta_1 - a|} > 0$. ∎

There are two natural properties, one-sided classes can take: being simple and maximal. The first one was already introduced above: a simple class is one-sided and intersects every infinite one-sided class. The second one is the following: A one-sided class $\mathcal{A}$ is maximal if it is coinfinite and has the property that either $\mathcal{A} \cup \mathcal{B}$ or $\mathcal{A} \cup \overline{\mathcal{B}}$ is cofinite for every one-sided class $\mathcal{B}$. The easiest way to construct a maximal class is just to convert a maximal set into it: Let U be a set which is maximal relative to K, i.e., U is enumerable relative to K, coinfinite and no further set which is enumerable relative to K can split the complement of U into two infinite parts. The class $\{A : |A| \neq 1 \vee (A = \{x\} \wedge x \in U)\}$ is maximal. So maximal classes exist. Simple and maximal classes are not only large in the sense that they intersect every infinite one-sided class. They are also large w.r.t. measure theory.

Theorem 5.6 *If $\mathcal{A}$ is simple then $\mu(\mathcal{A}) > 0$. If $\mathcal{A}$ is maximal then $\mu(\mathcal{A}) = 1$.*

Proof First it is shown that no one-sided class $\mathcal{A}$ of measure 0 is simple. So let $\mathcal{A}$ be a one-sided class of measure 0. $\mathcal{A}$ then also has recursive measure 0 and there is a computable martingale m witnessing this fact. The class $\mathcal{B}_c = \{A : (\exists n)\,[m(A(0)A(1)\ldots A(n)) > c]\}$ of all sets on which m obtains some value greater than c has measure at most $\frac{1}{c}$. So the class $\overline{\mathcal{B}_2}$ has at least measure $\frac{1}{2}$ and is therefore infinite. $\overline{\mathcal{B}_2}$ is disjoint to $\mathcal{A}$ since the martingale succeeds on every set in $\mathcal{A}$. Furthermore $\overline{\mathcal{B}_2}$ is one-sided via guessing 1 on input A as long as $m(A(0)A(1)\ldots A(n)) \leq 2$ and then making a mind change to 0. So the infinite one-sided class $\overline{\mathcal{B}_2}$ is disjoint to $\mathcal{A}$ and $\mathcal{A}$ is not simple.

Second it is shown that $\mu(A) = 1$ for all maximal classes $\mathcal{A}$. Given a maximal class $\mathcal{A}$ some kind of "kernel" B of $\mathcal{A}$ is constructed as follows: for each n one of the classes $\{A \notin \mathcal{A} : A(n) = 0\}$ and $\{A \notin \mathcal{A} : A(n) = 1\}$ is finite and the other one is infinite; so let $B(n)$ take that value b for which $\{A : A(n) = b\}$ is infinite. It follows that for every n the class $\{A \notin \mathcal{A} : A(n) \neq B(n)\}$ is finite and thus their union is countable. So $\overline{\mathcal{A}}$ has at most countably many members: those just mentioned plus perhaps B itself. Therefore the complement of any maximal class is countable and so every maximal class has measure 1. ∎

Theorem 5.6 has two limitations: first it is only claimed that maximal classes have measure 1 but not that they have recursive measure 1. Indeed this is not possible

since by Example 5.2 there is a cosingle class $\mathcal{A}$ not having a recursive measure 1 and taking any maximal class $\mathcal{B}$, the new class $\mathcal{A} \cap \mathcal{B}$ does also not have recursive measure 1 but is still maximal.

The second restriction is that there are simple classes with measure below 1 as will be proven below. Indeed this tradeoff between the size of maximal and simple classes has an analogon in recursion theory given by the fact that a simple set can be arbitrary thin – there is for every given computable function f a simple set which has only n elements among the numbers $0, 1, \ldots, f(n)$ – while no similar result holds for maximal sets.

Example 5.7 *For each $\epsilon > 0$ there is a simple class $\mathcal{A}$ with $\mu(\mathcal{A}) < \epsilon$.*

Proof Each string σ generates a open class $\sigma \cdot \{0,1\}^\infty$ of sets. This open class is said to meet the one-sided class $\mathcal{H}_e$ generated by H_e effectively iff there is a set $A \in \sigma \cdot \{0,1\}^\infty$ such that $H_e(A(0)A(1)\ldots A(n)) = 1$ for all $n \geq |\sigma|$. This condition is coenumerable, i.e., for each class $\mathcal{H}_e$ given by H_e the set

$$N_e = \{\sigma : \sigma \cdot \{0,1\}^\infty \text{ does not meet } \mathcal{H}_e \text{ effectively}\}$$

is enumerable. Furthermore almost all sets $A(0)A(1)\ldots A(n) \cdot \{0,1\}^\infty$ meet $\mathcal{H}_e$ effectively whenever $A \in \mathcal{H}_e$. There is an algorithm which generates a three-dimensional array $\sigma_{e,i,j}$ of strings such that

(1) each string $\sigma_{e,i,j}$ has length at least j;

(2) if $\mathcal{H}_e \neq \emptyset$ then $\sigma_{e,j} = \lim_{i\to\infty} \sigma_{e,i,j}$ exists;

(3) if $\sigma_{e,j}$ exists then the open class $\sigma_{e,j} \cdot \{0,1\}^\infty$ meets $\mathcal{H}_e$ effectively.

This algorithm works after a simple schema: $\sigma_{e,i,j}$ is just the first string (w.r.t. some given ennumeration of all strings) whose length is at least j and which is not enumerated to N_e within i computation steps. So this algorithm converges to some $\sigma_{e,j}$ if $\mathcal{H}_e \neq \emptyset$ and diverges otherwise. Let E denote the index set of all nonempty classes $\mathcal{H}_e$, i.e., the set of all e where the sequences $\sigma_{e,0,j}, \sigma_{e,1,j}, \ldots$ converge.

Given ϵ there is a number n such that $2^{1-n} < \epsilon$. Now $\mathcal{A}$ is taken to be the union of all classes $\sigma_{e,n+e} \cdot \{0,1\}^\infty$ with $e \in E$. The measure of $\mathcal{A}$ is bounded by $\mu(\mathcal{A}) \leq \Sigma_{e\in E}\, 2^{-e-n} \leq 2^{1-n} < \epsilon$ and so the requirement on the measure of $\mathcal{A}$ is satisfied. So it remains to show that $\mathcal{A}$ is one-sided. A one-sided classifier for $\mathcal{A}$ is given as follows:

> H outputs on A at least n 0s iff there is $i \geq n$ such that
> $\sigma_{e,i,j} \neq A(0)A(1)\ldots A(m)$ for all $m \leq n$ and $e \leq n$.

If $A \in \mathcal{A}$ then some $\sigma_{e,i,j}$ converges to a prefix of A. So there are m, k such that $\sigma_{e,i,j} = A(0)A(1)\ldots A(m)$ for all $i \geq k$. It follows that H does not output more than $e + k + m$ 0s and H accepts A.

If $A \notin \mathcal{A}$ then for each n there is a stage i such that all strings $A(0)A(1)\ldots A(m)$

with $m \leq n$ are enumerated to all N_e with $e \leq n$. It follows that all $\sigma_{e,i,j}$ with $e \leq n$ are different from all $A(0)A(1)\ldots A(m)$ with $m \leq n$ and H outputs on A eventually at least n 0s. Since this holds for each n, H rejects A.

So H is a one-sided classifier for $\mathcal{A}$. By construction, $\mathcal{A}$ meets every nonempty set $\mathcal{H}_e$, so $\mathcal{A}$ is a simple class. Furthermore $\mu(\mathcal{A}) < \epsilon$ and so $\mathcal{A}$ satisfies all conditions of the theorem. ∎

6 Classifying Recursive Sets Only

Smith, Wiehagen and Zeugmann [24, 27] looked at classification tasks where only the behaviour on computable sets is considered. Case, Kinber, Sharma and Stephan [5] extended this work. Many of the anomalies of classification (compared to the setting of enumerable versus computable sets) disappear, if classification of only computable is considered. In this model, every one-sided class is two-sided relative to a high oracle and cosingle classes are already two-sided without any help of an oracle. This section now looks at the relation of the general model where all sets are classified versus the restricted model where only computable sets are classified. The next theorem shows that there is a class which is two-sided in the restricted model but does not have any two-sided classifier relative to any oracle in the general model.

Theorem 6.1 *There is a class $\mathcal{A}$ of computable sets such that some computable M classifies all computable sets w.r.t. $\mathcal{A}$ but there is even no non-recursive classifier which converges on every input-set and classifies all computable sets w.r.t. $\mathcal{A}$.*

Proof Let S be a simple set and $\mathcal{A} = \{\text{finite } A : A \cap S = \emptyset \wedge |A| \text{ is odd}\}$. First it is shown that some M classifies S one-sided where M converges on every computable set. This M is given by

$$M(\sigma) = \begin{cases} 0 & \text{if } \{x : \sigma(x)\downarrow = 1\} \text{ meets } S_{|\sigma|} \text{ or has even cardinality;} \\ 1 & \text{otherwise.} \end{cases}$$

That means that M outputs 0 or 1 depending on the cardinality of the 1s in σ until M discovers that some x with $\sigma(x)\downarrow = 1$ is enumerated into S – then M switches to 0 forever.

Now it is shown that M converges on every computable set. If A is finite then M changes only finitely often its mind – either when M finds a new element or when some already found element is enumerated into S. If A is computable and infinite then M also makes only finitely often a mind change since M eventually discovers that there is an $x \in A \cap S$ and from this time on only outputs 0.

The second part is to show that every classifier N which is correct on all finite sets w.r.t. $\mathcal{A}$ diverges on some infinite set $A = \{a_0, a_1, \ldots\}$ where this set is defined inductively starting with $n = 0$ and some $a_0 \notin S$.

> Take some $\sigma_n \preceq \{a_0, a_1, \ldots, a_n\}$ such that $|\sigma_n| > a_n$ and $N(\sigma_n) = 1$ iff $n+1$ is odd. Then take some $a_{n+1} \notin S \cup dom(\sigma_n)$.

The σ_n is found since N classifies $\{a_0, a_1, \ldots, a_n\}$ w.r.t. $\mathcal{A}$. The a_{n+1} exists since S is coinfinite and $dom(\sigma_n)$ is finite. So the construction works for all n and the resulting set A is infinite and disjoint to S. Furthermore $N(\sigma_n) = 0$ for all even n, $N(\sigma_n) = 1$ for all odd n and the σ_n are all different prefixes of A, so N does not converge on A. ∎

So sometimes two-sided classifiers for computable sets can not be extended to two-sided classifiers for all sets. But as the next theorem shows, they can be extended to one-sided classifiers for all sets such that the corresponding class has measure 1.

Theorem 6.2 *Every one-sided class $\mathcal{A}$ has a one-sided "extension" $\mathcal{B}$ of measure 1 such that $A \in \mathcal{A} \Leftrightarrow A \in \mathcal{B}$ for every computable set A.*

Proof Let H be a one-sided classifier for $\mathcal{A}$. Now a new one-sided classifier N is constructed such that the class $\mathcal{B}$ defined via N has the desired properties. N just slows down the output of the 0s and meets the following definition:

> N outputs on A at least n 0s if H outputs on A at least n 0s and there is $m \geq n$ such that $\varphi_m(x)\!\downarrow = A(x)$ for all $x \leq m$.

So whenever A is recursive, A has infinitely many indices and in particular for each n there is an index $m \geq n$ of A. This m satisfies of course $\varphi_m(x)\!\downarrow = A(x)$ for all $x \leq m$. Therefore M outputs on A infinitely many 0s if H does and N classifies A to be in $\mathcal{B}$ iff H classifies A to be in $\mathcal{A}$. So $\mathcal{A}$ and $\mathcal{B}$ coincide on the computable sets.

Let $\mathcal{A}_m = \{A : (\forall x \leq m)\,[\varphi_m(x)\!\downarrow = A(x)]\}$. Each class $\mathcal{A}_m$ has measure 2^{-m-1} provided that φ_m is defined on the input $0, 1, \ldots, m$. So whenever N outputs on a set A at least n 0s, then A belongs to some $\mathcal{A}_m$ with $m \geq n$. Since $\mu(\mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \ldots) \leq \mu(\mathcal{A}_n) + \mu(\mathcal{A}_{n+1}) + \ldots \leq 2^{-n-1} + 2^{-n-2} + \ldots = 2^{-n}$, it follows that the measure of the class of all A on which N outputs at least n 0s has the upper bound 2^{-n}. Thus the measure of $\overline{\mathcal{B}}$ which is the class of all sets on which N outputs infinitely many 0s is bounded by each number 2^{-n} and so is 0. It follows that $\mathcal{B}$ has measure 1. ∎

A similar Theorem does not hold with measure 1 in place of measure 0. Taking the one-sided class $\mathcal{A} = \{0,1\}^\infty$ of all sets, every one-sided class $\mathcal{B}$ which agrees with $\mathcal{A}$ on all computable sets just has to contain every computable set. The measure of $\mathcal{B}$ can not be 0 since then $\mathcal{B}$ had recursive measure 0 and so there would be a recursive martingale succeeding on all computable sets - which does not exist.

7 Conclusion

The paper deals with the notion of one-sided and two-sided classes. It is shown that there are several similarities between the relation of one-sided to two-sided classes on one hand and the well-studied relation of enumerable to computable sets on the other hand.

Similar to the study of Turing degrees of enumerable sets, the Turing complexity

of one-sided classes is defined as – roughly spoken – the amount of information which is needed to construct a two-sided classifier for a one-sided class. It turns out that the classes with the highest Turing complexity which are not two-sided relative to any oracle are intractable more because of topological reasons than because of computational difficulty.

Studying the classes with intermediate Turing complexity (which are not two-sided but have a two-sided classifier operating with some oracle) are linked to hyperarithmetic Turing degrees. In particular if a class is two-sided relative to a cone above a Turing degree **a** but not relative to any other Turing degree then this Turing degree is generated by a Π^0_2 singleton. On the other hand every Π^0_2 singleton A generates the one-sided class $\{B : B \neq A\}$ which has exactly the same Turing degree as A. Furthermore there are intermediate one-sided classes which do not have a Turing degree but are linked to a collection of Turing degrees, e.g., there is a one-sided class which is two-sided relative to an oracle iff that has high Turing degree. Within this area the following questions remained open.

- Is there a one-sided and relatively two-sided class which has no hyperarithmetical two-sided classifierΓ
- Is the structure of the one-sided classes concerning 1-reduction isomorphic to any well-known degree-structureΓ

Further research deals with index sets $\{e : \mathcal{H}_e \text{ belongs to } \mathcal{G}\}$ where $\mathcal{G}$ is a collection of some classes. Most natural index sets as those of the two-sided classes, of the class $\{0,1\}^\infty$ and of the 1-complete sets are Π^1_1 complete. Some as the problem whether a class is empty have the complexity Π^0_2 and are therefore easier.

Every one-sided class is Borel (in the standard topological sense) and therefore also measurable. So the question was investigated how effective this measure is. Due to the asymmetric definition of one-sided classification, one-sided classes of measure 0 have already recursive measure 0 while some cosingle one-sided classes does not have a recursive measure.

Arun Sharma [5] proposed to study the classification of only recursive sets. In this world, topological constraints are weakened and the concept becomes quite more similar to the scenario of enumerable versus computable sets. E.g., every one-sided class is two-sided relative to a high oracle in this world. Section 6 is therefore dedicated to the relation between these two worlds. The main result is that a two-sided classifier for the world of classifying computable sets only can not be extended to one for the world of classifying all sets – even not with oracles.

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