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Inherently nonfinitely based lattices ☆

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Abstract

We give a general method for constructing lattices **L** whose equational theories are inherently nonfinitely based. This means that the equational class (that is, the variety) generated by **L** is locally finite and that **L** belongs to no locally finite finitely axiomatizable equational class. We also provide an example of a lattice which fails to be inherently nonfinitely based but whose equational theory is not finitely axiomatizable. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A **variety** is a class of algebras which can be axiomatized by a set of equations (that is, by a set of universal sentences whose quantifier-free parts are equations between terms). According to a classical result of Garrett Birkhoff [5] the varieties are exactly those classes of algebras which are closed with respect to the formation of homomorphic images, subalgebras, and arbitrary direct products. The **variety generated** by the algebra **A** is the smallest variety to which **A** belongs. An algebra **A** is **finitely based** if and only if there is a finite set Σ of equations, each true in **A**, such that each equation true in **A** is a logical consequence of Σ (that is, Σ axiomatizes the variety generated by **A**). **A** is said to be **inherently nonfinitely based** provided **A** belongs to some locally finite variety, but **A** belongs to no finitely based locally finite variety. Plainly, inherently nonfinitely based algebras are not finitely based. Recall that an algebra

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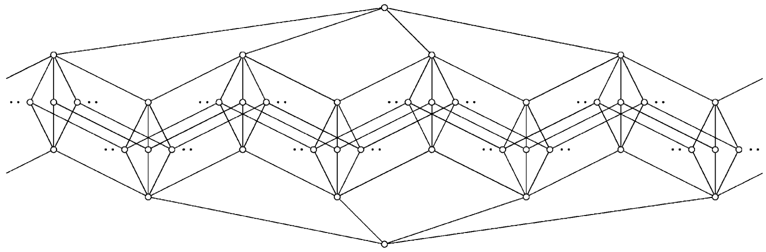

Fig. 1. The lattice $\mathbf{L}_f \bigstar_F \mathbf{M}_\omega$.

is **locally finite** provided each of its finitely generated subalgebras is finite, and a variety is **locally finite** if all the algebras in the variety are locally finite. But observe that there are locally finite algebras which belong to no locally finite variety—it is even an easy matter to construct such a locally finite lattice. Inherently nonfinitely based algebras, introduced independently by Murskiĭ [31] and Perkins [37], have been widely exploited, particularly in the construction of finite algebras which are not finitely based.

In this paper we offer a general method for constructing inherently nonfinitely based lattices. Such lattices must, of course, be infinite, since McKenzie [26] has proven that every finite lattice is finitely based. An inherently nonfinitely based lattice constructed by our method is illustrated in Fig. 1 below.

Our first examples of inherently nonfinitely based lattices were inspired by the lattices used in Nation [32] to refute the finite height conjecture, while the general method we use is obtained from that found in Baker et al. [2] modified by a variant of the doubling construction of Alan Day [10].

Lattices without finite equational bases were constructed by Kirby Baker [1] and [4], Ralph Freese [11], Christian Herrmann [18], Ralph McKenzie [26], and Rudolf Wille [43]. The lattices produced by the methods of this paper have the stronger inherent nonfinite basis property.

Finite axiomatizability has proven to be a subtle problem for varieties, even for varieties generated by a finite algebra. Using a difficult result of Emil Post [39], in 1951 Roger Lyndon [23] demonstrated that all 2-element algebras are finitely based. On the other hand, in 1954 he made the surprising discovery in [24] of an algebra with seven elements that is not finitely based. This led Alfred Tarski to pose the following problem:

Tarski's finite basis problem: *Does there exist a recursive algorithm which when presented with an effective description of a finite algebra will determine if the algebra is finitely based?*

In groundbreaking work, Ralph McKenzie [29] has shown that no such algorithm exists. (See also [41] for a second route to McKenzie's result.)

Despite this negative result much has been discovered concerning which finite algebras are finitely based. Among the finitely based algebras one finds

- All finite groups [34];
- All finite rings [22, 21];
- All finite lattices with operators [26];
- All commutative semigroups [36];
- All finite algebras with only finitely many basic operations which belong to congruence distributive varieties [3];
- All finite algebras with only finitely many basic operations which belong to congruence modular varieties with finite residual bounds [28];
- All finite algebras with only finitely many basic operations which belong to congruence meet semidistributive varieties with finite residual bounds [42]

The list of nonfinitely based algebras must seem more pathological, but it includes:

- The natural numbers endowed with addition, multiplication, exponentiation, and 1 [25];
- The natural numbers endowed with addition, multiplication, and exponentiation [16];
- Certain infinite groups [35];
- Certain infinite lattices as noted above;
- Certain finite nonassociative rings [38];
- Certain finite groups with an element distinguished by a new constant [6];
- The semigroup consisting of the following six matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

  where the operation is matrix multiplication [36].

In [40], Mark Sapir gave an algorithmic characterization of the inherently nonfinitely based finite semigroups, allowing us to considerably expand the last item above. In addition, Isaev [19] has constructed finite nonassociative rings which are inherently nonfinitely based.

Our methods for constructing inherently nonfinitely based lattices, as will become apparent, invariably produce nonmodular lattices. So our techniques do not directly address the following open problem.

Problem 1. *Is there an inherently nonfinitely based modular lattice?*

It is unknown whether there are inherently nonfinitely based groups. As noted by Kharlampovich and Sapir in [20] it seems unlikely that such groups exist. They point out that by a celebrated result of Zel'manov [44] there are no inherently nonfinitely based groups of prime exponent. Perhaps a solution to the following problem is accessible.

Problem 2. *Is there an inherently nonfinitely based group?*

The monograph by McKenzie et al. [27] provides notation and background information for the general theory of algebras and varieties. The books of Grätzer [15] and Burris and Sankappanavar [7] are also valuable references. More information on lattices can be found in the books of Crawley and Dilworth [8], Davey and Priestley [9] Grätzer [14], Freese et al. [12], and the forthcoming text of Nation [33].

The authors are grateful to the referee for pointing to parts of our proofs that required greater explanation.

2. Extending Day's doubling construction

Let $\mathbf{L}$ be a lattice, let F be a convex subset of L, and let $\mathbf{G}$ be a lattice with greatest element 1 and least element 0. We use $L \bigstar_F G$ to denote the disjoint union $(L-F)\cup(F\times G)$. Order $L \bigstar_F G$ by $x \leqslant y$ if one of the following holds:
(1) $x, y \in L-F$ and $x \leqslant y$ holds in $\mathbf{L}$,
(2) $x, y \in F\times G$ and $x \leqslant y$ holds in $\mathbf{F}\times\mathbf{G}$,
(3) $x\in L-F$, $y=\langle u,g\rangle \in F\times G$, and $x\leqslant u$ holds in $\mathbf{L}$, or
(4) $x=\langle v,g\rangle\in F\times G$, $y\in L-F$, and $v\leqslant y$ holds in $\mathbf{L}$.
There is a natural map λ from $L\bigstar_F G$ back onto L given by

$$\lambda(x) = \begin{cases} x & \text{if } x\in L-F,\\ v & \text{if } x=\langle v,g\rangle\in F\times G.\end{cases}$$

According to the theorem below, under this order $L\bigstar_F G$ is a lattice. We denote it by $\mathbf{L}\bigstar_F\mathbf{G}$, or by $\mathbf{L}\bigstar\mathbf{G}$ when no ambiguity arises. We say that $\mathbf{L}\bigstar_F\mathbf{G}$ is the **inflation** of $\mathbf{L}$ at F by $\mathbf{G}$.

Theorem 3. *Let F be a convex subset of a lattice $\mathbf{L}$ and let $\mathbf{G}$ be a lattice with a greatest element and a least element. Then $\mathbf{L}\bigstar_F\mathbf{G}$ is a lattice and $\lambda:\mathbf{L}\bigstar_F\mathbf{G}\to\mathbf{L}$ is an epimorphism.*

The condition that F should be convex is needed to establish that the ordering defined above is transitive. The least and greatest elements of $\mathbf{G}$ are needed to define the join and meet operations. (If $x, y\in L-F$ and $x\vee y=z\in F$, then $x\vee y=\langle z,0\rangle$ in $\mathbf{L}\bigstar_F\mathbf{G}$, and dually.) The proof of Theorem 3 differs in no important way from the proof of the corresponding result concerning Day's original doubling construction. See, for example, the proof of Theorem 1.1 in [12]. For a development of Day's doubling construction using the techniques of concept lattices see the work of Geyer [13].

Let $\mathbf{L}$ be a lattice. A subset F of L is called a **full lattice fence** in $\mathbf{L}$ if and only if there are two disjoint infinite antichains $\dots,t_{-1},t_0,t_1,\dots$ and $\dots,b_{-1},b_0,b_1,\dots$ of $\mathbf{L}$ such that
(1) $b_k\vee b_{k+1}=t_k$ for all $k\in\mathbb{Z}$,
(2) $t_k\wedge t_{k+1}=b_{k+1}$ for all $k\in\mathbb{Z}$, and
(3) $x\in F$ if and only if $b_k\leqslant x\leqslant t_j$, for some $k,j\in\mathbb{Z}$.

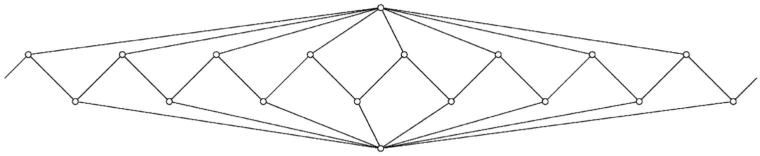

Fig. 2. The lattice $\mathbf{L}_f$.

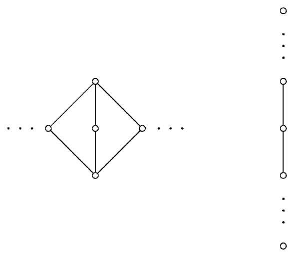

Fig. 3. The lattices $\mathbf{M}_\omega$ and $\mathbf{B}$.

For $a, b \in L$, we say that W is a **weak lattice fence** between a and b, whenever W is a finite set, say of cardinality n, whose elements can be listed as $w_0, \ldots, w_{n-1}$ so that
(1) $\{w_i : i < n \text{ and } i \text{ is even}\}$ and $\{w_i : i < n \text{ and } i \text{ is odd}\}$ are disjoint antichains,
(2) $a = w_0$ and $w_{n-1} = b$, and
(3) The condition displayed below or its dual holds:

$$w_{i+1} = \begin{cases} w_i \wedge w_{i+2} & \text{if } i \text{ is even,} \\ w_i \vee w_{i+2} & \text{if } i \text{ is odd} \end{cases} \qquad \text{for all } i < n-2.$$

We take $\{a\}$ to be a weak lattice fence for all $a \in L$.

The notion of a full lattice fence differs in some significant ways from the notion of a fence commonly used in the literature of ordered sets. Every full lattice fence in $\mathbf{L}$ is convex in $\mathbf{L}$. Moreover, in full lattice fences a given bottom element may lie below many top elements (and dually). Also, joins and meets are subject to certain restrictions in full lattice fences that need not be met in order-theoretic fences. Even weak lattice fences are more like order-theoretic fences still permit comparabilities and restrict joins and meets in ways that make them different from order-theoretic fences.

The lattice $\mathbf{L}_f$ illustrated in Fig. 2 is the least complex lattice with a full lattice fence F. This lattice was shown to be nonfinitely based in [26]. In Section 5 below we will show that this lattice fails to be inherently nonfinitely based by proving that it belongs to a certain finitely based locally finite variety of lattices. The lattice displayed in Fig. 1 is just $\mathbf{L}_f \bigstar_F \mathbf{M}_\omega$, where $\mathbf{M}_\omega$ is the lattice obtained by adjoining a greatest and a least element to a countably infinite antichain. $\mathbf{M}_\omega$ is displayed in Fig. 3.

3. A method for constructing inherently nonfinitely based lattices

An element of a lattice that is neither the greatest element nor the least element of the lattice is said to be a **proper** element.

Theorem 4. *Let* $\mathbf{L}$ *be a locally finite lattice, let* F *be a full lattice fence of proper elements in* $\mathbf{L}$, *and let* σ *be an automorphism of* $\mathbf{L}$ *such that*

(1) σ *partitions the set of proper elements into only finitely many* σ*-orbits, each infinite,*
(2) σ *preserves* F (*i.e.*, $\sigma(f) \in F$ *for all* $f \in F$), *and*
(3) *there is a natural number* d *such that no proper element of* L *is comparable to more than* d *others.*

Let $\mathbf{G}$ *be a lattice with a greatest element and a least element such that* $\mathbf{G}$ *belongs to a locally finite variety, and suppose that* $\mathbf{G}$ *has an automorphism with an infinite orbit. Then* $\mathbf{L} \bigstar_F \mathbf{G}$ *is inherently nonfinitely based.*

Proof. As observed in [30], a locally finite variety $\mathscr{V}$ of finite type is inherently nonfinitely based if and only if for infinitely many natural numbers N, there is a nonlocally finite algebra each of whose N-generated subalgebras belongs to $\mathscr{V}$. Thus under the hypotheses of our theorem, we must prove that $\mathbf{L} \bigstar_F \mathbf{G}$ generates a locally finite variety $\mathscr{V}$ for which we can construct the required nonlocally finite lattices, which we will denote by $(\mathbf{L} \bigstar_F \mathbf{G})_N$.

Our proof is based on a geometrical intuition that can be seen by examining the lattice $\mathbf{L}_f \bigstar \mathbf{M}_\omega$ displayed in Fig. 1. We view this lattice as laid out flat across the plane. To construct $(\mathbf{L}_f \bigstar \mathbf{M}_\omega)_N$ our plan is to wrap this flat lattice onto a cylinder with a large enough (as determined from N) circumference in such a way that the middle rows of the $\mathbf{M}_\omega$'s become one infinite zigzag spiral. This infinite spiral turns out to be generated by a finite set, so $(\mathbf{L}_f \bigstar \mathbf{M}_\omega)_N$ will not be locally finite. On the other hand, any N elements of $(\mathbf{L}_f \bigstar \mathbf{M}_\omega)_N$ lie on some relatively thin vertical section of our cylinder, which is isomorphic to the corresponding part of $\mathbf{L}_f \bigstar \mathbf{M}_\omega$. So the subalgebras of $(\mathbf{L}_f \bigstar \mathbf{M}_\omega)_N$ generated by N elements will belong to the variety $\mathscr{V}$ generated by $\mathbf{L}_f \bigstar \mathbf{M}_\omega$. In the course of establishing all this, the fact that $\mathscr{V}$ is locally finite will drop out. The hypotheses of Theorem 4 are sufficient to guarantee that this argument applies to $\mathbf{L} \bigstar_F \mathbf{G}$.

Let N be any natural number larger than 1.

Claim 5. *The* σ*-orbit of any element is an antichain. Thus* $\mathbf{L}$ *is a lattice of finite height.*

Proof of Claim 5. Suppose a is a proper element and that $\sigma(a)$ is comparable with a. It does no harm to suppose that $a \leqslant \sigma(a)$. Since the orbit of a is infinite, we must have $a < \sigma(a) < \sigma(\sigma(a)) < \cdots$. This is an infinite ascending chain $a = a_0 < a_1 < a_2 < a_3 < \cdots$ of proper elements. This makes a comparable to infinitely many elements, which cannot happen. Therefore, a and $\sigma(a)$ are incomparable, whenever a is a proper element.

More generally, we see that the orbit of a is an antichain. Therefore, the set of proper elements is the union of finitely many pairwise disjoint antichains. Hence $\mathbf{L}$ is a lattice of finite height. $\square$

Let m denote the number of σ-orbits of proper elements of $\mathbf{L}$. Make an arbitrary selection $a_0, a_1, \ldots, a_{m-1}$ of representatives, one from each orbit. Fix the following arrangement of the set of proper elements:

$$\ldots, a_{-m}, a_{-m+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{m-1}, a_m, \ldots, a_{2m-1}, \ldots$$

where $a_{qm+r} = \sigma^q(a_r)$ for all integers q and all $r \in \{0, 1, \ldots, m-1\}$. Thus, the σ-orbits of proper elements correspond to the congruence classes of indices modulo m. The words "consecutive", "interval", "distance", etc. applied to proper elements are to be understood by reference to these indices. Observe that the notions just mentioned (consecutive, interval, and distance) are invariant under σ.

We say that two proper elements a and b are **operationally related** provided either $a \vee b$ is proper or $a \wedge b$ is proper. Suppose that a and b are operationally related proper elements, and that $a \vee b$ is proper. According to the absorption law of lattices, $a = a \wedge (a \vee b)$. It follows that a is operationally related to $a \vee b$. Likewise, b is operationally related to $a \vee b$. Dually, in the case that $a \wedge b$ is proper, we find that a and b are both operationally related to $a \wedge b$.

Let a, b be any pair of elements of $\mathbf{L}$. Pick a nonnegative integer $i < m$ and $k \in \mathbb{Z}$ so that $\sigma^k(a_i) = a$. Since distance is invariant under σ, we see that the distance from a to b is the same as the distance from a_i to $b' = \sigma^{-k}(b)$. Now suppose that a and b are operationally related. Then there are at most d^2 choices for b' and at most m choices for i. This means that there are at most md^2 different numbers that can be distances between operationally related elements. Let M be the maximum possible distance between any two operationally related proper elements.

Consider an interval of M consecutive proper elements. Up to automorphisms of $\mathbf{L}$ (actually, up to powers of σ), there are only finitely many such intervals. Since $\mathbf{L}$ is locally finite, each of these intervals generates a finite sublattice. Among all these finite sublattices, let w be the greatest distance an element can be to the left or right of its generating interval.

Claim 6. *Let X be a set of proper elements all lying to the left of the proper element e. Any proper element in the sublattice generated by X can be no further to the right of e than the distance w.*

Proof of Claim 6. Without loss of generality, we suppose that X is the set of *all* elements to the left of e. Suppose that c is to the right of e and that c belongs to the sublattice generated by X. Let $t(x_0, x_1, \ldots)$ be a shortest term such that $t(a_0, a_1, \ldots) = c$ for some $a_0, a_1, \ldots \in X$. We contend that each a_i must lie no further to the left of e than the distance M. Otherwise, suppose a_i lies further to the left. Now there is a term s so that one of the following terms is a subterm of $t(x_0, x_1, \ldots)$: $x_i \vee s, x_i \wedge s, s \vee x_i, s \wedge x_i$.

Let us agree that $x_i \vee s$ is such a subterm. Let $b = a_i \vee s(a_0, a_1, \ldots)$. It follows from the minimality of t that a_i is operationally related to b, and hence the distance from a_i to b is at most M. But then b must lie to the left of e, and so $b \in X$. Now let $t'(y, x_0, x_1, \ldots)$ be obtained from $t(x_0, x_1, \ldots)$ by replacing $x_i \vee s$ by the new variable y. But then $t'(y, x_0, x_1, \ldots)$ is shorter than $t(x_0, x_1, \ldots)$ and nevertheless $t'(b, a_0, a_1, \ldots) = c$, a contradiction. It follows that c is generated by the interval of length M immediately to the left of e. So c must lie within the distance w of e. $\square$

Now notice that σ^2 partitions the set of proper elements into $2m$ orbits. Indeed, by selecting k sufficiently large, σ^k will partition the set of proper elements into a number of orbits exceeding any given bound.

Denote by ρ a power of σ so that ρ partitions the proper elements into more than $N(M + 2w)$ ρ-orbits. Notice that our selection of ρ *depends on the parameter* N although we have not made this dependence explicit in the notation.

A sublattice $\mathbf{S}$ of $\mathbf{L}$ is said to be ρ-**decomposable** provided there is a sublattice $\mathbf{S}_0$ of $\mathbf{L}$ such that $S = \bigcup_{k \in \mathbb{Z}} \rho^k(S_0)$ and no element of S_0 is operationally related to any element of $\rho^k(S_0)$ for any $k \in \mathbb{Z}$ with $k \neq 0$.

Claim 7. *The union of any N or fewer ρ-orbits generates a ρ-decomposable sublattice of* $\mathbf{L}$.

Proof of Claim 7. Let Y be the union of no more than N ρ-orbits. Examining the indices of the elements of Y we see there is some element of Y followed on the right by a gap of length at least $M + 2w + 1$ before the next element of Y is encountered. Indeed, Y is the union of pieces, each of the same cardinality (no more than N), each contiguous relative to Y, each a ρ-translate of the piece to its left, and each separated from the piece to its left by a gap of length at least $M + 2w + 1$. Let Y_0 be one of these pieces and let $\mathbf{S}_0$ be the sublattice generated by $Y_0 \cup \{0, 1\}$. By the choice of w, we know that S_0 extends to the left of Y_0 by at most w, and to the right by at most w as well. The corresponding observations hold for all the translates of S_0 by powers of ρ. Thus these translates are separated from each other by gaps of length at least $M + 1$. It follows that no element of one such translate can be operationally related to any element of any other translate. From this it follows that the union of all these translates is a sublattice of $\mathbf{L}$. So it is the sublattice generated by Y, as desired. $\square$

Since the fence F is fixed throughout the proof of our Theorem, we simplify the notation by letting $\mathbf{L} \bigstar \mathbf{G}$ stand for $\mathbf{L} \bigstar_F \mathbf{G}$.

Now let τ be an automorphism of $\mathbf{G}$ with an infinite orbit. Let $\zeta : L \bigstar G \to L \bigstar G$ be the map defined via

$$\zeta(x) = \begin{cases} \rho(x) & \text{if } x \in L - F, \\ \langle \rho(x'), \tau(g) \rangle & \text{if } x = \langle x', g \rangle \in F \times G. \end{cases}$$

ζ is an automorphism of $\mathbf{L} \bigstar \mathbf{G}$, as can be easily checked.

By a **partial algebra** we mean a structure $\langle A, Q, P, \ldots\rangle$ such that A is a nonempty set and $Q, P, \ldots$ are finitary partial operations on A (that is, the domains of $Q, P, \ldots$ are subsets of various finite direct powers of A while the ranges of $Q, P, \ldots$ are included in A). By a **partial lattice P** we mean a partial algebra $\langle P, \wedge, \vee\rangle$ for which there is an order $\leqslant$ on P such that if $a \wedge b$ exists then it is the greatest lower bound of a and b with respect to $\leqslant$, and dually, if $a \vee b$ exists, then it is the least upper bound of a and b

For any lattice $\mathbf{M}$, we denote by $\mathbf{M}^\flat$ the partial lattice obtained by removing the greatest and the least elements of $\mathbf{M}$, if they are present. Also, for any partial lattice $\mathbf{P}$ we form the algebra $\mathbf{P}^\sharp$ as follows. The universe of $\mathbf{P}^\sharp$ is obtained by adjoining two new elements 0 and 1 to P. The operations are defined below.

$$u \vee^{\mathbf{P}^\sharp} v = \begin{cases} u \vee^{\mathbf{P}} v & \text{if this join is defined in } \mathbf{P}, \\ v & \text{if } u = 0, \\ u & \text{if } v = 0, \\ 1 & \text{otherwise.} \end{cases}$$

$$u \wedge^{\mathbf{P}^\sharp} v = \begin{cases} u \wedge^{\mathbf{P}} v & \text{if this meet is defined in } \mathbf{P}, \\ v & \text{if } u = 1, \\ u & \text{if } v = 1, \\ 0 & \text{otherwise.} \end{cases}$$

An equivalence relation φ on P is called a **congruence** of $\mathbf{P}$ provided that

(1) $a \vee b \,\varphi\, a' \vee b'$ if $a \,\varphi\, a'$, $b \,\varphi\, b'$, and both $a \vee b$ and $a' \vee b'$ are defined, and
(2) $a \wedge b \,\varphi\, a' \wedge b'$ if $a \,\varphi\, a'$, $b \,\varphi\, b'$, and both $a \wedge b$ and $a' \wedge b'$ are defined.

This is precisely the condition needed in order to be able to carry out the construction of the quotient $\mathbf{P}/\varphi$.

Let θ be the equivalence relation of $(L \star G)^\flat$ induced by the ζ-orbits. So $x \,\theta\, y$ if and only if x and y belong to the same ζ-orbit. We will use π to denote the quotient map induced by θ.

Claim 8. *θ is a congruence of* $(\mathbf{L} \star \mathbf{G})^\flat$.

Proof of Claim 8. We show that joins behave correctly for θ; meets can be handled dually.

Let B and C be two ζ-orbits with $b, b' \in B$ and $c, c' \in C$. Suppose that both $b \vee c$ and $b' \vee c'$ are defined. (This implies $B \neq C$ by the choice of ρ.) We must show that these joins belong to the same ζ-orbit. Recall that λ is a homomorphism mapping $\mathbf{L} \star \mathbf{G}$ onto $\mathbf{L}$. Let $\mathbf{S}$ be the sublattice of $\mathbf{L}$ generated by the union of the ρ-orbit of $\lambda(b)$ and the ρ-orbit of $\lambda(c)$. Since we have taken $N \geqslant 2$, we know from Claim 7 that $\mathbf{S}$ is ρ-decomposable into operationally unrelated ρ-translates of a sublattice $\mathbf{S}_0$. Since $\lambda(b)$, $\lambda(c)$, and $\lambda(b \vee c)$ are operationally related, they must belong to a single translate of S_0. For the same reason $\lambda(b')$, $\lambda(c')$, and $\lambda(b' \vee c')$ must belong to a single translate of S_0. Hence there is an integer k so that $\rho^k(\lambda(b)) = \lambda(b')$ and $\rho^k(\lambda(c)) = \lambda(c')$. It follows that $\zeta^k(b)=b'$ and $\zeta^k(c)=c'$. Hence $b\vee c$ and $b'\vee c'$ belong to the same ζ-orbit. $\square$

Let $(\mathbf{L}\bigstar\mathbf{G})_N$ denote $((\mathbf{L}\bigstar\mathbf{G})^\flat/\theta)^\sharp$. Roughly speaking, this algebra is obtained by removing the 0 and 1 of $\mathbf{L}\bigstar\mathbf{G}$, wrapping the resulting partial lattice onto a cylinder using a helical covering via ζ, and then adding a new greatest and least element to the result. It will be a consequence of the reasoning below that this resulting algebra is indeed a lattice itself, provided $N\geqslant 3$.

Claim 9. $(\mathbf{L}\bigstar\mathbf{G})_N$ *is not locally finite.*

Proof of Claim 9. Recall that $\dots,t_{-1},t_0,t_1,\dots$ is the listing of the top elements of our full lattice fence F and that $b_i=t_i\wedge t_{i+1}$ for $i\in\mathbb{Z}$ provides a listing of the bottom elements of F. Note that these listings may have little to do with our fixed arrangement of the proper elements of L. By Claim 5, we know that $\mathbf{L}$ has finite height. So among the elements of F only finitely many heights are represented. Those elements of F with the greatest height must be top elements of F. It does no harm to assume that t_0 is such an element. Since σ preserves F, it follows that $\rho(t_0)$ is again an element of greatest height in F. Hence it is a top element. We assume without loss of generality that $\rho(t_0)=t_\ell$ where $1<\ell$. Let $s(x,y_0,z_1,\dots,y_{\ell-1},z_\ell)$ be the term $((\dots((x\wedge y_0)\vee z_1)\wedge\dots)\wedge y_{\ell-1})\vee z_\ell$. Then for any $g\in G$, in $\mathbf{L}\bigstar\mathbf{G}$ we have

$$s(\langle t_0,g\rangle,\langle b_0,1\rangle,\langle t_1,0\rangle,\dots,\langle b_{\ell-1},1\rangle,\langle t_\ell,0\rangle)=\langle t_\ell,g\rangle=\langle\rho(t_0),g\rangle.$$

Recall that π denotes the quotient map induced by θ. Thus $\pi(\rho(a),g)=\pi(a,\tau^{-1}(g))$ for all $a\in F$ and $g\in G$. Let $q(x)$ be the following unary polynomial of $(\mathbf{L}\bigstar\mathbf{G})_N$:

$$s(x,\pi(b_0,1),\pi(t_1,0),\dots,\pi(b_{\ell-1},1),\pi(t_\ell,0)).$$

So in $(\mathbf{L}\bigstar\mathbf{G})_N$ we have $q(\pi(t_0,g))=\pi(\rho(t_0),g)=\pi(t_0,\tau^{-1}(g))$ for all $g\in G$. Also note that $\pi(t_0,g)\neq\pi(t_0,\tau^{-1}(g))$. Now pick $g\in G$ belonging to an infinite τ-orbit. Then for all natural numbers k, we have that $\pi(t_0,\tau^{-k}(g))$ belongs to the sublattice generated by the set consisting of $\pi(t_0,g)$ and the 2ℓ elements that play the role of constants in $q(x)$. Consequently, $(\mathbf{L}\bigstar\mathbf{G})_N$ is not locally finite. $\square$

We extend the map π to $\pi^\sharp: L\bigstar G\to(L\bigstar G)_N$ by setting $\pi^\sharp(1)=1$ and $\pi^\sharp(0)=0$. This map may not be a homomorphism.

Claim 10. *Each N-generated subalgebra of $(\mathbf{L}\bigstar\mathbf{G})_N$ is the isomorphic image under $\pi^\sharp$ of a sublattice of $\mathbf{L}\bigstar\mathbf{G}$.*

Proof of Claim 10. Pick N elements of $(L\bigstar G)_N$. It does no harm to suppose they are all proper. Thus, we have selected N ζ-orbits of $\mathbf{L}\bigstar\mathbf{G}$. Each of these project, via λ, onto a ρ-orbit of $\mathbf{L}$. Let $\mathbf{S}$ be the sublattice of $\mathbf{L}$ generated by the union of these N ρ-orbits. By Claim 7 there is a sublattice $\mathbf{S}_0$ of $\mathbf{L}$ so that S is the union of the

ρ-translates of S_0 and the proper parts of all these ρ-translates are pairwise disjoint, with elements of one translate operationally unrelated to elements of any other translate.

Let $T=\{\langle a,g\rangle : a\in S_0\cap F$ and $g\in G\}\cup(S_0\cap(L-F))$. Evidently **T** is a sublattice of $\mathbf{L}\bigstar\mathbf{G}$. It is also clear from the choice of $\mathbf{S}_0$ that $\pi^{\#}$ is one-to-one on T. To see that $\pi^{\#}$ restricted to T is a homomorphism, we only have to consider proper elements of T which join to 1 (and, dually those that meet to 0). So suppose x and y are proper elements of T and that $x\vee y=1$. Assume, for the sake of a contradiction, that $x'\vee y'$ is proper, that x' and y' are proper elements, that x and x' belong to the same ζ-orbit, and that y and y' belong to the same ζ-orbit. It follows that $\lambda(x)$ and $\lambda(x')$ belong to the same ρ-orbit and that $\lambda(y)$ and $\lambda(y')$ belong to the same ρ-orbit. Since $\lambda(x')$ and $\lambda(y')$ are operationally related, they must belong to the same ρ-translate of S_0. So x' and y' belong to the same ζ-translate of T. This means that there is an integer k so that $\zeta^k(x)=x'$ and $\zeta^k(y)=y'$. It follows that $x'\vee y'=\zeta^k(x\vee y)$ is not proper, the contradiction we desired. Meets of proper elements of T can be handled similarly.

Therefore $\pi^{\#}$ embeds **T** into $(\mathbf{L}\bigstar\mathbf{G})_N$. The N elements we originally selected belong to $\pi^{\#}(T)$. $\square$

Since $\mathbf{L}\bigstar\mathbf{G}$ is a lattice, we see that every N-generated subalgebra of $(\mathbf{L}\bigstar\mathbf{G})_N$ is a lattice. Since lattices are defined by equations involving only three variables, we conclude that $(\mathbf{L}\bigstar\mathbf{G})_N$ is a lattice whenever $N\geqslant 3$.

Claim 11. *The N-generated sublattices of* **L** *have no more that Np elements, where p is the number of ρ-orbits* (*which also depends on N*).

Proof of Claim 11. Our N elements are contained in the union of no more than N ρ-orbits. The sublattice **S** generated by this union is ρ-decomposable by Claim 7. Thus, there is a sublattice $\mathbf{S}_0$ of cardinality no larger than p so that S is the union of the ρ-translates of S_0 and the different translates are operationally unrelated as before. Our original N elements belong to at most N of these translates. The union of these translates of S_0 is a sublattice of cardinality at most Np. $\square$

It is well known that an algebra belongs to some locally finite variety if and only if for arbitrarily large natural numbers N, there is a finite upper bound on the cardinality of its N-generated subalgebras. Thus Claim 11 implies that **L** belongs to a locally finite variety.

Claim 12. $\mathbf{L}\bigstar\mathbf{G}$ *belongs to a locally finite variety.*

Proof of Claim 12. We will prove that $Np(g_N+2)$ is an upper bound on the cardinalities of the N-generated sublattices of $\mathbf{L}\bigstar\mathbf{G}$, where g_N is an upper bound on the cardinalities of the N-generated sublattices of **G** and p is the number of ρ-orbits.

So select some N elements of $L\bigstar G$. Let Q be the set of all elements of L of the form $\lambda(x)$ where x is one of our N selected elements. Let **R** be the sublattice of **L** generated by Q. So $|R| \leqslant Np$ by Claim 11. Let S be the set of all $g \in G$ such that $\langle a, g\rangle$ is one of our N selected elements for some a. Let $\mathbf{T}$ be the sublattice of $\mathbf{G}$ generated by S with the 0 and 1 of $\mathbf{G}$ adjoined. So $|T| \leqslant g_N + 2$. Now $R \cap (L - F) \cup \{\langle a, g\rangle\colon a \in R \cap F$ and $g \in T\}$ is the universe of a sublattice of $\mathbf{L} \bigstar \mathbf{G}$. Plainly, this sublattice has cardinality no larger than $Np(g_N + 2)$ and it contains the N elements we selected. □

We have now completed all phases of the proof that $\mathbf{L} \bigstar_F \mathbf{G}$ is inherently nonfinitely based. □

Plainly, our Theorem 4 is very closely related to Theorem 1.1 in [2]. The same can be said for the proof of our theorem and the proof in [2]. The main difference is that we draw the conclusion that $\mathbf{L} \bigstar_F \mathbf{G}$ is inherently nonfinitely based, not that $\mathbf{L}$ itself is inherently nonfinitely based. $\mathbf{L} \bigstar_F \mathbf{G}$ need not be in the variety generated by $\mathbf{L}$, whereas the corresponding algebra in the proof of Theorem 1.1 in [2] is in the original variety.

The lattice $\mathbf{L}$ in Theorem 4 was required to have a full lattice fence and an automorphism satisfying certain conditions. Actually, the automorphism can be used to build a full lattice fence with the required properties.

Theorem 13. *Let* $\mathbf{L}$ *be a lattice and let* σ *be an automorphism of* $\mathbf{L}$ *such that*
(1) *the* σ*-orbit of any proper element of* L *is infinite*,
(2) $\mathbf{L}$ *has finite height*, *and*
(3) *there is a proper element* a *such that* a *and* $\sigma(a)$ *are incomparable*, *and either* $a \vee \sigma(a)$ *is proper or* $a \wedge \sigma(a)$ *is proper.*

Then $\mathbf{L}$ *has a proper full lattice fence preserved by* σ.

Proof. Without loss of generality, we will suppose that $a \wedge \sigma(a)$ is proper. Let $b_0 = a \wedge \sigma(a)$. For each $k \in \mathbb{Z}$, let $b_k = \sigma^k(b_0)$. So $\sigma^k(a) \wedge \sigma^{k+1}(a) = b_k$. Let $t_k = b_k \vee b_{k+1}$ for each $k \in \mathbb{Z}$. Now observe that $b_{k+1} = \sigma^{k+1}(a) \wedge \sigma^{k+2}(a) \geqslant t_k \wedge t_{k+1} \geqslant b_{k+1}$. Hence, $b_{k+1} = t_k \wedge t_{k+1}$ for all $k \in \mathbb{Z}$. To see that $\ldots, b_{-1}, b_0, b_1, \ldots$ is an infinite antichain, observe that it is infinite since it is the orbit of the proper element b_0. It must be an antichain, since otherwise it would contain an infinite chain (violating the finite height of $\mathbf{L}$). Likewise, each t_k is proper since $b_{k+1} = t_k \wedge t_{k+1}$, and thus $\ldots, t_{-1}, t_0, t_1, \ldots$ is an infinite antichain. Let $F = \{x\colon b_k \leqslant x \leqslant t_j$ for some $k, j \in \mathbb{Z}\}$. F is the desired fence. □

4. Examples of inherently nonfinitely based lattices

With the help of Theorem 4 it is easy to construct inherently nonfinitely based lattices. The lattices $\mathbf{M}_\omega$ and $\mathbf{B}$ in Fig. 3 are two of the least complicated lattices that can play the role of $\mathbf{G}$.

They both have greatest and least elements with the remaining proper elements constituting a single infinite orbit under some obvious automorphism. Both generate locally finite varieties.

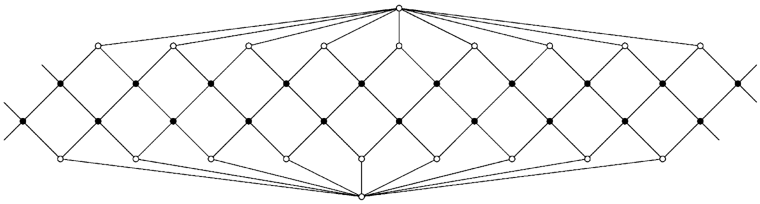

Fig. 4. The lattice **J**.

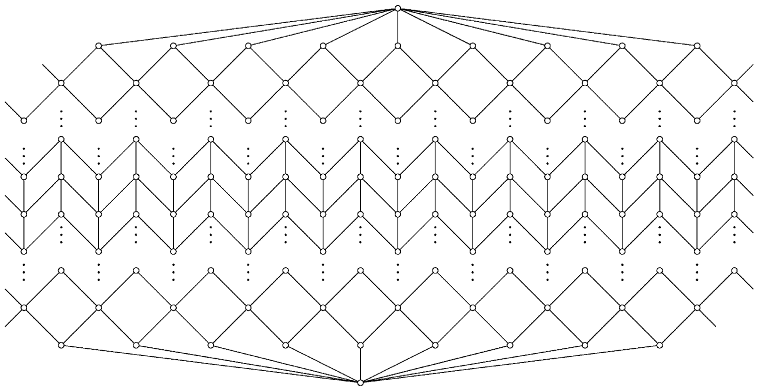

Fig. 5. The lattice $\mathbf{J} \star \mathbf{B}$.

Other lattices can play the role of **L**. The lattice **J** displayed in Fig. 4, with the fence elements indicated by •, has the required properties. The lattice $\mathbf{J} \star \mathbf{B}$, displayed in Fig. 5, is an inherently nonfinitely based lattice. The lattice $(\mathbf{J} \star \mathbf{B})_2$, displayed in Fig. 6, is the lattice Nation used in [32] to refute the finite height conjecture. In this figure points with the same labels should be identified. The whole figure suggests an infinitely tall cylinder with dually isomorphic finite caps at the top and bottom. There are 14 copies of the chain of integers arranged around the middle and wrapped with a single zigzag helix.

We can also let the lattice $\mathbf{L}_f$ of Fig. 2 play the role of **G**, since it has all the requisite properties. Thus $\mathbf{L}_f \star \mathbf{L}_f$ is an inherently nonfinitely based lattice.

5. $\mathbf{L}_f$ fails to be inherently nonfinitely based

By Theorem 14 just below, the lattice $\mathbf{L}_f$ belongs to every variety of lattices that can be shown to be inherently nonfinitely based by Theorem 4. Ralph McKenzie in [26] proved that $\mathbf{L}_f$ is not finitely based. Were we able to prove that $\mathbf{L}_f$ was *inherently* nonfinitely based, we would have a result stronger than Theorem 4. The main work of this section is to prove that $\mathbf{L}_f$ fails to be inherently nonfinitely based.

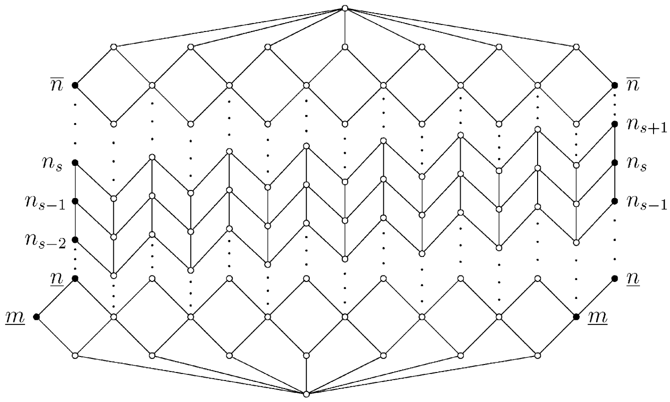

Fig. 6. The lattice $(\mathbf{J} \bigstar \mathbf{B})_2$.

Theorem 14. *Let* $\mathbf{L}$ *be a locally finite lattice, let* F *be a full lattice fence of proper elements in* $\mathbf{L}$, *and let* σ *be an automorphism of* $\mathbf{L}$ *such that*
(1) σ *partitions the collection of proper elements of* L *into only finitely may* σ*-orbits, each infinite*,
(2) σ *preserves* F (*i.e.* $\sigma(f) \in F$ *for all* $f \in F$), *and*
(3) *there is a finite uniform bound* d *such that every proper element of* L *is comparable with no more than* d *elements of* L.
Let $\mathbf{G}$ *be a lattice with a least and a greatest element. Then* $\mathbf{L}_f$ *belongs to the variety generated by* $\mathbf{L} \bigstar_F \mathbf{G}$.

Proof. According to Theorem 3, $\mathbf{L}$ is a homomorphic image of $\mathbf{L} \bigstar_F \mathbf{G}$. To complete the proof, we will argue that $\mathbf{L}_f$ is isomorphic to a subalgebra of $\mathbf{L}$.

With the help of (3), an easy induction on n shows that there is a finite $f(n)$ such that for any proper element a, there are no more than $f(n)$ elements b with a weak lattice fence from a to b of cardinality no more than n.

Now let a be an element of F of greatest height. So a is top element of F. It follows that $\sigma^k(a)$ is a top element of F for every natural number k. For each positive integer k pick a weak lattice fence G_k from a to $\sigma^k(a)$ of least possible cardinality. By the previous paragraph, $\{G_k : k \in \mathbb{Z}^+\}$ contains weak lattice fences of arbitrarily large finite cardinality.

Call a weak lattice fence W **good** provided the join of any two distinct nonadjacent bottom elements of W is 1 and the meet of any two distinct nonadjacent top elements of W is 0.

Claim 15. G_k *is a good weak lattice fence*, *for every positive integer* k.

Proof of Claim 15. To the contrary, suppose b_i and b_j are bottom elements of G_k with $j - i > 1$ such that $b_i \vee b_j = c$ is a proper element. Let $b_i' = t_{i-1} \wedge c$ and $b_j' = c \wedge t_j$, where

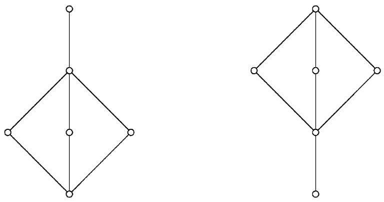

Fig. 7. $\mathbf{M}_3+\mathbf{1}$ and $\mathbf{1}+\mathbf{M}_3$.

t_{i-1} and t_j are the appropriate top elements of G_k. We obtain a new fence G_k' from a to $\sigma^k(a)$ by replacing the part of G_k extending from b_i to b_j by b_i', c, b_j'. G_k' has smaller cardinality than G_k, contradicting the choice of G_k. This argument has obvious modifications in case $\{a, \sigma^k(a)\}$ and $\{b_i, b_j\}$ are not disjoint. $\square$

Pick a representative from each σ-orbit of proper elements. Let $\mathscr{T}$ be the collection consisting of the empty set and all good weak lattice fences in $\mathbf{L}$ of odd cardinality, whose middle element is one of the selected representatives. $\mathscr{T}$ is made into a finitely branching infinite tree by declaring that the empty set is the root, the selected representatives its immediate successors, and that the good weak lattice fence W is a successor of W', where W' results from W by deleting both its left and right endpoints. The fact that $\mathscr{T}$ is infinite follows since the G_k's get arbitrarily long. By pruning off ends if necessary we arrive at arbitrarily long good weak lattice fences of odd length. By applying the correct power of σ we can translate this good weak lattice fence of odd length onto one whose middle element is one of the selected representatives.

By König's Infinity Lemma, $\mathscr{T}$ has an infinite branch. The union of this branch could be called a two-way infinite good weak lattice fence. Adjoining 1 and 0, we obtain a copy of $\mathbf{L}_f$. $\square$

To establish that $\mathbf{L}_f$ fails to be inherently nonfinitely based, we provide a finitely based locally finite variety $\mathscr{V}$ to which $\mathbf{L}_f$ belongs. Let $\mathscr{K}$ be the class of all lattices $\mathbf{L}$ of height at most 3 such that neither $\mathbf{M}_3+\mathbf{1}$ nor $\mathbf{1}+\mathbf{M}_3$ are sublattices of $\mathbf{L}$. These two lattices are displayed in Fig. 7. Note that $\mathbf{L}_f \in \mathscr{K}$. Let $\mathscr{V}$ be the variety generated by $\mathscr{K}$. We will prove that $\mathscr{V}$ is a finitely based locally finite variety.

Theorem 16. *$\mathscr{V}$ is locally finite.*

Proof. To prove that $\mathscr{V}$ is locally finite, it suffices to find a function $b(n)$ on the natural numbers such that every n-generated sublattice of $\mathbf{L}$ has no more than $b(n)$ elements for every $\mathbf{L} \in \mathscr{K}$. We argue that $b(n) = 2n+2$ will serve. In passing, we note that this choice of $b(n)$ is sharp in the sense that this bound is achieved for all $n \geqslant 3$ by a suitable choice of $\mathbf{L} \in \mathscr{K}$ and a suitable selection of n elements in L.

So let $\mathbf{L} \in \mathscr{K}$ and let X be a finite set of proper elements of L. Let

$$\begin{aligned}
&Y_0 = \{x \vee x' : x,\ x' \in X,\ x \text{ and } x' \text{ are incomparable, and } x \vee x' \neq 1\},\\
&Y_1 = \{x \wedge x' : x,\ x' \in X,\ x \text{ and } x' \text{ are incomparable, and } x \wedge x' \neq 0\},\\
&Y = Y_0 \cup Y_1, \quad \text{and}\\
&S = X \cup Y \cup \{0,1\}.
\end{aligned}$$

Claim 17. *S is closed under $\vee$ and $\wedge$.*

Proof of Claim 17. Let $a, b \in S$ with a and b incomparable proper elements. It suffices to prove that $a \vee b \in S$. In case $a \vee b = 1$ we are already finished. So consider the case that $a \vee b$ is proper. Since $\mathbf{L}$ has height at most 3, we see that a and b both cover 0, and that $a \vee b$ covers both a and b. In the event that $a, b \in X$, then $a \vee b \in Y$ and we are done. So, without loss of generality, we suppose that $a \in Y$. Since a covers 0, it follows that there are $x, x' \in X$, which are incomparable, so that $a = x \wedge x'$. Hence, x, x', and $a \vee b$ all cover a. But $\mathbf{L}$ has no sublattice isomorphic to $\mathbf{M}_3 + \mathbf{1}$, so we have $a \vee b \in \{x, x'\} \subseteq X \subseteq S$, as desired. $\square$

At this point a $O(n^2)$ bounding function is apparent. This is enough to establish the theorem. To obtain the tighter bound we impose the structure of a labeled graph on X. Given any two incomparable elements $x, x' \in X$ at most one of $x \vee x'$ and $x \wedge x'$ can be proper, since $\mathbf{L}$ has height at most 3. Draw an edge between x and x' provided either the join or the meet is proper and label that edge with whichever of the join or meet is proper. The resulting graph has vertex set X with edges labeled by the set Y. Call this graph $\mathbb{X}$. Notice that every element of Y occurs as the label of some edge of $\mathbb{X}$. Thus the number of edges in our graph is an upper bound on the cardinality of Y.

Claim 18. *Every vertex in $\mathbb{X}$ has degree at most 2.*

Proof of Claim 18. Suppose not. Pick four distinct elements $y, x_0, x_1, x_2 \in X$ so that y is adjacent to each of x_0, x_1, and x_2. We may suppose that $y \wedge x_0$ is proper. From height considerations it follows that 1 covers y. This entails that $y \wedge x_1$ and $y \wedge x_2$ must also be proper. Thus, y covers each of $y \wedge x_0$, $y \wedge x_1$, and $y \wedge x_2$. Since $\mathbf{L}$ has no sublattice isomorphic to $\mathbf{M}_3 + \mathbf{1}$, it follows that $y \wedge x_0$, $y \wedge x_1$, and $y \wedge x_2$ cannot be distinct. Suppose $y \wedge x_0 = y \wedge x_1$. But then each of the three distinct elements y, x_0, and x_1 covers $y \wedge x_0$. This forces an isomorphic copy of $\mathbf{1} + \mathbf{M}_3$ inside $\mathbf{L}$, a contradiction. $\square$

In any graph where 2 is an upper bound on the degree of any vertex, the number of edges cannot exceed the number of vertices. Hence $|Y| \leqslant |X|$. Consequently, $|S| \leqslant 2|X| + 2$ and so the sublattice generated by X can have at most $2|X| + 2$ elements, as desired. $\square$

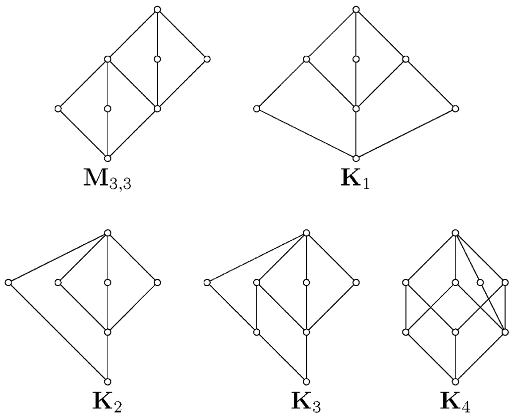

Fig. 8.

The next theorem essentially describes the varieties covering $\mathscr{V}$ within the variety generated by lattices of height at most 3 as the varieties generated by the lattices in Fig. 8 and their duals.

Theorem 19. *If* $\mathbf{L}$ *is a subdirectly irreducible lattice of height at most* 3, *then* $\mathbf{L}\in\mathscr{K}$ *if and only if none of the lattices* $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, $\mathbf{K}_4$, *or* $\mathbf{M}_{3,3}$ *nor their duals is a sublattice of* $\mathbf{L}$.

Proof. Clearly, each of the lattices listed in the theorem contains $\mathbf{M}_3+\mathbf{1}$ or $\mathbf{1}+\mathbf{M}_3$ as a sublattice, and hence is not in $\mathscr{K}$.

Conversely, assume that $\mathbf{L}$ is a subdirectly irreducible lattice of height 3 which is not in $\mathscr{K}$. Without loss of generality, we suppose that $\mathbf{L}$ has a sublattice isomorphic to $\mathbf{1}+\mathbf{M}_3$. Because $\mathbf{L}$ has height no larger than 3, this means that there is an element d of L which covers 0 such that at least 3 proper elements of $\mathbf{L}$ cover d. Let $A=\{s : s\in L$ and $d<s<1\}$. Let $|A|=\kappa$. We know that $\kappa\geqslant 3$ and that $A\cup\{d,1\}$ comprises a sublattice isomorphic to $\mathbf{M}_\kappa$.

Case I: d has no complement in $\mathbf{L}$.

Since $\mathbf{1}+\mathbf{M}_\kappa$ is not subdirectly irreducible, $\mathbf{L}$ must have elements besides those in $A\cup\{1,d,0\}$. Each such additional element must lie below exactly one element of A, for otherwise it would be a complement of d. Let g be such an additional element. It is easy to see that $A\cup\{1,d,g,0\}$ comprises a sublattice, which, however, is not subdirectly irreducible. So let h be yet another element of L. In the event that g and h lie beneath the same element of A, a copy of $\mathbf{M}_{3,3}$ is induced. In the event that g and h lie beneath distinct elements of A, a copy of $\mathbf{K}_1$ is induced. This finishes Case I.

Case II: d has a complement in $\mathbf{L}$.

Let e be a complement of d. Let $B=\{s : s\in L$ and $0<s<e\}$. If there are at least three elements of A which meet with e to 0, then a copy of $\mathbf{K}_2$ is induced. If perchance exactly two elements of A meet with e to 0, then a copy of $\mathbf{K}_3$ is induced. If only

one element of A meets with e to 0, then a copy of $\mathbf{K}_4$ is induced. We are reduced to the situation that the meet of e with any element of A is proper. Since $\mathbf{L}$ has height 3, consideration of the cover relations reveals that $A \cup B \cup \{1, e, d, 0\}$ comprises a sublattice isomorphic to $\mathbf{M}_\kappa \times \mathbf{2}$. Hence, this sublattice is not subdirectly irreducible. There must be additional elements in L. If some additional element is comparable to one of the elements in $A \cup B \cup \{e, d\}$, then it must be either below some element of A or above some element of B. In either case, a copy of $\mathbf{M}_{3,3}$ is induced. We are left with the case that each additional element is incomparable with every element of $A \cup B \cup \{e, d\}$. Let g be such an element. Let a, b, and c be three distinct elements of A. Observe that $\{1, g, a, b, c, d, 0\}$ comprises a sublattice. Consider, for example, $g \wedge a$. Being comparable with a, it must not be an additional element. Since g is also comparable to $g \wedge a$, it must be that $g \wedge a = 0$. The sublattice at hand is a copy of $\mathbf{K}_2$. This completes Case II and the proof of the theorem. □

Theorem 20. *$\mathscr{V}$ is finitely based.*

Proof. Christian Herrmann [17] has proven that the variety of lattices generated by the class of all lattices of height at most 3 is finitely based. Let Σ be a finite base for this variety. Now the class $\mathscr{K}$ is evidently closed under the formation of homomorphic images, sublattices, and ultraproducts. Thus, in view of Jónsson's Lemma, every subdirectly irreducible lattice in $\mathscr{V}$ actually belongs to $\mathscr{K}$. Therefore, the lattices displayed in Fig. 8 and their duals are subdirectly irreducible lattices of height 3 that fail to belong to $\mathscr{V}$. For each of these nine lattices pick an equation true in $\mathscr{V}$ which fails in the lattice. Let Δ denote the set of these nine equations. Then $\Sigma \cup \Delta$ is a finite set of equations true in $\mathscr{V}$. To see that it is a base for $\mathscr{V}$, we only need to argue that every subdirectly irreducible model of $\Sigma \cup \Delta$ actually belongs to $\mathscr{K}$. So suppose that $\mathbf{L}$ is a subdirectly irreducible model of $\Sigma \cup \Delta$. Since $\mathbf{L} \models \Sigma$, by Jónsson's Lemma $\mathbf{L}$ as height at most 3. Since $\mathbf{L} \models \Delta$, none of the lattices displayed in Fig. 8 nor their duals can be sublattices of $\mathbf{L}$. So it follows from Theorem 19 that $\mathbf{L} \in \mathscr{K}$, as desired. □

Corollary 21. *$\mathbf{L}_f$ fails to be inherently nonfinitely based.*

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