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Parry Syllogisms

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Abstract Parry discusses an extension of Aristotle's syllogistic that uses four *nontraditional* quantifiers. We show that his conjectured decision procedure for validity for the extended syllogistic is correct even if syllogisms have more than two premises. And we axiomatize this extension of the syllogistic.

1 Background and motivation Parry [2] discusses an extension of the syllogistic in which sentences are formed by using the quantifiers $\alpha, \iota, \eta, \omega$ in addition to the traditional quantifiers A, E, I, O. If a and b are terms and Q is a quantifier then Qab is a *sentence*. A sentence Qab is an *affirmative sentence* if Q is A, I, α, or ι; otherwise, it is a *negative sentence*. Our discussion of sentences Qab will be restricted to those in which $a \neq b$. As is customary Aab, Eab, Iab, and Oab are read as 'All a are b', 'No a are b', 'Some a are b', and 'Some a are not b', respectively. αab, ιab, ηab, and ωab may be read as 'There is exactly one a and all a are b, and there is exactly one b and all b are a', 'There is exactly one b and all b are a', 'It is not true that ιab', and 'It is not true that αab', respectively. (So ηab and ωab may be read as disjunctive sentences.)[1] Parry conjectures a decision procedure for validity for this extension of the traditional syllogistic, given that syllogisms have no more than two premises. We show that his decision procedure is correct even if syllogisms have more than two premises. And we axiomatize the *Parry syllogistic* and thus Aristotle's syllogistic as well.[2]

2 Preliminaries

Definition 2.1 $cd(Aab)$ (the contradictory of Aab) $= Oab$; $cd(Iab) = Eab$, $cd(\alpha ab) = \omega ab$; $cd(\iota ab) = \eta ab$; and $cd(cd(x)) = x$.

Definition 2.2 A pair $\langle W, v\rangle$ is a *model* if and only if W is a nonempty set and v a function that maps terms into nonempty subsets of W and maps sentences into $\{t, f\}$ where:

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(i)	$v(Aab)$	$=$	t	iff	$v(a) \subseteq v(b)$;
(ii)	$v(Iab)$	$=$	t	iff	$v(a) \cap v(b) \neq \varnothing$;
(iii)	$v(\iota ab)$	$=$	t	iff	$v(b)$ has exactly one member and $v(Aba) = t$;
(iv)	$v(\alpha ab)$	$=$	t	iff	$v(\iota ab) = t$ and $v(\iota ba) = t$; and
(v)	$v(x)$	$=$	t	iff	$v(cd(x)) = f$.

A set X of sentences is *consistent* if and only if there is a model $\langle W, v\rangle$ such that v assigns t to every member of X; otherwise X is *inconsistent*. $X \models x$ if and only if $X \cup \{cd(x)\}$ (or $X, cd(x)$ for short) is inconsistent.

Definition 2.3 Sentence x is *a superordinate of* sentence y if and only if $\langle x, y\rangle$ has one of the following forms: $\langle x, x\rangle$; $\langle \alpha ab, \alpha ba(\iota[ab], A[ab], I[ab])\rangle$ (where $Q[ab]$ is Qab or Qba); $\langle \iota ab, Aba(I[ab])\rangle$; $\langle Aab, I[ab]\rangle$; or $\langle Iab, Iba\rangle$; or $cd(y)$ is a superordinate of $cd(x)$ in virtue of one of the above forms. x is a *subordinate* of y if and only if y is a superordinate of x.

The following proofs use this fact: if x is a superordinate of y (or y is a subordinate of x) then $x \models y$ (which is short for $\{x\} \models y$).

3 Decision procedure for validity *Distribution* is defined by the following table:

	a	b
$Eab, \alpha ab$	d	d
$Aab, \eta ab$	d	
$Oab, \iota ab$		d
$Iab, \omega ab$		

So, for example, a is distributed in Eab and b is undistributed in ωab. The following proofs use this fact: if x is a superordinate of y and term a is distributed in y then a is distributed in x.

Definition 3.1 A set C of sentences is a *chain* if and only if it has form $Q_1[a_1a_2]$, $\ldots$, $Q_{n-1}[a_{n-1}a_n]$, $Q_n[a_na_1]$, where each term a_i occurs exactly twice and no term occurs twice in a sentence.

Theorem 3.2 *A chain C is inconsistent if and only if:*

(i) exactly one negative sentence occurs in C;
(ii) every term is distributed at least once in C; and
(iii) if η occurs in C so does α or ι.[3]

Proof: (Only if)

Case 1: (i) is not satisfied.

Subcase 1: No negative sentence occurs in C. Construct chain C' by replacing every affirmative quantifier in C with α. C' is consistent given model $\langle\{1\}, v\rangle$, where v assigns $\{1\}$ to every term. So C is consistent since $\alpha ab \models Qab$ if Q is affirmative.

Subcase 2: More than one negative sentence occurs in C.

Definition 3.3 A set X of sentences has form $\alpha[a_1 - a_n]$ if and only if either $X = \varnothing$ and $a_1 = a_n$ or X has form $\alpha[a_1 a_2], \ldots, \alpha[a_{n-1} a_n]$. A set X of sentences has form $\alpha a_1 - a_n$ if and only if either $X = \varnothing$ and $a_1 = a_n$ or X has form $\alpha a_1 a_2, \ldots, \alpha a_{n-1} a_n$.

Construct chain C' with form $\alpha[a_1 - a_2]$, $Ea_2 a_3, \ldots, \alpha[a_{n-1} a_n]$, $Ea_n a_1$, where each sentence in C is a subordinate of a sentence in C' and each sentence in C' is a superordinate of a sentence in C. C' is consistent given model $\langle\{1, 2, 3\}, v\rangle$, where $v(x) = \{1\}$ if x is a_2 or x is a term in a member of $\alpha[a_1 - a_2]$; $v(x) = \{2\}$ if x is a_{2n} or x is a term in a member of $\alpha[a_{2n} - a_{2n-1}]$, where n is even and $n > 2$; and $v(x) = \{3\}$ if x is any other term. So C is consistent.

So, for example, Eab, Acb, ηcd, Ide, Oef, αfg, ωgh, ιha is consistent since Eab, α, Ecd, αde, Eef, αfg, Egh, αha is consistent given model $\langle\{1, 2, 3\}, v\rangle$ such that $v(b) = \{1\}$, $v(c) = \{1\}$, $v(d) = \{2\}$, $v(e) = \{2\}$, $v(f) = \{3\}$, $v(g) = \{3\}$, $v(h) = \{2\}$, and $v(a) = \{2\}$.

Case 2: (i) is satisfied, but (ii) is not.

Subcase 1: A term a is undistributed in two affirmative sentences in chain C. Construct chain C' with form Ede, $\alpha[e - b]$, ιab, ιac, $\alpha[c - d]$, where each sentence in C is a subordinate of a sentence in C' and each sentence in C' is a superordinate of a sentence in C. C' is consistent given $\langle\{1, 2\}, v\rangle$, where $v(x) = \{1\}$ if $x = e$ or $x = b$ or x is a term in $\alpha[e - b]$, $v(x) = \{2\}$ if $x = c$ or $x = d$ or x is a term in $\alpha[c - d]$, and $v(x) = \{1, 2\}$ if x is any other term. So C is consistent.

Subcase 2: A term a is undistributed in an affirmative sentence and a negative sentence. Construct chain C' with form $\alpha[c - b]$, ιab, Oac, where each sentence in C is a subordinate of a sentence in C' and each sentence in C' is a superordinate of a sentence in C. C' is consistent given $\langle\{1, 2\}, v\rangle$, where $v(a) = \{1, 2\}$ and for every term x other than a, $v(x) = \{1\}$. So C is consistent.

Case 3: (i) and (ii) are satisfied, but (iii) is not. Then C has form ηab, $Ab - a$. So C is consistent given $\langle\{1, 2\}, v\rangle$, where for every term x, $v(x) = \{1, 2\}$.

(If) We show that every chain that satisfies conditions (i) to (iii) is "reducible" to a chain with two members that satisfies these conditions. Then we rely on the inconsistency of these two-membered chains. □

Definition 3.4 A chain x, y, X is 1-*reducible* to a set z, X if and only if $x, y \models z$.

So, for example, chain Eab, ιbc, ιac is 1-reducible to Eac, ιac.

Lemma 3.5 *If a chain C with n ($n \geq 3$) sentences satisfies conditions (i) to (iii) then it is 1-reducible to a chain with $n - 1$ sentences that satisfies conditions (i) to (iii).*

Proof: Assume the antecedent where $C = Q_1 ab$, $Q_2[bc]$, X. There are exactly four cases to consider since exactly one negative sentence occurs in C.

Case 1: $Q_1 = E$.

Subcase 1: c is distributed. Then $Q[bc]$ is a superordinate of Acb. Since Eab, $Acb \models Eac$, C is 1-reducible to Eac, $X(C')$. C' obviously satisfies conditions (i) and (iii). Condition (ii) is satisfied since any every term that occurs in C' that is distributed in C is also distributed in C'.

Subcase 2: c is undistributed. Then $Q[bc]$ is a superordinate of Ibc. $Eab, Ibc \models Oca$.

Case 2: $Q_1 = O$.

Subcase 1: c is distributed. Then $Q[bc]$ is a superordinate of Acb. $Oab, Acb \models Oac$.

Subcase 2: c is undistributed. Then $Q[bc]$ is $I[bc]$, Abc, or ιcb. Then there must be some affirmative sentence $Q_2[dc]$ in which c is distributed. Then $Q_2[dc]$ is a superordinate of Acd. Suppose d is distributed. $I[bc], \alpha[cd] \models \iota bd$. Suppose d is undistributed. $I[bc], Acd \models Ibd$.

Case 3: $Q_1 = \eta$.

Subcase 1: c is distributed. Then $Q[bc]$ is $\alpha[bc]$. $\eta ab, \alpha[bc] \models Eac$.

Subcase 2: c is undistributed. Then $Q[bc]$ is Abc or ιcb. Suppose the former. $\eta ab, Abc \models \eta ac$. Suppose the latter. $\eta ab, \iota cb \models Oca$.

Case 4: $Q_1 = \omega$.

Subcase 1: c is distributed. Then $Q[bc]$ is $\alpha[bc]$. $\omega ab, \alpha bc \models Oac$.

Subcase 2: c is undistributed. Then $Q[bc]$ is a superordinate of Abc. $\omega ab, Abc \models \omega ac$. □

Lemma 3.6 *Suppose $C_1, \ldots, C_n$ is a sequence of chains such that C_1 satisfies conditions (i) to (iii) and C_j is 1-reducible to C_{j+1}. Then C_n satisfies conditions (i) to (iii) and C_1 is inconsistent if C_n is inconsistent.*

Proof: Use induction, relying on the preceding lemma and this fact: if X, y is inconsistent and $Y \models y$ then X, Y is inconsistent. □

Lemma 3.7 *Every two-membered chain that satisfies conditions (i) to (iii) is inconsistent.*

Proof: The only two-membered chains that satisfy conditions (i) to (iii) are: Eab, Iab; Oab, Aab; $\eta ab, \iota ab$; $\omega ab, \alpha ab$; and their superordinates, where 'chains that are superordinates of chains' is defined in the natural way. It is easily shown that these chains are inconsistent. □

Corollary 3.8 (Smiley [3]) *A chain in which no quantifiers other than A, E, I, or O occur is inconsistent if and only if it has one of the following forms: (i) $Oab, Aa-b$, (ii) $Eab, Ac-a, Ac-b$, or (iii) $Eab, I[cd], Ac-a, Ad-b$.*

Definition 3.9 Suppose X is a set of sentences and x is a sentence. $\langle X, x\rangle$ is a *syllogism* if and only if $X, cd(x)$ is a chain. $X \models y$ ('X, so x' is *valid*) if and only if $X, cd(x)$ is inconsistent.

Theorem 3.10 *Theorem 3.2 provides a decision procedure for determining whether any syllogism is valid.*

Proof: Note that $X \models x$ if and only if $X, cd(x)$ is inconsistent. □

Theorem 3.11 *A syllogism is valid if and only if there is no countermodel with a three-membered domain.*[4]

Proof: Given the above proof of Theorem 3.2 every consistent chain can be shown to be consistent by using a three-membered model. □

Definition 3.12

(i) Dilution		If $X \vdash x$ then $X, Y \vdash x$.
(ii) Cut		If $X \vdash x$ and $Y, x \vdash y$ then $X, Y \vdash y$.
(iii) Antilogism		$X, x \vdash y$ then $X, cd(y) \vdash cd(x)$.
(iv) Reductio		If $X, x \vdash y$ and $X, x \vdash cd(y)$ then $X \vdash cd(x)$.
(v) Superordination		$x \vdash y$ if x is a superordinate of y.
(vi) Basic syllogisms		$Aab, Abc \vdash Aac$; $Iab, Abc \vdash Iac$; $Iab, \alpha bc \vdash \iota ac$; $Aab, \alpha bc \vdash \alpha ac$; and $Aab, \iota cb \vdash \iota ca$.
(vii)		$X \vdash y$ iff $X \vdash y$ in virtue of (i) to (vi).

Theorem 3.13 *If X, x is a syllogism then $X \models x$ if and only if $X \vdash x$.*

Proof: (If) Straightforward. (Only if) Assume the antecedent. Given the proof of Theorem 3.2 there is a sequence of chains $C_1(X, cd(x)), \ldots, C_n(\{y, z\})$ such that $\{y, z\}$ satisfies conditions (i) to (iii) of Theorem 3.2 and C_j is 1-reducible to C_{j+1}. $C_n \vdash y$ and $C_n \vdash cd(y)$ (by Superordination and Dilution); $C_j \vdash C_{j+1}$ (by Basic syllogisms, Superordination, Antilogism, and Cut); and $C_1 \vdash y$ and $C_1 \vdash cd(y)$ (by repeated uses of Cut). So $X \vdash x$ (by Reductio). □

We illustrate the algorithm for showing that $X \vdash x$ given $\langle X, x\rangle$ is a valid syllogism by considering the Pseudo-Scotus valid syllogism mentioned in note 1: $\iota\iota I - 2$. $Eab \vdash Eab$ (by Reflexivity). $\iota ab \vdash cd(Eab)$ (by Superordination). So $Eab, \iota ab \vdash Eab$ and $Eab, \iota ab \vdash cd(Eab)$ (by Dilution). $Eac, \iota cb \vdash Eab$ (by Antilogism) since $Iab, \iota cb \vdash Iac$ (by Cut) since $\iota cb \vdash Abc$ (by Superordination) and $Iab, Abc \vdash Iac$ (by Basic syllogisms). So $Eac, \iota cb, \iota ab \vdash Eab$ and $Eac, \iota cb, \iota ab \vdash cd(Eab)$ (by Cut). So $\iota cb, \iota ab \vdash Iac$ (by Reductio).

Departing from the algorithm, $\iota cb, \iota ab \vdash Iac$ (by Cut) since $\iota cb \vdash Abc$ and $\iota ab \vdash Iab$ (by Superordination) and $Abc, Iab \vdash Iac$ (by Basic syllogisms).

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NOTES

1. See Parry [2] for alternative ways of reading the nontraditional sentences and for a useful discussion of the history of them. He reads ιab as 'Some b is every a.' And he claims that the earliest known example of a syllogism with such nontraditional sentences is due to Pseudo-Scotus: "Something that moves in a circle is every moon. Something shining is every moon. So something shining moves in a circle." An alternative formulation of the syllogism is: 'There is exactly one moon and all moons are things that move in a circle. There is exactly one moon and all moons are things that shine. So some things that shine are things that move in a circle.' So this nontraditional syllogism has mood and figure $\iota\iota I - 2$.

2. Our axiomatization of Aristotle's syllogistic is similar to Smiley [3].

3. Parry [2] conjectures that this theorem holds for chains with three members given the following fourth condition is added: if ω occurs then α or ι occurs. Given our theorem the fourth condition is superfluous.

4. Johnson [1] proves the special case of this theorem that involves only the Aristotelian quantifiers.

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