On the Renormalisation Group Explanation of Universality

Alexander Franklin

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Abstract

It is commonly claimed, both by physicists and philosophers that the universality of critical phenomena is explained through particular applications of the Renormalisation Group (RG). This paper has three aims: (i) to clarify the nature of the explanation of universality; (ii) to discuss the physics of such renormalisation group explanations; (iii) to examine the extent to which universality is thus explained.

The derivation of critical exponents proceeds via a real-space or a field-theoretic approach to the RG. Following Mainwood (2006) I argue that these approaches ought to be distinguished: while the real-space approach fails adequately to explain universality, the field-theoretic approach succeeds in the satisfaction of this goal.

'Universality' is the technical term for a striking kind of multiple realisability. This occurs when diverse systems exhibit similar scaling behaviour on the approach to a critical phase transition. While first-order phase transitions are abrupt variations in macroscopic behaviour – such as the transformation from water to steam – critical phase transitions mark the point (the critical temperature) beyond which systems no longer undergo first-order phase transitions.

It turns out that the similar scaling behaviour exhibited on approach to the critical phase transition can be very well described by power laws of the form $a_i(t) \propto t^\alpha$ where t is proportional to the temperature deviation from the critical temperature.[1] Physical systems can be categorised into universality classes according to their behaviour as they approach the critical point: members of the same class have identical critical behaviour – the same set of critical exponents $\{\alpha, \beta, ...\}$ for several power laws – while they may have radically diverse microphysical structures and behaviour away from the critical point.

A paradigm example of universality is that the liquid-gas critical phase transition and the (uniaxial) ferromagnetic-paramagnetic critical phase transition share critical exponents. Both of these types of systems may be described by equivalent power laws as they transition from certain ordered states (liquid or ferromagnetic respectively) to critical states. These systems are examples of the 3D Ising universality class.

Hundreds of papers have been published in Physics journals over the last fifty years on this topic. On the one hand a great deal of experimental evidence is available which classifies many different physical systems into a few universality classes, and finds the critical exponents for these classes to ever greater accuracy; see Sengers and Shanks (2009) and references therein. On the other, theoretical work is continually under way to refine and develop the theoretical models for each universality class; see Pelissetto and Vicari (2002). It is now the case that both through computer modelling (Monte Carlo simulations) and through field-theoretic derivations (using perturbation theory) critical exponents derived match very closely those discovered empirically.

[1] E.g. the specific heat (in zero magnetic field) (c) scales as $c \sim \alpha^{-1}(t^{-\alpha})$ as $T \to T_c$. $t = \frac{T - T_c}{T_c}$.

This paper has three aims: (i) to clarify the nature of the explanation of universality; (ii) to discuss the physics invoked in renormalisation group explanations; (iii) to examine the extent to which the physics of the renormalisation group explains universality.

(i) In §1 I outline a range of different explananda, and distinguish the kinds of explanation which may satisfy each. This is important because some confusion in the philosophical debate over the explanation of universality has resulted from implicit appeal to different explanatory standards in the literature.

(ii) §2 details the physics of the real-space and field-theoretic approaches to the renormalisation group. The two approaches, which have different mathematical structures, are often elided although, as I argue, the explanations on offer are distinct. Paying attention to the physics further reveals that there are various technical lacunae in the renormalisation group explanation of universality which have been neglected in the philosophical literature.

(iii) In §3 I develop reasons for thinking that, despite various technical lacunae, the field-theoretic approach to the renormalisation group is sufficient to explain universality along the lines developed in §1. In this section I express doubts that a similar argument could be run in the context of the real-space approach. This latter conclusion is particularly worth highlighting because the philosophical literature, insofar as the approaches are distinguished, focusses primarily on the real-space approach.

1 Structure of the Explanation of Universality

The discussants in the philosophical literature on the explanation of universality[2] seem to have different standards of explanation in mind. As such, I set out what I take to be the various phenomena which need explaining (explananda) and the explanations offered in response (explanantia).

Throughout the rest of the paper I refer back to this table and explain

[2]E.g. Batterman (2000; 2014), Butterfield and Bouatta (2011), Callender and Menon (2013), Reutlinger (2014) and references therein.

the various boxes in more detail. The explanantia on the right hand side explain the corresponding explananda on the left hand side.

Explanandum (L)	Explanans (R)
1L: System A has critical exponents $\{\alpha\}$ & system B has critical exponents $\{\alpha\}$ & ... & system E has critical exponents $\{\alpha\}$.	1R: $\{\alpha\}$ derived from Hamiltonian for system A & $\{\alpha\}$ derived from Hamiltonian for system B & ... & $\{\alpha\}$ derived from Hamiltonian for system E.
2L: Systems A-E have certain behaviour in common (or the differences between A-E are irrelevant to their behaviour).	2R: Identify common features shared by A-E and show that they are sufficient for common behaviour (or demonstrate the irrelevance of heterogeneities).
3L: There's a generic commonality in behaviour (or a generic irrelevance of certain details).	3R: The trajectories in the abstract space converge.

In the table above, 1L is just a conjunction of seemingly independent facts about the critical exponents of independent systems where each fact is offered an independent explanation by 1R. 1L is distinguished from 2L to emphasise that 2L requires a deeper explanation. The common behaviour ought not to be explained by distinct explanations for each physical system; an explanation of the form of 2R is required which adduces a similarity in the systems and demonstrates its sufficiency for their common behaviour.[3]

Consider an analogy: a traveller visits a foreign country and goes from house to house observing the local customs. She observes an oddity in the locals' behaviour: in each family she visits the youngest child sleeps in a bed angled such that their head is vertically lower than their feet. At each visit she asks for an explanation of this phenomenon and every family offers a different answer: 'because he is short and this way he will grow taller'; 'because greater blood flow to her head will increase her intelligence'; 'because it is cooler and his head otherwise becomes hot'; 'because

[3]A similar point is made in Batterman (2016).

that is the only way to avoid the awakening smell of dinner' ...

Our traveller will likely be dissatisfied with this range of distinct explanations – analogous to our dissatisfaction with 1R. There is an unusual commonality and she will seek a unified explanation along the lines of 2R.

A straightforward way to offer an explanation of universality follows 2R: one explains the common behaviour by isolating and pointing to an aspect of the underlying description which is shared by the different systems in the same class. While merely pointing to the commonality is inadequate to a full explanation of universality, a 2R-type explanans explains if one additionally demonstrates that the common features lead to the observed common behaviour in each case. In §§3.1-3.2 I respectively claim that the field-theoretic approach provides a 2R-type explanation while the real-space approach does not.

What about 3R? If we consider our traveller, 3R might correspond to the general claim that 'communities tend to share cultural practices'. This would only be satisfying if our traveller were assured that the general claim applied to this community and that there was some mechanism through which the families' sleeping practices were aligned.

Batterman appeals to an explanation along the lines of 3R: commonality is to be expected generically because of the convergence of flows in the abstract space:

> It turns out that different physical Hamiltonians can flow to the same fixed point. Thus, their critical behaviors are characterized by the same critical exponents. This is the essence of the explanation for the universality of critical behavior.
> [Batterman (2000, p.127)]

Of course this explanation is insufficient if the converging trajectories are not linked to the systems which exhibit the commonality. In large part the goal of the rest of this paper is to explore whether these trajectories are so linked. Batterman claims that the link is due to the flow of 'physical Hamiltonians', as such much of this paper will explore how physical Hamiltonians may be defined in the context of critical phenomena. That is, I explore whether we can link the abstract convergent flows to the description of distinct systems which exhibit universality.

Without a link to 2L, 3R remains a claim not grounded by reference to real physical systems. By analogy the claim that communities tend to share cultural practices is explanatorily insufficient if it cannot be demonstrated that *this* community shares cultural practices. Importantly in both the analogy and the universality of critical phenomena 3R-type claims may play an important explanatory role where the link to 2L is available. I argue in this paper that the physics is much less worked out than the literature seems to suggest. Nonetheless we seem to have a framework available for a 3R-type explanation in the field-theoretic approach to the RG: in §3.1 I show that there are sound theoretical arguments whereby the field-theoretic approach implies that convergent flows can be linked to trajectories in phase space which may represent the different physical systems which display common behaviour.

Note that in the real-space RG context the formalism of convergent flows in a phase space can also be written down. There is little reason to think that the distinct convergent trajectories represent the different physical systems of interest; although there are limited exceptions discussed in §3.2. As such I argue that the 3R explanation fails in the real-space RG because it does not have the resources to imply 2L; the real-space RG does nonetheless allow for the prediction of the critical exponents for certain (archetypal) systems in each universality class. Thus it is important to reiterate that the real-space RG approach is predictively useful despite its inadequacy *qua* explanation of universality.

There is a further issue which ought to be discussed here – this pertains to the parenthetical statements in the table above. Batterman highlights the fact that the details which distinguish systems are irrelevant to their universal behaviour:

> *In effect the renormalization group transformation eliminates degrees of freedom (microscopic details) that are inessential or irrelevant for characterizing the system's behavior at criticality.*
> [Batterman (2000, p.127), original emphasis]

The renormalisation group (RG) formalism allows one to demonstrate the irrelevance of aspects of our physical systems quite generally. We may view this as offering two equivalent explanations: adducing the details which are in common and sufficient for behaviour of interest; or demonstrating that the details which distinguish the systems are irrelevant to said behaviour. What's important for Batterman is that the RG provides a general demonstration of robustness at the critical point.[4] He argues that robustness with respect to microphysical perturbations implies that all systems which go to the critical point have a representation which is independent of microphysical details. This would thus establish that the distinguishing details are irrelevant and, by implication, that the common details are sufficient for common behaviour. This robustness demonstration is rather like that of type 3R. In order for it to explain universality we need an additional argument to the effect that the details which distinguish the universally behaving systems are those details which are demonstrably irrelevant according to the RG analysis. Throughout the rest of this essay I discuss the extent to which such arguments might succeed.

Whilst other authors seem to ignore the conceptual links between irrelevance of details and universality, Batterman's work is very important for its role in highlighting these connections. My contention in this paper is that the field-theoretic RG is the proper context for gleaning such insights. The upshot of this section is that we have a framework for explaining universality. We may either proceed via 2R, or via 3R with an appropriate link to 2L made explicit.

In the following I outline the background physics (§2) and analyse the explanation on offer by the field-theoretic and the real-space approaches to the Renormalisation Group (RG), see §3.1 and §3.2 respectively.[5]

In the physics literature it is standard practice to distinguish these approaches; I will argue, following Mainwood (2006), that the distinction is also significant when assessing the RG explanation of universality. I, like Mainwood, endorse the field-theoretic explanation of universality while

[4] In some places Batterman refers to the core mathematical representation which is robust with respect to perturbations in microphysical details as a 'minimal model'; see for example Batterman and Rice (2014). The literature on this issue is rather thorny and I will not discuss it further here.

[5] The field-theoretic approach is also known as the 'momentum-space', 'k-space' or 'Wilsonian RG' approach. The field-theoretic approach, primarily developed by Wilson and Fisher is not merely the Fourier transform of the real-space approach, spearheaded by Kadanoff. Rather each approach involves a different, though related, set of techniques and physical models; see §2.2 for further discussion.

arguing that the real-space explanation is inadequate. However, my reasons for believing this are distinct from those endorsed by Mainwood.

2 The Physics

The following two sections involve some technical detail; overall I claim that the two approaches to the RG provide different putative explanations of universality and that, as such, they ought to be distinguished.

In the real-space approach critical exponents are derived based only on a representative model for each universality class. A model is not provided for each member of the same class and it is not demonstrated that the details which distinguish each member of the same class are irrelevant to that system's critical behaviour. In addition I argue that the mathematical model employed is insufficiently general to represent the common aspects of all members of the universality classes; thus an explanation of type 2R is not available and there is no general way to construct a 3R-2L mapping.

For the real-space case, universality is *not explained but assumed*: no justification is given for the application of the single model to the other members of the class. This conclusion is reached through consideration of the models and a sketch of the RG methods by which the critical exponents are derived for each such model.

In §3.2 I consider three responses to this assumed-not-explained objection: first one might claim that the observation of self-similarity explains universality; second there is an argument due to Kadanoff (1971) which develops the real-space RG argument; third liquid-gas systems and uniaxial magnetic systems may exhibit common behaviour because of a structural mapping (the lattice-gas analogy) between them. In all three cases I express doubts that the behaviour of the broad range of systems which exhibit universality could be thus explained.

I also describe the field-theoretic RG approach. The field-theoretic RG makes use of a renormalisable Hamiltonian. I argue that this provides it with the tools to describe the commonalities in the various systems sufficient to their common behaviour. The RG techniques then allow one to class all differences between systems so represented as irrelevant to their critical behaviour. As such the field-theoretic RG explains universality by showing that all systems in the same class will have common behaviour due to the irrelevance of the details which distinguish them and their shared representation by a renormalisable Hamiltonian at the critical point – this is a 2R-type explanation.

The standard account of this explanation implicitly depends on physics which has not been worked out, as such it includes certain technical lacunae. These correspond to our inability explicitly to formulate Hamiltonians which represent the details which distinguish systems within the same universality class. Nonetheless, unlike in the real-space case, we have theoretical justification for the claim that such distinguishing details are irrelevant. In addition we may derive a link (the order parameter) between each system and its mathematical representation. In §3.1 I further discuss the gaps in the physics and adduce reasons to consider the field-theoretic RG explanation nonetheless adequate.

Overall: the real-space RG provides an explanation of type 1R for a few individual systems but does not achieve 2R. It looks like we can draw diagrams which provide 3R as well in this case, but the abstract picture of convergent trajectories fails to correspond to the real physical systems in the same universality class. Conversely the field-theoretic RG explains along the lines of 2R (where the common features are representation by the LGW Hamiltonian and the order parameter), and allows one to justify 3R-type explanations.

2.1 The Models

It turns out that the critical behaviour of the different universality classes can be derived from a range of simple model systems. I briefly describe the Ising model, and its extension to the n-vector model which defines a broad range of models classified according to their values for two variables. This model is crucial to understanding the real-space RG, and is abstracted to provide the basis for the field-theoretic RG. Microphysical models are not defined for multiple members of the same universality class, rather a representative model is used for each class.

Niss (2005) describes the early history of the Lenz-Ising model.[6] This history demonstrates that the Ising model was specifically designed to represent the physical characteristics of magnetic systems rather than the broader range of systems which display critical phenomena. Niss observes that it was commonplace in the 1920s to model magnetic materials as composed of a lattice of numerous micromagnets – often idealised as compass needles – which mutually interact. The model was proposed to represent the transition between the ferromagnetic and paramagnetic states of certain materials.[7] The major innovations due to Lenz and Ising were to define a particular interaction between neighbouring micromagnets and to restrict their possible orientations to a discrete range. This latter assumption arose out of a combination of empirical data, knowledge of the structural and symmetry properties of solid matter and considerations from early quantum mechanics. Note that the Ising model provides crude approximations to the properties of real ferromagnets but captures their key qualitative features.

In modern formulations the Ising model is described as an array of spins. It consists of a D-dimensional cubic lattice with $\{\mathbf{e_i}\}$ basis vectors with sites labelled $\mathbf{k} = (k_1\mathbf{e_1}, ..., k_D\mathbf{e_D})$. At each site there is a spin variable $\sigma_\mathbf{k} \in \{-1, 1\}$, though in extensions to this model the spin variable can take a greater range of values. A Hamiltonian is defined:

$$\mathcal{H} = -J \sum_{\mathbf{k},\mathbf{k}+\boldsymbol{\mu}} \sigma_\mathbf{k}\sigma_{\mathbf{k}+\boldsymbol{\mu}} - B \sum_\mathbf{k} \sigma_\mathbf{k} \tag{1}$$

The coupling constant J takes a positive value and is assumed to be independent of all variables other than the system volume. The Ising model interaction is generally defined over nearest, or next-nearest neighbours, thus $\boldsymbol{\mu}$ is a lattice vector which takes any vector to the relevant neighbour in the positive direction. B is an external magnetic field.

The Hamiltonian of a system corresponds to the energy of the system in a particular configuration, thus we see (as is confirmed empirically) that the Ising Hamiltonian will take a lower value when the spins are

[6] I will henceforth refer to it as the 'Ising model' – as it is generally known – although Lenz and Ising jointly proposed it in papers in 1920 and 1924 respectively.

[7] The Ising model also predicts the spontaneous magnetisation below the critical temperature, though this was not discovered until Peierls did so in 1936.

aligned, and a higher value when spins are disordered. The ferromagnetic-paramagnetic transition can be defined over this lattice as the transition from the spin configuration with all spins aligned to that where there is no general correlation between the spin directions. This transition will take place at the Curie temperature (T_c). In 1944 Lars Onsager published a paper which derived the specific heat of a two dimensional Ising model in the absence of an external magnetic field. He demonstrated that this system will display power law behaviour with a particular critical exponent. However, despite much effort, no-one has succeeded in an analytic derivation of critical behaviour for any three dimensional model.

Behaviours characteristic of systems approaching T_c are termed 'critical phenomena' and it is with respect to the power laws which describe such behaviour that universality can be observed. Current mathematical procedures to describe such behaviour involve the Renormalisation Group (RG) which I describe below. First I note the n-vector model which generalises the Ising model to various universality classes. As Stanley (1999, p. S361) notes: "empirically, one finds that all systems in nature belong to one of a comparatively small number of such universality classes".

The n-vector model includes spins which can take on a continuum of states.

$$\mathcal{H}(d,n) = -J \sum_{\mathbf{k},\boldsymbol{k}+\boldsymbol{\mu}} \boldsymbol{\sigma}_{\mathbf{k}} \cdot \boldsymbol{\sigma}_{\boldsymbol{k}+\boldsymbol{\mu}} - B \sum_{\mathbf{k}} \boldsymbol{\sigma}_{\mathbf{k}} \tag{2}$$

Here, the spin $\boldsymbol{\sigma}_{\mathbf{k}} = (\boldsymbol{\sigma}_{\mathbf{k},1}, \boldsymbol{\sigma}_{\mathbf{k},2}, ..., \boldsymbol{\sigma}_{\mathbf{k},n})$ is an n-dimensional unit vector. The two parameters which determine the universality class are the system dimensionality d (which will determine the set of nearest neighbours) and the spin dimensionality n. The standard, 3D Ising model corresponds to $\mathcal{H}(3,2)$.

I now turn to a discussion of the renormalisation group derivation of critical exponents. A full exposition would require more space than we have here but I sketch the procedure below.[8] RG transformations are constructed to preserve thermodynamical properties of the system of interest (those derived from the partition function) while increasing the mean size

[8]There are many textbooks and review articles which describe these techniques, see for example Binney et al. (1992), Cardy (1996) and Fisher (1998).

of correlations. Thus, for example, the RG transformations take a ferromagnetic system towards the critical point (where the order parameter fluctuates wildly).

2.2 Field-theoretic and Real-space Renormalisation

I mentioned above that there are competing methods for deriving the critical exponents for each universality class. These correspond to different RG approaches:

Real-space RG: Consider the Hamiltonian of a system on a lattice (e.g. in the Ising model). The higher energy interactions will probe the structure of the lattice, and in order to consider the system probed at a larger length-scale, we average over the higher energy contributions to the Hamiltonian. This can be done by increasing the effective lattice size and constructing a new Hamiltonian for a system on a larger lattice; see figure 2 on p.24; this is sometimes referred to as 'coarse-graining' or 'zooming out'. This can be thought of as a blocking procedure, whereby some group of particles is replaced by one particle which represents the group through an average or suchlike.[9] On this model the RG flow represents the changes in parameters which leave the form of the Hamiltonian, and certain qualitative properties of the system unchanged (i.e. those which are derived from the partition function) while increasing the lattice size. Monte Carlo computer based methods allow for the derivation of the critical exponents from the n-vector Hamiltonian (equation (2)) via the real-space RG.

Field-theoretic RG: The Hamiltonian (equation (7)) considered in this case is more abstract (technically it is a functional of the order parameter) and depends for its construction on Ising-type models – I discuss its derivation below. The calculation of this Hamiltonian for real systems involves integration over a range of scales and energies. The highest energy (smallest scale) cut-off (denoted Λ) corresponds to the impossibility of fluctuations on a scale smaller than the distance between the particles in the physical system. The RG transformation in this case involves decreasing the cut-off, thus increasing the minimum scale of fluctuations considered. This procedure is analogous to increasing the lattice size and will

[9] A variety of acceptable blocking methods are discussed by Binney et al. (1992).

similarly generate a flow through parameter space designed to maintain the Hamiltonian form and qualitative properties of the system in question.

The following is an expanded version of the paragraphs above with more technical detail, the aim is to provide a clear sketch of the derivation of the critical exponents for each approach. The RG transformation $\mathcal{R}$ transforms a set of (coupling) parameters $\{K\}$ to another set $\{K'\}$ such that $\mathcal{R}\{K\} = \{K'\}$. $\{K^*\}$ is the set of parameters which corresponds to a fixed point, defined such that the RG transformation will have no effect on the set of parameters transformed, as such $\mathcal{R}\{K^*\} = \{K^*\}$. If we assume that $\mathcal{R}$ is differentiable at the fixed point this leads us to a version of the RG equations.

$$K'_a - K^*_a \sim \sum_b T_{ab}(K_b - K^*_b) \tag{3}$$
$$\text{where } \; T_{ab} = \frac{\partial K'_a}{\partial K_b}\Big|_{K=K^*}$$

There are now two more steps before we can define relevance and irrelevance. Firstly we define the eigenvalues of the matrix T_{ab} as $\{\lambda^i\}$ and its left eigenvectors as $\{e^i\}$. Now we can define scaling variables which are linear combinations of the deviations from the fixed points: $u_i \equiv \sum_a e^i_a(K_a - K^*_a)$. By construction these scaling variables will transform multiplicatively near the fixed point such that $u'_i = \lambda^i u_i$. The second (trivial) step is to redefine the eigenvalues as $\lambda^i = b^{y_i}$ where b is the renormalisation rescaling factor and y_i are known as the renormalisation group eigenvalues.

If $y_i > 0$ then u_i is relevant; if $y_i < 0$, u_i is irrelevant; and if $y_i = 0$, u_i is marginally relevant. The relevant scaling variables will increase in magnitude after repeated RG transformations while the irrelevant scaling variables will tend to zero after multiple iterations. (The behaviour of the marginal scaling variables requires more analysis to determine.) Thus, given the Hamiltonian of one of our models one can define an RG transformation which will allow one to: (i) classify certain of the coupling parameters of the system in question as (ir)relevant to its behaviour near the fixed point, (ii) extract the critical exponents from the scaling behaviour near the fixed point. Up to this point the description is generic.[10]

The real-space RG depends on the application of a blocking transformation, a standard example is depicted in figure 2, though almost any blocking transformation would do equally well. It is required that the Hamiltonian form is stable across these transformations. Since the Hamiltonians are not renormalisable this involves the application of a transformation and subsequent truncation of the Hamiltonian.[11] This procedure is generally carried out using computer methods.

The field-theoretic RG approach derives the critical exponents using diagrammatic perturbation theory – I do not have space to elaborate this here. The Hamiltonian in this context is macroscopic and depends on the order parameter ($\boldsymbol{\phi}$) which, in the Ising model context, is a sum of the spins in a small region of volume δV at $\mathbf{x}$: $\boldsymbol{\phi}(\mathbf{x}) = \frac{\mu}{\delta V}\sum_{i\in\delta V}\sigma_i$.[12] We require that $a \ll \delta V \ll l$ where a is the physical lattice spacing and l is the dominant statistical length (often the correlation length). One can approach its construction from the Ising model as follows (see Klein, Gould, and Tobochnik (2012) for more details):[13]

[10]Note that in the real-space approach the coupling parameters to the Ising-type Hamiltonians are marked as relevant or irrelevant while in the field-theoretic approach it's the operators – functions of the order parameter – which are so labelled.

[11]It is these truncations which motivate Mainwood (2006)'s dismissal of the explanation on offer by the real-space RG. I discuss this further on p.18. See §3.2 for my distinct critique of the real-space RG explanation.

[12]The symbol ϕ is used to refer to the thermal average of the order parameter $\phi(\mathbf{x}, t)$. This quantity has a system-dependent definition. For example in liquid-gas transitions $\phi(\mathbf{x}) \equiv \rho(\mathbf{x}) - \rho_{\mathrm{gas}}(\mathbf{x})$ where $\rho(\mathbf{x})$ is the average density in a volume centred on $\mathbf{x}$ i.e. is a fluctuating quantity and $\rho_{\mathrm{gas}}(\mathbf{x})$ is the time-averaged density for the gas at the temperature at $\mathbf{x}$. Clearly, below T_c for gaseous systems and above T_c in general $\phi \approx 0$, but below T_c for liquid systems $\phi > 0$. Analogously at the ferromagnetic-paramagnetic transition, where the magnet is well modelled by the Ising model the order parameter is as defined above. Thus for ferromagnetic systems (at $T < T_c$) $\boldsymbol{\phi} \neq 0$ and for paramagnetic systems ($T > T_c$) $\boldsymbol{\phi} = 0$.

The order parameter is defined for many other systems: for the binary fluid $\phi(\mathbf{x}) = X'(\mathbf{x}) - X''$ where $X'(\mathbf{x})$ is the local molar density of one of the fluids and X'' its thermally averaged value when the fluids have separated; for Helium I - Helium II transitions the order parameter is $\psi(\mathbf{x})$ which is the quantum amplitude to find a particle of He II at $\mathbf{x}$; similarly for conductor-superconductor transitions where $\psi(\mathbf{x})$ is the quantum amplitude to find a Cooper pair at $\mathbf{x}$.

[13]There are many different derivations of this Hamiltonian which speaks to its generality. See Binney et al. (1992), Goldenfeld (1992) for some alternatives.

Start with the Ising model (equation (1)); then postulate a form for the Helmholtz free energy $\mathcal{F}(\phi)$ of a system in contact with a heat bath. The terms in equation (4) correspond (a) to the interaction of the coarse grained Ising spins with an external magnetic field, (b) the interactions between the coarse grained spins which depends only on the distance between the blocks and (c) an approximation of the entropy (using Stirling's approximation). $F = U - TS$.

$$\begin{aligned}\mathcal{F}(\phi) = &-\overbrace{B\int \phi(\mathbf{x})d\mathbf{x}}^{\text{(a)}} - \overbrace{\frac{1}{2}\iint J(|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{x})\phi(\mathbf{y})d\mathbf{x}d\mathbf{y}}^{\text{(b)}} \\ &- k_B T\Big(\underbrace{\int [1+\phi(\mathbf{x})]\ln(1+\phi(\mathbf{x}))d\mathbf{x} + \int [1-\phi(\mathbf{x})]\ln(1-\phi(\mathbf{x}))d\mathbf{x}}_{\text{(c)}}\Big) \end{aligned} \tag{4}$$

Assuming $\phi(\mathbf{x})$ is small allows the logarithms to be expanded and truncated after the second order (on the assumption that the spin blocks only vary significantly over large distances). Using Parseval's theorem, expanding $J(|\mathbf{x}-\mathbf{y}|)$ in Fourier space, truncating after the second derivative, converting back to real-space and then integrating by parts leads to (b) becoming

$$\hat{J}(0)\int \phi(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} + \frac{1}{2}R^2\int [\boldsymbol{\nabla}\phi(\mathbf{x})]^2 d\mathbf{x} \tag{5}$$

This results in a modified version of equation 4:

$$\mathcal{F}(\phi) = \int d\mathbf{x}[R^2[\boldsymbol{\nabla}\phi(\mathbf{x})]^2 + \epsilon\phi^2(\mathbf{x}) + \phi^4(\mathbf{x}) - B\phi(\mathbf{x})] \tag{6}$$

This is the Landau-Ginzburg free energy, where $R^2 \propto \int \mathbf{x}^2 J(|\mathbf{x}|)d\mathbf{x}$. This has the same form as the Landau-Ginzburg-Wilson (LGW) Hamiltonian (equation (7)).[14] I will not discuss the few remaining steps as the physical underpinnings have been outlined. The form changes only slightly when it is read as a functional integral and the system is considered in the

[14]In Statistical Mechanics $\mathcal{F} = \frac{\mathrm{Tr}\{He^{-\beta H}\}}{Z} = \langle H \rangle$.

absence of an external magnetic field ($B = 0$). The integral is generalised to dimension d.

$$\mathcal{H} = \int \mathrm{d}^d \boldsymbol{x} \Big[\frac{1}{2}\zeta^2 |\boldsymbol{\nabla}\boldsymbol{\phi}(\mathrm{x})|^2 + \frac{1}{2}\theta|\boldsymbol{\phi}(\mathrm{x})|^2 + \frac{1}{4!}\eta|\boldsymbol{\phi}(\mathrm{x})|^4 \Big] \tag{7}$$

Note however that the LGW Hamiltonian is not the Ising model effective Hamiltonian. This latter object is more complicated, however it is demonstrated in Binney et al. (1992, Appendix K), (and is plausible given its derivation) that equation (7) is a good approximation to a truncated form of the Ising Hamiltonian near the critical point.

The construction of equation (7) is quite different from equations (1-2). It builds on these models but abstracts from them. More details can be found in (e.g.) Fisher (1974). There he demonstrates the field theoretic methods which allow one to derive expressions for the critical exponents as functions of d and n, see equation (8) for the first few terms of the exponent α; this will give a value for various universality classes. This derivation depends on the functional integration of the LGW Hamiltonian over all functions $\boldsymbol{\phi}(\boldsymbol{x})$.

$$\alpha = \frac{4-n}{2(n+8)}(4-d) + \frac{(n+2)^2(n+28)}{4(n+8)^3}(4-d)^2 + ... \tag{8}$$

Crucially, it can be shown that the addition of certain terms to the LGW Hamiltonian will lead to irrelevant contributions which do not affect the values for critical exponents describing the approach to a given fixed point. In Binney et al. (1992, Ch.14) the criteria for relevance and irrelevance are derived. An operator O_p is relevant if $p - d(p-2)/2 > 0$ and irrelevant if $p - d(p-2)/2 < 0$ where d is the dimension of the system under investigation.[15] O_p specifies the power of ϕ under consideration.[16]

This serves to establish that for the LGW Hamiltonian, for $d = 3$, any

[15] Note that the Ising-type Hamiltonians used in the real-space RG approach are not renormalisable, as such criteria for relevance and irrelevance of additions to those Hamiltonians cannot be specified in this generality.

[16] It is formally defined as follows: $O_p \equiv \int \mathrm{d}^d\boldsymbol{x}\, \lambda_p \sum_{m=0}^{p/2-1} (-1)^m \frac{C_m}{(p-2m)!}\phi^{p-2m}$ where where $C_m \equiv \frac{1}{2^m m!}\big(\int^\Lambda \frac{\mathrm{d}^d\boldsymbol{q}}{\zeta^2 q^2}\big)^m$.

O_p with $p > 6$ will be irrelevant at the appropriate fixed point.[17] This is an important result for the discussion in the remainder of this paper. Its generality depends on the justification for the applicability of the LGW Hamiltonian to various models. As we will see in what follows this will depend in part on the order parameter assigned to each member of each universality class.

The theory behind this result is relatively involved, but the idea is simple: the LGW Hamiltonian is renormalisable. This means that applying an RG transformation to the Hamiltonian will not add terms which cannot be absorbed into the parameters ζ, θ, η in equation (7). Thus the Hamiltonian is in some sense scale-invariant: its renormalisability means that it is independent of the details of the cut-off (Λ).

The fixed point – which describes the location of the critical phase transition – is itself a point of scale invariance as it is unaffected by RG transformations.[18] Thus at the fixed point the only elements which are relevant and contribute to the behaviour at the fixed point are those in the renormalisable Hamiltonian. All other terms which may be added to that Hamiltonian will consequently be irrelevant or marginally relevant (see §3.1). By contrast the Hamiltonians employed in the real-space approach are not renormalisable and the description of their behaviour near the critical point relies on the imposition of scale invariance by truncating the Hamiltonian after each iteration of the RG transformation.

The next section will explore the extent to which each RG approach can be considered to explain the universality of critical phenomena.

3 Universality Explained?

Universality is explained if we are able to show that each member of each universality class has features in common *and* to demonstrate that it is sufficient to have those features to generate the universal behaviour – that is

[17] Odd powers of ϕ are generally excluded for reasons of symmetry. For $d = 3$ it can be established perturbatively (at least to low orders) that O_6 is also irrelevant.

[18] This corresponds physically to the divergence of the correlation length in critical systems.

an explanation along the lines of 2R; see §1 for my taxonomy of explanations. Universality may be equally well explained by the convergence of flows in an abstract space, so long as we can provide a map between these flows and the physical systems they purport to represent. In this section I build upon the details of physics given thus far. I argue that the field-theoretic explanation is adequate (§3.1) but that the real-space explanation is inadequate both to 2R- and 3R-type explanations (§3.2).

My claims here follow those of Mainwood (2006, pp. 152-187) who argues that the real-space and field-theoretic approaches should be distinguished when assessing the RG explanation of universality. Mainwood claims that the real-space approach fails to provide an adequate explanation because the RG transformation needs to be tailored to each model under consideration – which follows from the non-renormalisability of the Hamiltonians used. As such he considers the real-space approach inadequate to the identification of common aspects between members of the same universality class.

I suggest that the real-space approach cannot explain universality for a more basic reason: it fails to represent the diverse range of systems which fall into the same class and thus does not demonstrate a flow of different systems into the same fixed point; I discuss this further in §3.2. Mainwood's claims may bolster my own to the extent that even were the real-space RG to model each distinct system one would still have some grounds for doubting the explanation of universality.

In addition to my worries about the real-space RG, I argue that the standard characterisation of the field-theoretic RG explanation of universality adverts to physics which is somewhat less developed than first appears. I highlight these technical lacunae and demonstrate that they ought not overly to bother us.

Where the explanation of universality is presented, it is commonly claimed that the explanation demonstrates the irrelevance of the heterogeneous details of physical systems. E.g.:

> The distinct sets of inflowing trajectories reflect their varying physical content of associated irrelevant variables and the corresponding non-universal rates of approach to the asymptotic power laws dictated by $\mathcal{H}$.

[Fisher (1998)]

Similar arguments can be found in (e.g.) Kadanoff (2013), Batterman (2014) and various textbooks. Such arguments are often represented pictorially, see figure 1. This explanatory sketch implies that we are able to include irrelevant details of diverse physical systems in the mathematical representations.

Multiple systems in the same universality class are represented at the critical point by the LGW Hamiltonian (eqn. 7). In order to represent the details which distinguish such systems we would need to add irrelevant operators to that equation. This is not done for the specific systems which fall into the same universality class. It is not known how to construct Hamiltonians which represent all the peculiarities of the various systems which exhibit universal behaviour. Thus no map can be *explicitly* defined which relates the distinct convergent trajectories of figure 1 to real, physical systems.

3.1 Field-theoretic RG

We have sound theoretical reasons to think that the LGW Hamiltonian represents a wide range of physical systems. The physical analysis behind this claim is the renormalisability of the LGW Hamiltonian and the demonstration that certain classes of operators are irrelevant, as discussed on p.16. Binney et al. (1992, p.366) express this as follows:

> to the accuracy of our calculation we have shown that any three-dimensional physical system whose Hamiltonian can be written as an even functional of a one-component scalar field should have the same critical behaviour as the Landau-Ginzburg model.

Thus we need to show, for each system of interest, that its order parameter can be written as a one-component scalar field. Paying close attention to the order parameter of each system will also ground the various assignations of systems to different universality classes. The order parameter

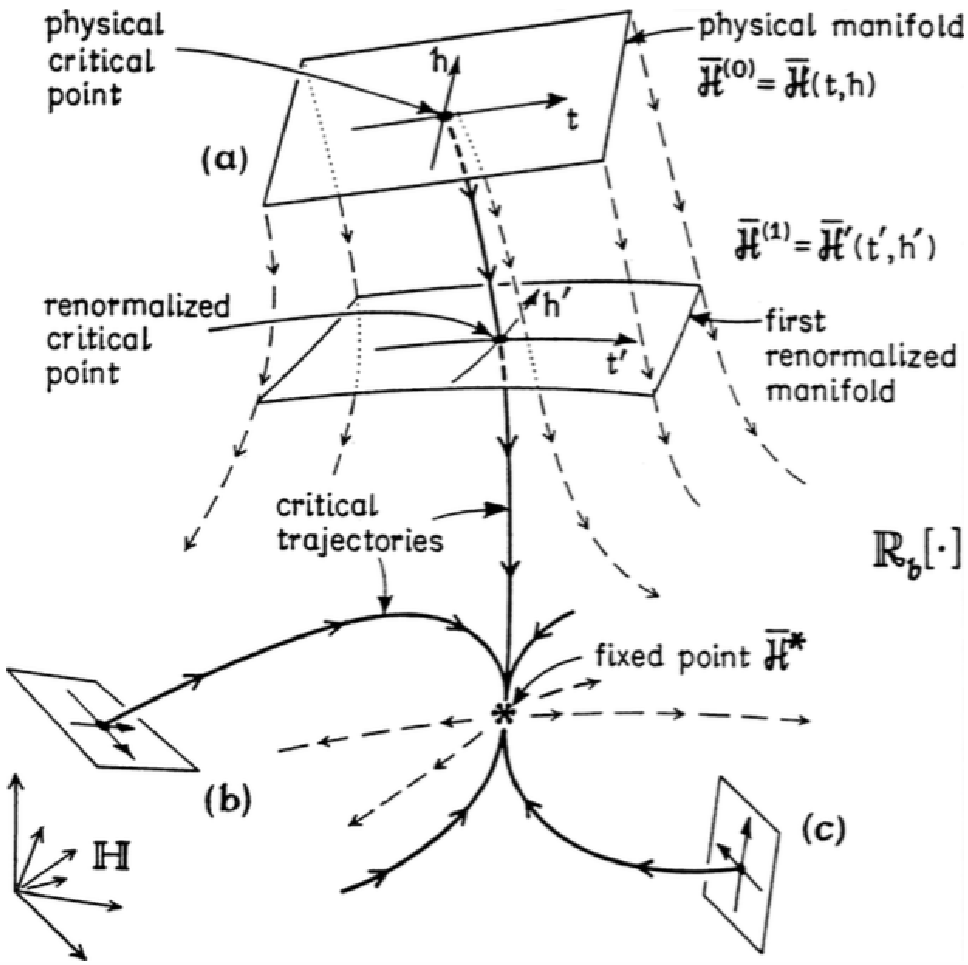

Figure 1: The RG flow in the abstract space of Hamiltonians (or, more precisely, the space of couplings for a fixed Hamiltonian form). Figure from Fisher (1998).

accounts for the symmetry group (i.e. the n of the n-vector model) and the dimensionality. Defining the order parameter for a condensed matter system is not a straightforward process. It depends subtlely on the kind of phase transition the systems undergoes, and which macroscopic features change at such a phase transition. Footnote 12 on p.14 provides some examples of various order parameters.

The field-theoretic approach provides a 2R-type explanation of universality: all systems in the same class are represented both by order parameters with the same symmetry and dimensionality and by the central operators of the LGW Hamiltonian; additionally we have a general argument that at the critical point such commonality is sufficient for common critical behaviour.

One might worry that this explanation assumes universality: once the order parameter has been specified, common representation is assured for the different systems in the same class. As such, specification of the order parameter might be said to do all the explanatory work.

I don't think that this argument goes through. The reason is that the different systems which have common order parameters are only guaranteed to have common behaviour at the critical point; this is exactly what we wanted: that's the statement of universality. What the field-theoretic RG framework does not guarantee, nor should it, is that systems which share order parameters (and thus symmetry and dimensionality) have common properties away from the critical point – differences away from the critical point are represented by the irrelevant operators. The explanatory force of the RG lies in exactly this principle: that it shows common behaviour at the critical point for systems whose behaviour is otherwise different. Part of this explanation does lie in the matching of order parameters to systems, but the RG framework and the LGW Hamiltonian also play important roles.

A further concern with this explanation rests on the observation that the order parameter and LGW Hamiltonian correspond to large-scale features of the systems of interest, thus our explanation is not tied in detail to the microscopic heterogeneities of systems of interest.

So long as we have reason to believe that our model represents the different systems which exhibit universality the explanation needn't include all the details of each system; it has long been acknowledged that good explanations may abstract from underlying details. Observations of scale symmetry coupled with a microscopic justification of the choice of order parameter provide reasons to link the model to the universally behaving systems; discussion of irrelevant operators supplement such reasons. Further justification akin to that provided by the derivation of the LGW Hamiltonian from the Ising model – see §2.2 – may provide a bottom-up account of critical phenomena. Such accounts, though interesting, are not essential to adequate explanations.

The central operators of the LGW Hamiltonian represent the common aspects which unite the various systems in the same class at the critical point. Such systems are distinguished at most by operators which are demonstrably irrelevant to the behaviour at the critical point. Thus the common aspects are sufficient for the systems' exhibiting universal behaviour and we have a 2R-type explanation. Furthermore this, in principle, grounds a 3R-type explanation: the various systems are represented by Hamiltonians distinguished by irrelevant operators and the flows of the distinct systems converge at the fixed point.

That we cannot write down the irrelevant operators may be worrying. It may be thought that evidence is scant for the claim that such operators indeed represent the heterogeneities which distinguish physical systems away from the critical point.

I hope partially to alleviate such worries by briefly considering crossover theory.[19] The theoretical description of crossover tells us that in some cases we may derive a correspondence between certain operators and the details of physical systems. By showing that certain operators may represent the details which distinguish systems away from the critical point, this ought to bolster the analysis of the field-theoretic RG explanation presented just above.

Systems undergoing crossover display critical behaviour characteristic of some universality class as they approach T_c, but under repeated iterations of the RG transformations (read: as the temperature moves closer to

[19]This has been appealed to by Mainwood (2006), Callender and Menon (2013), and Butterfield and Bouatta (2011) with a view to deflating claims of emergence in the context of critical phase transitions. My claims here are distinct from those and ought to be far less contentious.

T_c) they deviate from that behaviour and *cross over* to a different universality class. For example a system near the Heisenberg fixed point may have an additional relevant operator, we might thus define a Heisenberg type ($n = 3$) Hamiltonian including operators for isotropic and anisotropic couplings. It turns out that a system so described will cross over to Ising-type behaviour; for further details see Fisher (1974) and Cardy (1996).

Crossover theory is empirically successful, and such successes are predicated on deriving a relationship between operators and the details of physical systems. Although the operators for which such a correspondence can be shown are not irrelevant – these are relevant or marginally relevant operators – such correspondences help to establish that operators may play the required role in the field-theoretic RG explanation of universality.

For most instances of universality we have yet to discover irrelevant operators which are physically interpreted as representing those features which distinguish multiple members of the same class. The phenomenon of crossover does suggest that such differences can be modelled. This in turn justifies the claim that the field-theoretic approach explains the universality of critical phenomena: it identifies shared features in our systems of interest (represented by the LGW Hamiltonian) sufficient to predict their display of the critical exponents. The expanded Hamiltonians with the irrelevant operators, together with the flow induced by the RG, may be depicted as in figure 1 and thus explain universality.

The question remains: is universality thus explained? I claimed above that one way to explain universality is by constructing a map between convergent flows and real physical systems (akin to 3R on p.4). However, as noted above, no map can be explicitly constructed in this case since we do not know how to write down the irrelevant operators for the various systems of interest. Thus an explanation of type 3R with the necessary link to 2L may be found only in principle; in practice the 2R-type explanation goes through.

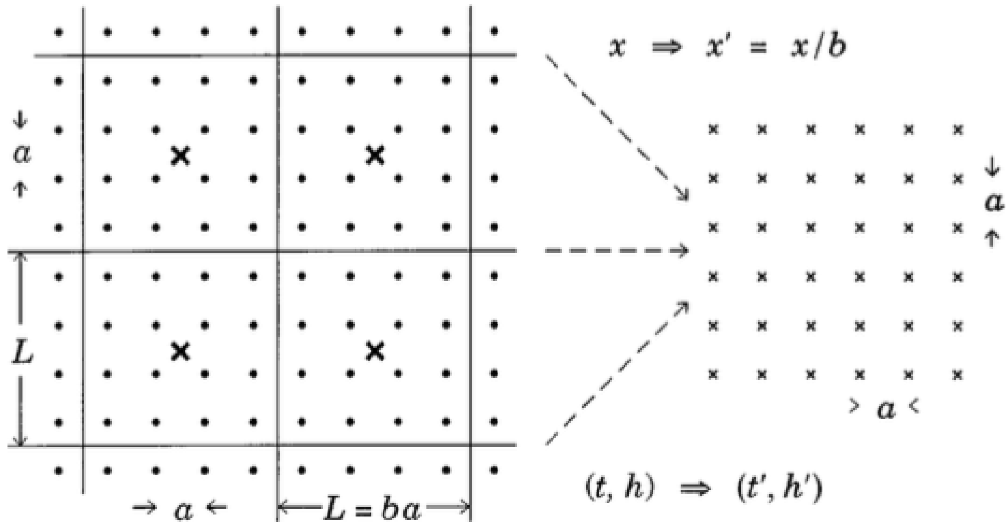

Figure 2: This demonstrates a single application of the real-space RG where a block of spins is replaced by a single larger spin. Figure from Fisher (1998).

3.2 Real-space RG

The real-space RG may be understood by appeal to simple diagrams like that in figure 2. It is thus unfortunate that, as I argue in this section, the putative explanation provided by the real-space RG is inadequate.

The real-space approach allows for the derivation of critical exponents consistent with empirical observation for various models. Furthermore we have an account of relevance and irrelevance and the claim that: "In general, for fixed points describing second-order critical points, there are two relevant parameters: the temperature and the field conjugate to the order parameter (for the magnet it is the magnetic field)" (Cheung (2011, p.51)). Why is this explanation of universality not sufficient?

On p.4 I categorised a few options for how universality may be explained. I claimed that 1R was insufficient but 2R or a supplemented 3R could do the job. I think that neither latter option is live in the real-space case. This is because the mathematical model employed does not have the tools to represent systems other than the archetypal system for each universality class.

While the field-theoretic approach makes use of a Hamiltonian derived from the Ising model, the Hamiltonian used in that approach is renormalisable. As such it includes a scale-invariant core at the critical point which represents a range of different systems. That's how 2R is achieved: by showing what's in common and the general demonstration that all possible distinguishing features are irrelevant.

The real-space approach does not make use of a renormalisable Hamiltonian, nor does it have a formalism which establishes generally that the Hamiltonians apply across a wide range of systems. Likewise there seems to be little hope that a picture like that in figure 1 can be shown to correspond to distinct systems in the same universality class having convergent flows. Thus no core commonality can be adduced for a 2R-type explanation, nor can a 3R-2L link be established. Three responses to this claim ought to be considered:

Firstly, it could be noted that irrelevant couplings are discussed in the real-space context, and we know that only a few, relevant couplings determine the critical exponents, thus perhaps these relevant couplings provide the 2R explanation. My response here is that the model is still tied to the details of the system it was created to represent. Thus the irrelevant couplings are those aspects of that system which will not affect its critical exponents.

To show that some aspects of a given system are irrelevant to its behaviour in a given context is quite different from showing that all systems with the relevant properties (and with different irrelevant properties) will display the behaviour. The former, system-relative claim is established by the real-space RG but the latter more general claim is not. The success of the field-theoretic RG explanation is due to the fact that we can categorise operators as relevant or irrelevant quite generally. In the real-space RG potential couplings are only categorised for a given model; as such 3R-2L is unavailable.

Secondly, one might claim that the blocking transformation is tailored specifically to the behaviour of systems at criticality thus perhaps the blocking itself represents the common features for 2R. The blocking is constructed so as to mirror the self-similarity of such systems and its application to systems at criticality is thus justified. This would mean that the real-space RG explains universality by appeal to the fact that all these systems have some commonality, i.e. their self-similarity, which justifies the use of these techniques to derive their critical exponents.

The problem with this claim is that the real-space RG approach does not simply derive the exponents from the blocking techniques. In fact, such exponents are derived by applying the blocking transformation and then truncating the Hamiltonian so that it will retain its original form. As such, the original Hamiltonian significantly determines the application of the real-space RG and the exponents derived. It is thus not sufficient to claim that the blocking techniques are justified by the self-similarity exhibited by the systems.

This can be seen by considering the phenomenon of crossover mentioned above. How may we establish theoretically whether a given model will display crossover phenomena? This depends on the terms in the Hamiltonian for that system. I raise this here as evidence that the initial Hamiltonian is crucial to the real-space derivation and explanation. As such, the appeal to a common blocking RG transformation cannot provide a 2R-type explanation.

Thirdly, for the remainder of this section I consider a pair of specific elaborations of the real-space RG approach which provide limited explanations of universality. The first demonstrates that critical exponents derived on the real-space approach do not depend on certain couplings for the Ising model. The second – the lattice-gas analogy – relates liquid-gas to magnetic systems. Neither, I argue, provides a general account of the range of classes and systems which behave universally. Such general accounts are presently unavailable: one might claim that for any system we could *in principle* write down its full microscopic Hamiltonian and deduce its critical exponents using real-space RG methods. While no-one can actually do that and there is no general argument that such a methodology would be successful I see no reason to presume that the real-space RG provides a general explanation of universality.

Batterman (2016) highlights an argument found in Kadanoff (1971) to the effect that one can introduce a parameter λ into the free energy function for the Ising model, and it can be demonstrated that the critical exponents do not depend on the value of this parameter. In Kadanoff's example this parameter corresponds to the ratio of the couplings for nearest neighbour to next nearest neighbour models. As such we may be assured of a particular kind of generality of the Ising model representation. The independence of the critical exponents from such parameters is, however, insufficient to establish the requisite generality for the real-space approach. If a similar argument were available for the variation between a liquid-gas system and a magnetic system then this explanation would be far more convincing.

The lattice-gas analogy exemplifies a possible mapping which may provide reasons to view the Ising model as representing liquid-gas systems in addition to magnetic systems. If this succeeds it would justify an explanation of type 2R for the limited case at hand. However it would not justify an RG explanation because the mapping is not one sourced in the relevance and irrelevance criteria of the RG. Rather it would provide a distinct explanation of universality.

In addition the lattice-gas explanation would not explain why liquid-gas systems behave like anisotropic magnets in the critical region but not outside that region. Since anisotropic magnets and liquid-gas systems exhibit radically different behaviour outside such regions it is not appropriate to claim that they are both well described by the Ising model in all contexts.

Furthermore it is not generalisable to other examples of universality, e.g. those systems which fall into the Heisenberg universality class. In both respects the field-theoretic approach outdoes the real-space approach even with the lattice-gas analogy.

The lattice gas model is summarised as follows:

> Consider the Hamiltonian
>
> $$\mathcal{H} = -4J \sum_{\langle ij \rangle} \rho_i \rho_j - \mu \sum_i \rho_i, \tag{9}$$
>
> where $\rho_i = 0, 1$ depending if the site is empty or occupied, and μ is the chemical potential. If we define $\sigma_i = 2\rho_i - 1$, we reobtain the Ising-model Hamiltonian with $B = 2qJ + \mu/2$, where q is the coordination number of the lattice. Thus, for $\mu = -4qJ$, there is an equivalent transition separating the gas phase for $T > T_c$ from a liquid phase for $T < T_c$.
>
> [Pelissetto and Vicari (2002, p.554)]

This mapping is clear enough, but merely shifts the burden of justification. As Pelissetto and Vicari acknowledge "The lattice gas is a crude approximation of a real fluid" (ibid.). Their justification for this approximation is empirical: "Nonetheless, the universality of the behavior around a continuous phase-transition point implies that certain quantities, e.g., critical exponents ... are identical in a real fluid and in a lattice gas, and hence in the Ising model." The model is provided the following rationale in the context of its original presentation:

> The question naturally arises as to the relationship between a lattice gas and a real gas in which the atoms are not confined to move on lattice points. If one replaces the configurational integral in the partition function of the real gas by a summation over lattice sites, one would obtain the partition function of the lattice gas. Theoretically speaking, by making the lattice constant smaller and smaller one could obtain successively better approximations to the partition function of the real gas.
> [Lee and Yang (1952, p.412)]

This is rather odd. Although gases may often be modelled as continuum gases, this is itself an idealisation which requires a physical justification. Furthermore, the problem with the application of the Ising model to a physical gas is not that the Ising model is discretised – we expect gases to contain finitely many particles. Rather one should be concerned that the molecules have more degrees of freedom available to them than the components of uniaxial magnets. The move towards continuum is an idealising step: we sought a de-idealisation to justify the application of the Ising model to liquid-gas systems.[20]

If we were to accept this justification of the lattice-gas model further questions would be raised: for magnets and liquid-gas systems do not display the same behaviour away from the critical point. It is precisely because the systems behave so differently much of the time that universality is startling. Thus, even if it turns out that the lattice-gas analogy

[20]This is not to suggest that idealised models are intrinsically problematic, I am just sceptical that the physical justification of this analogy is sufficient to explain universality.

gives a good account of liquid-gas systems, additional details are needed to explain the limited applicability of the Ising model to such systems.[21]

Do we have an explanation why these different systems undergo similar behaviour near the critical point? We are told that most of a system's features are irrelevant to its critical behaviour. It turns out, and this is surprising and interesting, that uniaxial magnets and fluids have some behaviour which is approximately described by the same model: namely the Ising model. But this result is a consequence of careful mapping between the systems; it was not an RG result. While a lattice gas model may explain universality to some degree, a generalisable RG explanation is not available on the real-space approach: the RG was used for the derivation of the critical exponents from the models, not in the justification of the applicability of the models to various physical systems.

On the real-space approach we only have an account for what's in common between systems with diverse microphysics when we have a well-motivated mapping between the Ising models and a model for the system in question. The lattice-gas analogy *may* provide one such mapping. However the real-space RG does not allow for a generalised explanation of universality because it cannot underwrite the flow of various different systems into the same fixed point.

4 Conclusion

Batterman characterises the RG explanation of universality as follows:

> One constructs an enormous abstract space each point of which might represent a real fluid, a possible fluid, a solid, etc. Next one induces on this space a transformation that has the effect, essentially, of eliminating degrees of freedom by some kind of averaging rule.

[21]Even at the critical point Vause and Sak (1980) argues for a failure of the lattice-gas analogy: while magnets display a symmetry under global spin inversion (in the absence of an external magnetic field) which implies a symmetry of the magnetisation-temperature curve about the temperature axis, liquid-gas systems will not display analogous symmetries of the density-temperature curve.

> ... Those systems/models (points in the space) that flow to the same fixed point are in the same universality class–the universality class is delimited–and they will exhibit the same macro-behavior. That macro-behavior can be determined by an analysis of the transformation in the neighborhood of the fixed point.
>
> [Batterman (2014, pp. 13-14)]

My aim in this paper has been to spell out the physical details which underpin the quote above. In so doing I argued that the picture Batterman provides of the RG explanation is not workable on the real-space approach, but that it is consistent with the field-theoretic approach. However I claimed that, due to certain outstanding technical lacunae, the field-theoretic RG explanation is better conceived as providing an explanation which: adduces common aspects of the various systems which exhibit universality, and demonstrates that such common aspects are sufficient for universal behaviour.

The real-space approach starts with a model and derives the critical exponents on the basis of that model. It is difficult to see how this approach adequately explains the phenomenon that heterogeneous systems have identical critical behaviour.

The field-theoretic approach, on the other hand, explains universality by positing an effective Hamiltonian and deriving the critical exponents from that. That this Hamiltonian is demonstrably general grounds the explanation of universality. Thus the primary moral of my paper is that these two approaches ought to be distinguished

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