

A New Approach to Spinors and Some Representations of the Lorentz Group on Them

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We give a geometric realization of space-time spinors and associated representations, using the Jordan triple structure associated with the Cartan factors of type 4, the so-called spin factors. We construct certain representations of the Lorentz group, which at the same time realize bosonic spin-1 and fermionic spin- $\frac{1}{2}$ wave equations of relativistic field theory, showing some unexpected relations between various low-dimensional Lorentz representations. We include a geometrically and physically motivated introduction to Jordan triples and spin factors.

1. INTRODUCTION

A determination of all unitary representations of the inhomogeneous Lorentz group is equivalent to a determination of all possible relativistic wave equations.⁽¹⁾ Moreover, the formalism of relativistic quantum theory is based on a unitary (possibly reducible) representation of the restricted Poincaré group. The representation provides the quantum mechanical Hilbert space, and the infinitesimal generators of its Lie algebra are interpreted as the basic physical observables (energy, momentum, angular momentum).⁽²⁾ The irreducible representations of the Lorentz group were first determined by E. Wigner in the 1930s.

Geometric characterizations, with quantum mechanical significance, of the state spaces of Jordan algebras⁽³⁾ have been known for some time. These characterizations are expressed axiomatically in the category of ordered Banach spaces. However, a geometric model for quantum mechanics

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based on the premise that states are just the unit vectors of a normed space resulted in the characterization of the state spaces of atomic JB^* -triples.⁽⁴⁾

An important example which corresponds to “two state” physical systems gives rise to the JB^* -triple called a Cartan factor of type 4, or a spin factor. JB^* -triples occur in the study of bounded symmetric domains in finite and infinite dimensions. In recent years, bounded symmetric domains and JB^* -triples have appeared naturally in physics, for example in transmission line theory and special relativity.^(5,6) In this paper we show how the JB^* -triple-structure occurs in the representation theory of the Lorentz group.

More precisely, we propose a triple product representation of the canonical anticommutation relations which does not make use of the associative Clifford algebra. Imposition of these commutation relations on the natural basis of C^n defines a triple product making C^n into a Cartan factor of type 4 that we denote by \mathcal{S}^n . This Jordan structure is used to represent the Lorentz group on \mathcal{S}^3 and \mathcal{S}^4 . The irreducible representation on \mathcal{S}^3 corresponds to the relativistic transformations of the electro-magnetic field. The irreducible spin-1 representation on \mathcal{S}^4 extends the Lorentz space-time transformation. By taking the self-adjoint part of this representation with respect to the spin conjugation, a reducible spin- $\frac{1}{2}$ representation on \mathcal{S}^4 results. The latter is shown to induce two spin-1 representations in the space of determinant preserving maps on \mathcal{S}^4 showing that the same spin factor could be used to represent the two types of elementary particles: bosons and fermions.

This paper is organized as follows. Except for a review of some physical concepts in Subsec. 2.1, Secs. 2–4 are concerned with JB^* -triples. The definition of JB^* -triple and background on the connection with bounded symmetric domains is given in Sec. 2.2. In Sec. 3 we introduce a triple product structure on C^n and show that it satisfies the algebraic part of the definition of JB^* -triple. Then we construct two linear Lie groups, as well as their Lie algebras, which are derived from this Jordan structure. These spaces are used in the constructions of the representations of the Lorentz group in Sec. 5. We shall see that although these finite dimensional representations cannot be unitary, they do have an algebraic invariant, namely the *determinant*. In Sec. 4, using the determinant, a simple characterization of the tripotents (the basic building blocks of the triple product) is given. Also, a crucial decomposition, interesting in its own right, which we call the *facial decomposition* (Theorem 1), of any element into a linear combination of two maximal tripotents, is used to define the norm which together with the triple product makes C^n into a JB^* -triple (Theorem 2) denoted \mathcal{S}^n . We also use the facial decomposition to obtain an analog of the polar decomposition for an arbitrary element (Theorem 3).

Section 5 contains the construction of the representations corresponding to the relativistic transformations of the electromagnetic field and to the Lorentz space-time transformation. The latter is a spin-1 representation. The Jordan theoretic relation between the spin-1 and spin- $\frac{1}{2}$ representations is described in Theorem 4.

2. PRELIMINARIES

2.1. Lorentz Group in Relativity and in Electromagnetic Fields

The Lorentz group and its representations pervade any advanced textbook of physics. Most often the Lorentz group acts on four-vectors, which form a four dimensional real space. Such vectors may represent space-time, energy-momentum, scalar and vector potentials of the electromagnetic field and other physical quantities. The Lorentz group also introduces nonlinear maps of the ball in \mathbf{R}^3 of all possible velocities, and acts on the intensities of the electromagnetic field, which are a pair of three dimensional vectors. In Relativistic Quantum Mechanics, this group is represented on Dirac bispinors which are complex four tuples and are use to represent the state of an electron. We will now recall some of these facts.⁽⁷⁾

The *space-time* is a four dimensional real Euclidean space with one coordinate representing time and three coordinates representing position. A *reference frame* is an orthonormal basis in the spacetime. A reference system is called *inertial* if a moving body which is not acted on by external forces moves with constant velocity, that is, is *freely moving*. The *principle of relativity* states that all laws of nature are identical in all inertial frames of reference, that is, the equations expressing these laws are invariant with respect to appropriate transformations of coordinates and time from one system to another. The principle of relativity of Einstein leads to relativistic mechanics, in which the interval between two events is identical for all inertial frames.

The *interval* between two events (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is defined as

$$s = [c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2]^{1/2}$$

If we set

$$x_0 = ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$

then the interval becomes

$$s = [\Delta x_0^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2]^{1/2}$$

The group generated by all possible transformations from one inertial frame to another is called the Poincaré group. This group contains the subgroups of spatial and time translations and the Lorentz group. The Lorentz group consists of all space-time transformations preserving the interval and the origin of the reference frame.

The Lorentz group is a six dimensional Lie group. It contains as a subgroup the group of *spatial rotations*, which is isomorphic to the group $SO(3)$ consisting of all 3 by 3 orthogonal real matrices with determinant one. The rotation group is a three dimensional Lie group. The remaining three dimensions of the Lorentz group come from the *boosts* representing transformations from one inertial system to another that is moving in parallel to the first one with constant velocity V .

The Lie group of rotations consists of spatial rotations about some axis in space by an angle φ . For example the rotation matrix R_1 about the x -axis by an angle t in coordinates x_0, x_1, x_2, x_3 is,

$$R_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{bmatrix} \quad (1)$$

The infinitesimal generator of $R_1(t)$, denoted by J_1 , represents the angular momentum and in the above coordinates is given by

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (2)$$

Similarly, the infinitesimal generators of rotations about the y -axis and z -axis are

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

The Lie algebra of the Lie group $SO(3)$, is linearly spanned by these three matrices, which satisfy the commutation relations

$$[J_1, J_2] = -J_3, \quad [J_2, J_3] = -J_1, \quad [J_3, J_1] = -J_2 \quad (4)$$

To determine the Lorentz transformations from one inertial reference system to another obtained by boosts, consider the boost in the x-direction. It is easy to show [7, p. 10] that the linear transformation which preserves the intervals for this case leads to

$$x_0 = \frac{x'_0 + \frac{V}{c} x'_1}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad x_1 = \frac{x'_1 + \frac{V}{c} x'_0}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad x_2 = x'_2, \quad x_3 = x'_3$$

where V is the velocity of the second reference frame in direction of the x-axis with respect to the first. Let K_1 be the infinitesimal generator corresponding to this boost. Then the matrix of K_1 in the above mentioned coordinates is

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, the infinitesimal generators K_2, K_3 of the Lorentz boosts in the y and z directions are given by

$$K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Direct calculation show that the remaining commutation relations of the Lie algebra (the real span of J_k, K_k with $k = 1, 2, 3$) of the Lorentz group are

$$[J_1, K_1] = 0, \quad [J_1, K_2] = -K_3, \quad [J_1, K_3] = K_2 \quad (5)$$

$$[J_2, K_1] = K_3, \quad [J_2, K_2] = 0, \quad [J_2, K_3] = -K_1 \quad (6)$$

$$[J_3, K_1] = -K_2, \quad [J_3, K_2] = K_1, \quad [J_3, K_3] = 0 \quad (7)$$

$$[K_1, K_2] = J_3, \quad [K_2, K_3] = J_1, \quad [K_3, K_1] = J_2 \quad (8)$$

The above action of the Lorentz group is called the *realization on the four-vectors*. Transformation from one inertial system to another induces also a nonlinear transformation on the ball of possible velocities. These transformations are affine and preserve segments. We will discuss them in a future paper.

The equations of motion of a charge in a given electromagnetic field involve the electric field intensity \mathbf{E} and the magnetic field intensity \mathbf{H} . The transformation of the electric field intensity from one inertial system to another one moving with velocity V in the x -direction is given [7, p. 62] by

$$E_x = E'_x, \quad E_y = \frac{E'_y + \frac{V}{c} H'_z}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad E_z = \frac{E'_z - \frac{V}{c} H'_y}{\sqrt{1 - \frac{V^2}{c^2}}}$$

and similar equations for the magnetic field, where \mathbf{E} has coordinates E_x, E_y, E_z .

If we let \mathbf{F} denote $\mathbf{E} + i\mathbf{H}$, it is a fact that the complex valued quantity

$$\mathbf{F}^2 = F_x^2 + F_y^2 + F_z^2 = \|\mathbf{E}\|^2 - \|\mathbf{H}\|^2 + 2i\mathbf{E} \cdot \mathbf{H} \quad (9)$$

is invariant under the transition from one inertial frame to another. The above transformations of electric and magnetic fields take the form

$$F_x = F'_x, \quad F_y = F'_y \cosh \psi - iF'_z \sinh \psi, \quad F_z = iF'_y \sinh \psi + F'_z \cosh \psi$$

and are thus described by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \psi & i \sinh \psi \\ 0 & -i \sinh \psi & \cosh \psi \end{bmatrix} \quad (10)$$

where $\tanh \psi = V/c$. Thus, the natural model for the electromagnetic field intensity is the vector \mathbf{F} in \mathbf{C}^3 and the Lorentz group is represented by the linear maps preserving \mathbf{F}^2 .

2.2. JB^* -Triples

A *symmetric domain* is an open connected subset D of \mathbf{C}^n with the following property: for each point of D , there is an involutive holomorphic automorphism of D which has that point as an isolated fixed point. This is the same as the domain having the property at a single point, and being

homogeneous (the group of all holomorphic automorphisms acts transitively). For example, in the complex plane, the unit disk, and hence by the Riemann mapping theorem, every simply connected proper planar domain is a symmetric domain.

For a bounded symmetric domain D in \mathbb{C}^n , the set $\mathcal{G} = \text{aut } D$ of complete holomorphic vector fields on D is a real Lie algebra making the group $\mathcal{G} = \text{Aut}(D)$ of all holomorphic automorphisms of D into a real Lie group. From this the irreducible bounded symmetric domains in \mathbb{C}^n were classified. These results were accomplished in the 1930s.

In some unpublished lecture notes of 1969, M. Koecher established a correspondence between bounded symmetric domains in \mathbb{C}^n and finite dimensional JB^* -triples (defined below); for a thorough discussion of this see the lecture notes of Loos.⁽⁸⁾ A link was provided between infinite dimensional holomorphy and functional analysis by Harris:⁽⁹⁾ the open unit ball of a C^* -algebra is a bounded symmetric domain, a transitive family of automorphisms being provided by the Möbius transformations

$$z \mapsto (I - aa^*)^{-1/2} (z + a)(I + a^*z)^{-1} (I - a^*a)^{1/2}$$

Since this formula involves only the symmetrized triple product

$$\{abc\} = \frac{ab^*c + cb^*a}{2} \quad (11)$$

the result holds for norm closed subspaces of C^* -algebras which are stable for the map $(a, b, c) \mapsto \{abc\}$, equivalently $a \mapsto aa^*a$. Such spaces were called J^* -algebras by Harris but are now called JC^* -triples.

There are six types of Cartan factors. The types 1, 2, and 3 are very easy to describe. In finite dimensions, they consist respectively of the rectangular m by n complex matrices, the k by k anti-symmetric complex matrices, and the l by l symmetric complex matrices. Here, $m \geq 1$, $n \geq 1$, $k \geq 3$, $l \geq 2$. The types 5 and 6 are exceptional of dimensions 16 and 27 respectively, and we will not be concerned with them. The Cartan factor of type 4, sometimes called a *spin factor* will be a key ingredient of the rest of this paper. Cartan factors of types 1–4 are JC^* -triples under the triple product (11), and the above descriptions of them are valid in infinite dimensions.

We introduce now some facts about spin factors⁽¹⁰⁾ which will help us to motivate our paper. An even dimensional spin factor is a linear space with a basis $\mathcal{G} = \{v_j, \tilde{v}_j : j \in J\}$ and a (conjugate linear) conjugation $\#$ defined by $v_i^\# = \tilde{v}_i$ (and $\tilde{v}_i^\# = v_i$). Let u_i be either v_i or \tilde{v}_i . Define the triple product on the space by extending it linearly (conjugate linear in one of the

variables) from the basis elements as follows. The only non-zero triple products of the basis elements for any $i, j \in J$ with $i \neq j$ are the following

- (i) $\{u_i u_i u_i\} = u_i$ meaning that u_i is a *tripotent*;
- (ii) $\{u_i u_i u_j\} = \frac{1}{2} u_j$ meaning that u_i and u_j are *co-orthogonal* (*colinear* in the literature)
- (iii) $\{u_i u_j u_i^\# \} = -\frac{1}{2} u_j^\#$ meaning that $u_i, u_j, u_i^\#, u_j^\#$ form an *odd quadrangle*.

Note that from the above definition, it follows that $\{u_i u_i^\# u_k\} = 0$ for any u_k , meaning that the pairs v_i, \tilde{v}_i are *orthogonal*. Also, each u_i is a *minimal tripotent*, meaning that $\{u_i u_j u_i\} = 0$ for all $j \neq i$. We shall also make use of *maximal tripotents*, which are defined by $\{w w w\} = w$ and $\{w w a\} = a$ for every $a \in \mathcal{G}$. For example, for each $a \in \mathcal{G}$, $a + a^\#$ is a maximal tripotent. A basis of the above type for a spin factor is called a *spin grid*.

We give two examples of spin grids. The second will be important in Subsec. 5.4. The triple product used is given by (11).

Example 2.1. [2 by 2 matrices]

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{v}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \tilde{v}_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

Example 2.2. [4 by 4 anti-symmetric matrices]

$$v_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad v_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{v}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}; \quad \tilde{v}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad \tilde{v}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We note that in both Examples 2.1 and 2.2, there is an unexpected minus sign. This is due to the fact that in order for all the quadrangles $(v_i, v_j, \tilde{v}_i, \tilde{v}_j)$ to “glue” together correctly, as noted above, they must be *odd* quadrangles. Note that in Examples 2.1 and 2.2, the conjugation $\#$ is different from the usual adjoint for matrices, behaving more like the Hodge operator for differential forms.

A spin grid not only determines the triple product, but it can also be used to define an inner product by declaring the spin grid to be an orthonormal basis, that is, $\langle a | b \rangle = \sum a_j \bar{b}_j + \sum \tilde{a}_j \tilde{\bar{b}}_j$ if $a = \sum a_j v_j + \sum \tilde{a}_j \tilde{v}_j$ and $b = \sum b_j v_j + \sum \tilde{b}_j \tilde{v}_j$. Then the triple product, inner product, and conjugation are related by

$$2\{abc\} = \langle a | b \rangle c + \langle c | b \rangle a - \langle a | c^\# \rangle b^\# \tag{12}$$

A useful tool for studying spin factors is the so-called *determinant*, which is defined by

$$\det a = \frac{1}{2} \langle a | a^\# \rangle \tag{13}$$

For example, the spectral theorem says that if $\det a \neq 0$, then $a = s_1 e + s_2 f$, where e and f are orthogonal minimal tripotents, $s_1 \geq s_2 > 0$ and $\|a\| = \max\{s_1, s_2\} = s_1$. Moreover, for the dual norm, we have

$$\sup_{\|b\| \leq 1} |\langle a | b \rangle| = s_1 + s_2 \tag{14}$$

The definition of JB^* -triple will now be given. A complex Banach space U is said to be a JB^* -triple if it is equipped with a continuous triple product $(a, b, c) \mapsto \{abc\}$ mapping $U \times U \times U$ to U such that

- (i) $\{abc\}$ is linear in a and c and conjugate linear in b ;
- (ii) $\{abc\}$ is symmetric in the outer variables: $\{abc\} = \{cba\}$;
- (iii) for any $x \in U$, the operator $\Delta(x)$ from U to U defined by $\Delta(x) y = \{xxy\}$ is hermitian (that is, $\exp it\Delta(x)$ is an isometry for all real t) with non-negative spectrum;
- (iv) the triple product satisfies the following identity,

$$\Delta(x)\{abc\} = \{\Delta(x) a, b, c\} - \{a, \Delta(x) b, c\} + \{a, b, \Delta(x) c\} \tag{15}$$

- (v) $\|\{xxx\}\| = \|x\|^3$.

The identity (15) can be linearized to

$$\{x, y, \{abc\}\} = \{\{xya\}, b, c\} - \{a, \{yxb\}, c\} + \{a, b, \{xyc\}\}$$

The building blocks of the algebraic structure of a Jordan triple system are the tripotents (that is, elements e with $\{eee\} = e$) and their corresponding Peirce decompositions. There are important relations between pairs, triples, and quadruples of tripotents (orthogonal, colinear, governing, trangle,

quadrangle). These relations are fundamental tools in the classification⁽¹⁰⁾ of atomic JB^* -triples as well as in their abstract geometric counterparts, the facially symmetric spaces.⁽⁴⁾

3. TRIPLE PRODUCT REPRESENTATION OF CAR

The classical definition of the canonical anticommutation relations (CAR) involves a sequence p_k of elements of an associative algebra which satisfy the relations

$$p_l p_k + p_k p_l = 2\delta_{kl}$$

This trivially implies that

$$p_l p_k p_l = -p_k \quad \text{for } k \neq l; \quad \text{and} \quad p_k p_k p_l = p_l \quad \text{for any } k, l$$

The last relations we will call “the triple canonical anticommutation relations” (TCAR for short).

We note now that the TCAR occur naturally in a Cartan factor of type 4. Let $\mathcal{G} = \{v_j, \tilde{v}_j : j \in J\}$ be a spin grid (see Sec. 2.2) for a spin factor of dimension n which we assume finite. Define a basis of maximal tripotents w_1, \dots, w_n by

$$w_{2j} = v_j + \tilde{v}_j, \quad w_{2j+1} = i(v_j - \tilde{v}_j) \quad (16)$$

Then a direct calculation shows that $\langle w_p | w_q \rangle = 2\delta_{pq}$, $w_p^\# = w_p$ and

$$\{w_l w_k w_k\} = \{w_k w_k w_l\} = w_l; \quad \{w_l w_k w_l\} = -w_k + 2\delta_{kl} w_l$$

for any $1 \leq k, l \leq n$, and

$$\{w_k w_l w_m\} = 0 \quad \text{if } k, l, m \text{ are distinct}$$

This motivates the definitions in the next subsection.

3.1. A Triple Product Structure on \mathbf{C}^n

Now let \mathbf{C}^n be the n (finite or infinite) dimensional complex Euclidean space with the usual inner product $\langle \cdot | \cdot \rangle$, Euclidean norm $\|\cdot\|_2$, and natural basis

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$$

We define a triple product on \mathbf{C}^n by use of the TCAR for the elements of the natural basis and extend it to the whole space to be linear and symmetric in the outer variables and conjugate linear in the middle variable. Namely, define

$$\{e_l e_k e_k\} = e_l; \quad \{e_k e_k e_l\} = e_l; \quad \{e_l e_k e_l\} = -e_k + 2\delta_{kl} e_l \quad (17)$$

for any $1 \leq k, l \leq n$ and

$$\{e_k e_l e_m\} = 0 \quad \text{if } k, l, m \text{ are distinct} \quad (18)$$

Then for arbitrary $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, $c = (c_1, \dots, c_n) \in \mathbf{C}^n$ define

$$\{abc\} = \sum_{j,k,l} a_j \bar{b}_k c_l \{e_j e_k e_l\}$$

It is trivial to check that

$$\{abc\} = \langle a | b \rangle c + \langle c | b \rangle a - \langle a | \bar{c} \rangle \bar{b} \quad (19)$$

where $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$. This should be compared with (12).

We now define a *TCAR basis* to be any linearly independent set u_1, \dots, u_n in \mathbf{C}^n which satisfies (17) and (18), where the triple product is defined by (19). It follows that such a basis is an orthonormal basis in \mathbf{C}^n . Of course, an example of such a basis is the natural basis.

The straightforward proof of the following lemma will be omitted.

Lemma 3.1. An orthonormal basis u_1, \dots, u_n is a TCAR basis if and only if $u_j = \langle u_k | \bar{u}_k \rangle \bar{u}_j$ for all k, j . In particular, $\langle u_j | \bar{u}_k \rangle = 0$ for all $k \neq j$. Moreover, there is a $\lambda \in \mathbf{C}$, $|\lambda| = 1$ such that $\langle u_k | \bar{u}_k \rangle = \lambda$ and $u_k = \lambda \bar{u}_k$.

Corollary 3.2. An orthonormal set which satisfies the TCAR properties can be extended to a TCAR basis.

Proof. Let u_j , $1 \leq j \leq m$ satisfy TCAR. There is a λ with $|\lambda| = 1$ such that for all j , $u_j = \lambda \bar{u}_j$. Thus $u_j = \mu x_j$ where $\mu^2 = \lambda$ and x_1, \dots, x_m , is an orthonormal set of vectors with real coordinates. Now complete x_1, \dots, x_m to an orthonormal basis x_1, \dots, x_n of \mathbf{R}^n and set $u_j = \mu x_j$ for $m+1 \leq j \leq n$. \square

The space \mathbf{C}^n with the triple product defined by (19) will be denoted by \mathcal{S}^n . For any $x, y \in \mathcal{S}^n$ define a linear map $D(x, y)$ by

$$D(x, y) z = \{xyz\} \quad \text{for any } z \in \mathcal{S}^n$$

By straightforward calculation, using (19),

$$D(x, y)\{abc\} = \{D(x, y) a, b, c\} - \{a, D(y, x) b, c\} + \{a, b, D(x, y) c\} \quad (20)$$

which shows that \mathcal{S}^n satisfies the “algebraic part” of the definition of JB^* -triple, that is (i), (ii), and (iv). The analytic part, that is, conditions (iii) and (v) will be verified in Sec. 4 after constructing a norm different from the Hilbert space norm used so far.

3.2. The Linear Automorphism Group $\text{Taut}(\mathcal{S}^n)$ and Its Lie Algebra

The following proposition will be used to define a Lie group of linear transformations which will be instrumental in the construction of the representations in Sec. 4.

Proposition 3.3. Let T be a linear map of \mathcal{S}^n into \mathcal{S}^n . The following are equivalent:

(a) T is invertible and satisfies

$$T\{abc\} = \{TaTbTc\} \quad (21)$$

(b) If $\{u_1, \dots, u_n\}$ is any TCAR basis, then so is $\{Tu_1, \dots, Tu_n\}$.

(c) The matrix of T in the natural basis has the form λU where $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and U is a real orthogonal matrix.

Proof.

(a) \Rightarrow (b) Let $\{u_j\}$ be a TCAR basis and let $w_j = Tu_j$. Since T preserves the triple product, the w_j satisfy (17)–(18). We have already remarked that this implies that $\langle w_j | w_k \rangle = \delta_{jk}$.

(b) \Rightarrow (c) Since T takes an orthonormal basis to an orthonormal basis, it preserves the inner product. Since $\{Te_j\}$ is a TCAR basis, by Lemma 3.1 there is a $\mu \in \mathbb{C}$, $|\mu| = 1$ such that $\overline{Te_k} = \mu Te_k$. Define $\lambda = \bar{\mu}^{1/2}$ and $U = \bar{\lambda}T$ or $T = \lambda U$. Then

$$\overline{Ue_k} = \overline{\bar{\lambda}Te_k} = \lambda \overline{Te_k} = \lambda \mu T \bar{e}_k = \bar{\lambda}T(e_k) = Ue_k$$

This implies that the matrix of U in the natural basis is a real unitary matrix and thus orthogonal.

(c) \Rightarrow (a) Since the matrix for U is real, $\overline{Ua} = U\bar{a}$, so (21) follows. \square

A linear map T on \mathcal{S}^n satisfying one of the equivalent conditions of the above Proposition will be called a triple automorphism of \mathcal{S}^n . We denote by $\text{Taut}(\mathcal{S}^n)$ the Lie group of all such triple automorphisms. From Proposition 3.3 it follows that this group is the product of $O(n)$, the orthogonal group of dimension n , and the group $U(1)$ of rotations in the complex plane. Thus, the real dimension of $\text{Taut}(\mathcal{S}^n)$ is $n(n-1)/2+1$.

To describe the Lie algebra $\text{taut}(\mathcal{S}^n)$ of the group $\text{Taut}(\mathcal{S}^n)$ of triple automorphisms, note that the infinitesimal generators of such automorphisms are triple derivations, that is, linear maps δ on \mathcal{S}^n that satisfy

$$\delta\{abc\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}$$

For any u_k, u_l of a TCAR basis with $k \neq l$ we have $D(u_k, u_l) = -D(u_l, u_k)$ and thus from (20)

$$\begin{aligned} D(u_k, u_l)\{a, b, c\} \\ = \{D(u_k, u_l) a, b, c\} + \{a, D(u_k, u_l) b, c\} + \{a, b, D(u_k, u_l) c\} \end{aligned}$$

implying that $D(u_k, u_l) \in \text{taut}(\mathcal{S}^n)$. From (17) and (18) it follows that the matrix of $D(u_k, u_l)$ with respect to the basis u_1, \dots, u_n is a basic real anti-symmetric matrix. Since a real linear combination of triple derivations is also a triple derivation, the operator δ defined on \mathcal{S}^n by

$$\delta = \sum_{k < l} d_{kl} D(u_k, u_l) \quad \text{with } d_{kl} \in \mathbf{R}$$

is represented by an $n \times n$ real antisymmetric matrix and $\delta \in \text{taut}(\mathcal{S}^n)$.

Note that by properties of the triple product, it follows that the operator diI is also a triple derivation for any real d . Counting dimensions we arrive at the following proposition.

Proposition 3.4. The Lie algebra $\text{taut}(\mathcal{S}^n)$ is the direct sum of the algebra of real antisymmetric $n \times n$ matrices and $\mathbf{R}i$, that is,

$$\text{taut}(\mathcal{S}^n) = \left\{ \delta = diI + \sum_{k < l} d_{kl} D(u_k, u_l) : d, d_{kl} \in \mathbf{R} \right\}$$

This description of $\text{taut}(\mathcal{S}^n)$ is clearly independent of the choice of TCAR basis, up to isomorphism of Lie algebras.

3.3. The Determinant Preserving Group $\text{Dinv}(\mathcal{S}^n)$ and Its Lie Algebra

The finite dimensional representations that we are going to construct in Sec. 5 cannot be unitary because of the non-compactness of the Lorentz group. However, there is another algebraic invariant which will be preserved by our representations, and this invariant will be related to the relativistic invariant of intensity in an electromagnetic field. We introduce that invariant, called *determinant*, in this section. It will be used also in Sec. 4 to classify the tripotents of \mathcal{S}^n and to prove the important facial decomposition.

For any $a \in \mathcal{S}^n$ define a notion of *determinant* by

$$\det a = \langle a | \bar{a} \rangle = \sum a_j^2$$

After we have proved Theorem 2, we shall see that this definition agrees with the determinant in a spin factor, given by (13), and coincides with the ordinary determinant of a matrix in Examples 2.1 and 2.2. Now define the group $\text{Dinv}(\mathcal{S}^n)$ to be the set of all invertible linear maps of \mathcal{S}^n which preserve the determinant. Note that $SO(n) \subset \text{Dinv}(\mathcal{S}^n) \subset GL(\mathcal{S}^n)$ and if we denote the Lie algebra of $\text{Dinv}(\mathcal{S}^n)$ by $\text{dinv}(\mathcal{S}^n)$, we have $so(n) \subset \text{dinv}(\mathcal{S}^n) \subset gl(\mathcal{S}^n)$. By general principles, if $g(t)$ is a one-parameter subgroup of $\text{Dinv}(\mathcal{S}^n)$, then $X := g'(0) \in \text{dinv}(\mathcal{S}^n)$ and $g(t) = \exp tX$. Conversely, if $X \in \text{dinv}(\mathcal{S}^n)$, then $\exp tX$ is a one-parameter subgroup of $\text{Dinv}(\mathcal{S}^n)$. Hence $\text{dinv}(\mathcal{S}^n)$ consists of all infinitesimal generators of one-parameter subgroups of $\text{Dinv}(\mathcal{S}^n)$.

Note that for $\lambda U \in \text{Taut}(\mathcal{S}^n)$, $\langle \lambda U a | \overline{\lambda U a} \rangle = \lambda^2 \langle a | \bar{a} \rangle$ so that

$$\text{Taut}(\mathcal{S}^n) \cap \text{Dinv}(\mathcal{S}^n) = SO(n)$$

Proposition 3.5. If $g(t)$ is a one-parameter subgroup in $\text{Dinv}(\mathcal{S}^n)$ then its infinitesimal generator $X (= g'(0) \in \text{dinv}(\mathcal{S}^n))$ is an anti-symmetric matrix with respect to any TCAR basis of \mathcal{S}^n . Hence $\text{dinv}(\mathcal{S}^n) \subset A_n(\mathbf{C})$, where $A_n(\mathbf{C})$ denotes the Cartan factor of type 2 (n by n complex anti-symmetric matrices).

Proof. Differentiating the equation

$$\langle g(t) a | \overline{g(t) a} \rangle = \langle a | \bar{a} \rangle$$

at $t = 0$ yields $\langle X a | \bar{a} \rangle = 0$ for every $a \in \mathcal{S}^n$ and by polarization $\langle X b | \bar{a} \rangle + \langle X a | \bar{b} \rangle = 0$ for every $a, b \in \mathcal{S}^n$.

If X has the matrix $[x_{kj}]$ with respect to a TCAR basis u_1, \dots, u_n . Then

$$\langle Xu_j | u_k \rangle + \langle Xu_k | u_j \rangle = 0$$

so X is anti-symmetric. □

We shall see later (in Subsec. 5.2) that $\text{div}(\mathcal{S}^4) = A_4(\mathbf{C}) (= \mathcal{S}^6)$.

The problem of characterizing the linear maps which preserve the determinant has a long history going back to Frobenius in 1897. Frobenius showed that the linear maps of the n by n complex matrices which preserve determinant are of one of the forms $A \mapsto PAQ$ or $A \mapsto PA'Q$, where A' is the transpose of A and P, Q are matrices with $\det PQ = 1$. It would be interesting to find a characterization of determinant preserving linear maps on \mathcal{S}^n . It would also be interesting to explore the analog in our setting of the pseudo-orthogonal groups $SO(n, \mathbf{C})$ of determinant 1 linear maps of \mathbf{C}^n which preserve the form $z_1^2 + \dots + z_n^2$.

4. THE JB^* -TRIPLE STRUCTURE OF \mathcal{S}^n

In this section we construct the norm which makes \mathcal{S}^n with the triple product (19) into a JB^* -triple. To do this we need some decomposition theorems which are interesting in their own right. Moreover, the fact that \mathcal{S}^n is a JB^* -triple will be useful in our forthcoming study of *nonlinear* representations of the Lorentz group. Only Proposition 4.4 will be used explicitly in Sec. 4.

4.1. Facial Decomposition

We will describe first the *tripotents* u , meaning elements $u \in \mathcal{S}^n$ satisfying $u = \{uuu\}$, that are the building blocks of the triple product.

If $\det a \neq 0$ the argument of the determinant is defined as

$$\text{ard } a = \frac{\det a}{|\det a|}$$

and $\text{ard } a = 1$ if $\det a = 0$. Note that the elements with zero determinant are called null-vectors in the literature.

From (19) an element u is a tripotent in one of the following two cases:

$$\det u = 0 \quad \text{and} \quad \langle u | u \rangle = 1/2$$

in which case u is called a *minimal tripotent*, or

$$\bar{u} = \overline{(\text{ard } u)} u \quad \text{and} \quad \langle u | u \rangle = 1$$

in which case u is called a *maximal tripotent*.

Note that it follows from (19) that for any maximal tripotent u we have $D(u, u) = I$ and from (17) that each element u_j of a TCAR basis is a maximal tripotent. On the other hand, for a minimal tripotent v , we have $D(v, v) \bar{v} = 0$ and both $v + \bar{v}$ and $v - \bar{v}$ are maximal tripotents.

Lemma 4.1. Let w_1, w_2 be a pair of maximal tripotents in \mathcal{S}^n . The following are equivalent.

- (a) w_1, w_2 are linearly independent and $\{w_1 w_2 w_1\} = w_2$
- (b) $\text{ard } w_1 = -\text{ard } w_2$ and $\langle w_1 | w_2 \rangle = 0$
- (c) w_1, iw_2 are linearly independent and satisfy TCAR.

Proof.

(a) implies (b) From $\bar{w}_j = \overline{\text{ard } w_j} w_j$, and $2\langle w_1 | w_2 \rangle w_1 - \langle w_1 | \bar{w}_1 \rangle \bar{w}_2 = w_2$ we obtain

$$2\langle w_1 | w_2 \rangle w_1 - (\langle w_1 | \overline{\text{ard } w_1} w_1 \rangle \overline{\text{ard } w_2} + 1) w_2 = 0$$

(b) implies (c) Since w_1 and iw_2 are maximal tripotents, we need only to show that $\{w_1, iw_2, w_1\} = -iw_2$ and $\{iw_2, w_1, iw_2\} = -w_1$. For example

$$\begin{aligned} \{iw_2, w_1, iw_2\} &= -\{w_2 w_1 w_2\} = \langle w_2 | \bar{w}_2 \rangle \bar{w}_1 \\ &= \text{ard } w_2 \overline{\text{ard } w_1} w_1 = -w_1 \end{aligned}$$

(c) implies (a) $\{w_1 w_2 w_1\} = i\{w_1, iw_2, w_1\} = i(-iw_2) = w_2$. □

A pair of maximal tripotents w_1, w_2 will be called a *complementary pair* if one of the equivalent conditions in Lemma 4.1 is satisfied.

Theorem 1 (Facial Decomposition). For any element $a \in \mathcal{S}^n$ there exist a unique complementary pair w_1, w_2 of maximal tripotents and a unique pair of non-negative real numbers α_1, α_2 satisfying

$$\alpha_1 \geq \alpha_2 \geq 0$$

and

$$a = \alpha_1 w_1 + \alpha_2 w_2 \tag{22}$$

The formula (22) will be called the “facial decomposition” of a . This terminology will be justified later in this section.

Proof. For any element $a \in \mathcal{S}^n$ with $\det a \neq 0$, if λ is a complex number with $|\lambda| = 1$, then $\text{ard}(\lambda a) = \lambda^2 \text{ard } a$. Thus, if $\mu = (\text{ard } a)^{1/2}$, then the element b defined by $a = \mu b$ has a nonnegative determinant. If we decompose b as

$$b = x + iy, \quad x, y \in \text{span}_{\mathbf{R}}\{e_j\}$$

then (compare to (9))

$$\det b = \|x\|_2^2 - \|y\|_2^2 + 2i\langle x | y \rangle \quad (23)$$

and the condition $\det b \geq 0$ implies

$$\langle x | y \rangle = 0 \quad \text{and} \quad \|x\|_2 \geq \|y\|_2 \quad (24)$$

Therefore, in the above notation we have

$$a = \mu(x + iy) \quad \text{with} \quad \mu = (\text{ard } a)^{1/2}, \quad x, y \in \text{span}_{\mathbf{R}}\{e_j\} \quad (25)$$

and x, y satisfying (24).

For any element $a \in \mathcal{S}^n$ satisfying (25) we have

$$a = \mu \left(\|x\|_2 \frac{x}{\|x\|_2} + \|y\|_2 i \frac{y}{\|y\|_2} \right) = \alpha_1 w_1 + \alpha_2 w_2$$

where the positive real numbers $\alpha_1 = \|x\|_2$, $\alpha_2 = \|y\|_2$ and the elements $w_1 = \mu \frac{x}{\|x\|_2}$, $w_2 = i\mu \frac{y}{\|y\|_2}$ are complementary maximal tripotents with $\text{ard } a = \text{ard } w_1$. It is readily seen that w_1 and w_2 satisfy the condition (b) for complementary pair.

Let us now prove the uniqueness of the facial decomposition. If $a = \alpha_1 w_1 + \alpha_2 w_2 = \beta_1 v_1 + \beta_2 v_2$, with w_1, w_2 and v_1, v_2 complementary pairs of maximal tripotents and $\alpha_1 \geq \alpha_2 \geq 0$ and $\beta_1 \geq \beta_2 \geq 0$, there are, by Lemma 3.1, constants λ, μ of absolute value one with $w_1 = \lambda \bar{w}_1$, $w_2 = -\lambda \bar{w}_2$ and $v_1 = \mu \bar{v}_1$, $v_2 = -\mu \bar{v}_2$. From $\det a = \langle a | \bar{a} \rangle$ we obtain $(\alpha_1^2 - \alpha_2^2) \lambda = (\beta_1^2 - \beta_2^2) \mu$, and from $\|a\|_2^2 = \langle a | a \rangle$ we obtain $\alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2$ and so $\alpha_j = \beta_j$ and $\lambda = \mu$.

Now from

$$a = \alpha_1 w_1 + \alpha_2 w_2 = \alpha_1 v_1 + \alpha_2 v_2$$

and

$$\bar{a} = \alpha_1 \bar{\lambda} w_1 - \alpha_2 \bar{\lambda} w_2 = \alpha_1 \bar{\lambda} v_1 - \alpha_2 \bar{\lambda} v_2$$

we obtain

$$w_1 = \frac{a + \lambda \bar{a}}{2\alpha_1}, \quad w_2 = \frac{a - \lambda \bar{a}}{2\alpha_2} \quad (26)$$

and the same formulas hold for v_1 and v_2 . □

4.2. The JB^* -Triple Norm of \mathcal{S}^n ; Polar Decomposition

As consequences of the facial decomposition, we can now define a norm which will make \mathcal{S}^n into a JB^* -triple, and prove a polar decomposition for arbitrary elements of the JB^* -triple \mathcal{S}^n .

The norm is defined for any element $a \in \mathcal{S}^n$ with facial decomposition $a = \alpha_1 w_1 + \alpha_2 w_2$ by

$$\|a\| = \alpha_1 + \alpha_2 \quad (27)$$

We will show first that

$$\|\{aaa\}\| = (\alpha_1 + \alpha_2)^3 = \|a\|^3 \quad (28)$$

which is the analog of the defining identity $\|a^*a\| = \|a\|^2$ for a C^* -algebra. Thus we can refer to this as the “operator norm.”

As in the proof of Theorem 1, $\bar{a} = \bar{\mu}(x - iy) = \bar{\mu}^2(\alpha_1 w_1 - \alpha_2 w_2)$, $\langle a | a \rangle = \alpha_1^2 + \alpha_2^2$, and $\langle a | \bar{a} \rangle = \mu^2(\alpha_1^2 - \alpha_2^2)$.

From (19) it follows that

$$\{aaa\} = (\alpha_1^3 + 3\alpha_2^2\alpha_1) w_1 + (\alpha_2^3 + 3\alpha_1^2\alpha_2) w_2$$

which implies (28).

We still need to show that the function $\|\cdot\|$ is a norm. It is obvious from the uniqueness of the facial decomposition that

$$\|\lambda a\| = |\lambda| \|a\| \quad (29)$$

for every $\lambda \in \mathbb{C}$. The proof of the triangle inequality is more difficult. An elementary proof requires the solution of a optimization problem for which a direct proof seems intractable. Instead, we shall prove the following lemma using properties of spin factors.

Lemma 4.2. The function $\|\cdot\|$ defined by (27) satisfies the triangle inequality, and is therefore a norm on \mathcal{S}^n .

Proof. Let \mathcal{C} denote the Cartan factor of type 4 and dimension n . Assume for simplicity of notation that n is even and let $v_1, \tilde{v}_1, \dots, v_m, \tilde{v}_m$ be a spin grid for \mathcal{C} (see Subsec. 2.2). For $a = \sum a_j v_j + \sum \tilde{a}_j \tilde{v}_j \in \mathcal{C}$ let

$$\phi(a) = \frac{1}{2} (a_1 + \tilde{a}_1, -i(a_1 - \tilde{a}_1), a_2 + \tilde{a}_2, -i(a_2 - \tilde{a}_2), \dots, a_m + \tilde{a}_m, -i(a_m - \tilde{a}_m)) \in \mathcal{S}^n$$

One checks easily that

- $\langle \phi(a) | \phi(a) \rangle = \frac{1}{2} \langle a | a \rangle$
- $\langle \phi(a) | \overline{\phi(a)} \rangle = \sum a_j \tilde{a}_j$
- $\phi(a^*) = \overline{\phi(a)}$.

Hence, from (12) and (19), $\phi(\{aaa\}) = \{\phi(a), \phi(a), \phi(a)\}$ (and by polarization, ϕ is an isomorphism: $\phi(\{abc\}) = \{\phi(a), \phi(b), \phi(c)\}$).

Since the function $\|\phi(a)\|$ is continuous, there is a constant $k < \infty$ such that $\sup \{\|\phi(a)\| : \|a\| \leq 1\} \leq k$, and by (29), $\|\phi(a)\| \leq k \|a\|$. We now have $\|\phi(a)\|^3 = \|\{\phi(a) \phi(a) \phi(a)\}\| = \|\phi\{aaa\}\| \leq k \|\{aaa\}\| = k \|a\|^3$. By iteration $\|\phi(a)\| \leq \|a\|$, and by symmetry $\|\phi^{-1}(b)\| \leq \|b\|$. Finally $\|a + b\| = \|\phi^{-1}(a) + \phi^{-1}(b)\| \leq \|\phi^{-1}(a)\| + \|\phi^{-1}(b)\| = \|a\| + \|b\|$. \square

Theorem 2. \mathcal{S}^n is a JB^* -triple isomorphic to a Cartan factor of type 4.

Proof. Referring to the definition of JB^* -triple given earlier, it remains to show that the operator $\Delta(a)$ is Hermitian and positive. In the first place, if $a = \alpha_1 w_1 + \alpha_2 w_2$ is the facial decomposition of a , then the matrix of $\Delta(a)$ with respect to a TCAR basis $w_1, iw_2, u_3, \dots, u_n$ containing w_1, iw_2 is

$$\Delta(a) = \begin{bmatrix} \alpha_1^2 + \alpha_2^2 & 2i\alpha_1\alpha_2 & 0 & \cdots & 0 \\ -2i\alpha_1\alpha_2 & \alpha_1^2 + \alpha_2^2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_1^2 + \alpha_2^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^2 + \alpha_2^2 \end{bmatrix}$$

and this matrix has eigenvalues $\alpha_1^2 + \alpha_2^2$, $(\alpha_1 + \alpha_2)^2$, and $(\alpha_1 - \alpha_2)^2$.

Next let T denote the operator $\exp it\Delta(a)$ for real t . Since $it\Delta(a)$ is a derivation of the triple product, T is an automorphism. Thus, as in the proof of Lemma 4.2, from $\{TxTxTx\} = T\{xxx\}$ we obtain $\|Tx\|^3 =$

$\|\{TxTxTx\}\| = \|T\{xxx\}\| \leq \|T\| \|x\|^3$, so that by iteration $\|Tx\| \leq \|T\|^{3^{-k}} \|x\|$ and $\|T\| \leq 1$. Since T^{-1} is also an automorphism, $\|T^{-1}\| \leq 1$, so T is an isometry, which proves that $\Delta(a)$ is hermitian. \square

Now that we know that $\|\cdot\|$ is a norm, we can use the facial decomposition to prove the polar decomposition.

Theorem 3 (Polar Decomposition). For any $a \in \mathcal{S}^n$ with facial decomposition $a = \alpha_1 w_1 + \alpha_2 w_2$, there exists a positive Hermitian operator T on \mathcal{S}^n such that $a = T(w_1)$ and $\|T\| = \|a\|$. If w is any maximal tripotent and T is any Hermitian operator such that $a = Tw$, then $w = w_1$.

Proof. For any $a \in \mathcal{S}^n$ with the facial decomposition (22),

$$a = (\alpha_1 I + \alpha_2 D(w_1, w_2)) w_1$$

which we will call *the canonical polar decomposition* of a . We shall show that the operator $H := \alpha_1 I + \alpha_2 D(w_1, w_2)$ is an Hermitian operator with positive spectrum and with norm $\|a\|$. Note that $H = i(-i\alpha_1 I + \alpha_2 D(w_1, iw_2)) = i\delta$ with $\delta \in \text{taut}(\mathcal{S}^n)$.

The matrix of H with respect to a TCAR basis $w_1, iw_2, u_3, \dots, u_n$ which extends w_1, iw_2 is

$$H = \begin{bmatrix} \alpha_1 & i\alpha_2 & 0 & \cdots & 0 \\ -i\alpha_2 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1 \end{bmatrix} \tag{30}$$

which has eigenvalues $\alpha_1, \alpha_1 \pm \alpha_2$.

To show H is a hermitian operator with respect to the norm $\|\cdot\|$, it is required to show that $\exp itH$ is an isometry for all real t . As is well known for JB^* -triples, this is the same as $\exp itH$ being a (triple) automorphism. But we have already observed above that $iH \in \text{taut}(\mathcal{S}^n)$, and so $\exp itH \in \text{Taut}(\mathcal{S}^n)$.

Finally, since $\|\{abc\}\| \leq \|a\| \|b\| \|c\|$, if $x \in \mathcal{S}^n$, then $\|H(x)\| = \|\alpha_1 x + \alpha_2 \{w_1 w_2 x\}\| \leq (\alpha_1 + \alpha_2) \|x\|$. Since $H(w_1) = a$, $\|H\| = \|a\|$. This proves the first statement.

We now prove the second statement. Since T is hermitian, $iT \in \text{taut}(\mathcal{S}^n)$, so by Proposition 3.4,

$$T = i \sum_{j < k} d_{jk} D_{jk} - dI \quad \text{for some } d_{jk}, \quad d \in \mathbf{R}$$

where $D_{jk} = D(u_j, u_k)$ and u_1, u_2, \dots, u_n is any TCAR basis with $w = u_1$. In particular

$$a = Tu_1 = -du_1 - i \left(\sum_2^n d_{1j} u_j \right)$$

With $\bar{u}_j = \mu u_j$ and $\bar{w}_1 = \lambda w_1, \bar{w}_2 = -\lambda w_2,$

$$\bar{\lambda}(\alpha_1^2 - \alpha_2^2) = \det a = \bar{\mu} \left(d^2 - \sum_2^n d_{1j}^2 \right)$$

Therefore $\bar{\lambda}\mu = 1$ and

$$(\alpha_1^2 - \alpha_2^2) = \left(d^2 - \sum_2^n d_{1j}^2 \right)$$

Similarly, using $\langle a | a \rangle,$ we obtain

$$(\alpha_1^2 + \alpha_2^2) = \left(d^2 + \sum_2^n d_{1j}^2 \right)$$

so that $\alpha_1 = -d$ and $\alpha_2 = (\sum_2^n d_{1j}^2)^{1/2}.$

From (26)

$$w_1 = \frac{(1 + \bar{\lambda}\mu)(-du_1) - (1 - \bar{\lambda}\mu) i \sum_2^n d_{1j} u_j}{2\alpha_1} = u_1$$

so $w_1 = w.$

□

Although the phase w_1 in the polar decomposition is unique, the positive Hermitian operator T with $Tw_1 = a$ is not. Note that the matrix of any such T with respect to the TCAR basis $u_1, \dots, u_n,$ is

$$\begin{bmatrix} \alpha_1 & id_{12} & id_{13} & \cdots & id_{1n} \\ -id_{12} & \alpha_1 & id_{23} & \cdots & id_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -id_{1n} & -id_{2n} & -id_{3n} & \cdots & \alpha_1 \end{bmatrix}$$

whereas the matrix of $\alpha_1 I + \alpha_2 D(w_1, w_2)$ is (30), where $\alpha_2 = (\sum_2^n d_{1j}^2)^{1/2}.$

For example, for $n = 3,$ the eigenvalues of T are $\alpha_1, \alpha_1 \pm \sqrt{\alpha_2^2 + d_{23}^2}.$ Thus T is positive if and only if $\alpha_2^2 + d_{23}^2 \leq \alpha_1^2.$ On the other hand, since the

norm of a Hermitian element is its spectral radius, $\|T\| = \|a\|$ if and only if $d_{23} = 0$. Thus the operators

$$T_\theta = \begin{bmatrix} \alpha_1 & i\alpha_2 \cos \theta & i\alpha_2 \sin \theta \\ -i\alpha_2 \cos \theta & \alpha_1 & 0 \\ -i\alpha_2 \sin \theta & 0 & \alpha_1 \end{bmatrix}$$

are precisely the ones that occur in the possible polar decompositions of the element a with facial decomposition (22). The canonical polar decomposition occurs for $\theta = 0$.

Denote the unit ball of the JB^* -triple \mathcal{S}^n with respect to the operator norm by $B_{\mathcal{S}^n}$, namely

$$B_{\mathcal{S}^n} = \{a \in \mathcal{S}^n : \|a\| < 1\}$$

Thus $B_{\mathcal{S}^n}$ is a Cartan domain of type 4, called the *Lie ball* and any isometry of \mathcal{S}^n is a triple product automorphism.

From (28) it follows that the norm of any tripotent is one and from (22) that any element in the interior of the unit ball is a convex combination of zero and two complementary maximal tripotents. Thus an element of norm one lies on the line segment connecting w_1 and w_2 in the boundary of the unit ball. This is why we call (22) the facial decomposition of a .

4.3. The Trace Norm Unit Ball

We may consider any element $f \in \mathcal{S}^n$ as a functional by

$$f(a) = \langle a | f \rangle \quad \text{for any } a \in \mathcal{S}^n$$

with the norm $\|\cdot\|_*$ called the *trace norm*, defined by

$$\|f\|_* = \sup\{|f(a)| : a \in B_{\mathcal{S}^n}\}$$

Proposition 4.3. If the facial decomposition of f is $f = \alpha_1 w_1 + \alpha_2 w_2$ then

$$\|f\|_* = \alpha_1$$

Proof. Let ϕ be the isomorphism of \mathcal{C} onto \mathcal{S}^n used in the proof of Lemma 4.2. Recall that ϕ takes the elements $v_j + \tilde{v}_j$, $i(v_j - \tilde{v}_j)$ into the (natural) TCAR basis e_1, \dots, e_n of \mathcal{S}^n . Let u_1, \dots, u_n be a TCAR basis

extending w_1, iw_2 and let ψ be the isomorphism of \mathcal{C} onto \mathcal{S}^n which maps $v_j + \tilde{v}_j, i(v_j - \tilde{v}_j)$ onto u_1, \dots, u_n . We shall show that

$$\psi^*(f) = \frac{\alpha_1 - \alpha_2}{2} v_1 + \frac{\alpha_1 + \alpha_2}{2} \tilde{v}_1 \tag{31}$$

and therefore by (14),

$$\|f\|_* = \|\psi^*(f)\|_{\mathcal{C}^*} = \frac{\alpha_1 - \alpha_2}{2} + \frac{\alpha_1 + \alpha_2}{2} = \alpha_1$$

To prove (31), write $a = \sum a_j v_j + \sum \tilde{a}_j \tilde{v}_j$ so that $\psi(a) = a_1(u_1 - iu_2)/2 + \tilde{a}_1(u_1 + iu_2)/2 + \dots$. Now $f = \alpha_1 u_1 - i\alpha_2 u_2$ and therefore

$$f(\psi(a)) = \alpha_1 \frac{a_1 + \tilde{a}_1}{2} - \alpha_2 \frac{a_1 - \tilde{a}_1}{2}$$

On the other hand, writing $\psi^*(f) = \sum c_j v_j + \sum \tilde{c}_j \tilde{v}_j$, we have

$$\psi^*(f)(a) = \sum c_j a_j + \sum \tilde{c}_j \tilde{a}_j$$

so that

$$c_1 = \frac{\alpha_1 - \alpha_2}{2}, \quad c_2 = c_3 = \dots = 0$$

and

$$\tilde{c}_1 = \frac{\alpha_1 + \alpha_2}{2}, \quad \tilde{c}_2 = \tilde{c}_3 = \dots = 0 \quad \square$$

We will denote the unit ball in \mathcal{S}^n with respect to the trace norm by S . It is obvious from the facial decomposition that the boundary of S is the union of the sets

$$\begin{aligned} G_u &= \{u + \alpha w : 0 \leq \alpha \leq 1; \{www\} = w; \langle u | w \rangle = 0; \langle w | w \rangle = 1; \\ &\text{ard } w = -\text{ard } u\} \end{aligned} \tag{32}$$

where u varies over all maximal tripotents.

Proposition 4.4. The sets G_u are norm exposed faces of S , that is, $G_u = F_u$ where

$$F_u = \{a \in \mathcal{S}^n : \langle u | a \rangle = 1 = \|a\|_*\}$$

and are therefore $n-1$ dimensional unit balls. For $n=4$ this implies that the boundary of S is the union of real 3-dimensional balls.

Proof. We prove that $G_u = F_u$. If $a = u + \alpha w \in G_u$ then clearly $\langle u | a \rangle = 1 = \|a\|_*$, so that $a \in F_u$. Conversely, let $a \in F_u$ have facial decomposition $a = w_1 + \alpha_2 w_2$. Then $a \in F_{w_1}$ and the intersection $F_u \cap F_{w_1}$ is a face of rank one or two. In the latter case, $w_1 = u$ so that $a = w_1 + \alpha_2 w_2 = u + \alpha_2 w_2 \in G_u$. In the former case the intersection consists of a alone so that a is an extreme point and hence a minimal tripotent. Thus $\langle a | a \rangle = 1/2$ and $\det a = 0$. But $\det a = \lambda(1 - \alpha_2^2)$ for some complex number λ of absolute value one, so $\alpha_2 = 1$. But then $\langle a | a \rangle = 2$, a contradiction. So $F_u \subset G_u$, as stated. \square

5. SPIN-1 AND SPIN- $\frac{1}{2}$ REPRESENTATIONS OF THE LORENTZ GROUP

All of our representations will be defined on the Lie algebras $so(3)$ and $sl(2, C)$ and by simple connectedness of $SU(2)$ and $SL(2, C)$ we can pass freely to representations of those groups. The corresponding representations of the groups $SO(3)$ and the Lorentz group may therefore be double valued.

5.1. Lorentz Group Representations in $\text{dinv}(\mathcal{S}^3)$

Consider the rotation group $SO(3)$ consisting of all 3 by 3 orthogonal real matrices with determinant one. As was mentioned earlier, this group coincides with $\text{Taut}(\mathcal{S}^3) \cap \text{Dinv}(\mathcal{S}^3)$. The Lie algebra of the Lie group $SO(3)$, which is linearly spanned by the three 3 by 3 matrices obtained from (2) and (3) by omitting the first rows and columns, and satisfying the commutation relations (4) can be represented in $\text{taut}(\mathcal{S}^3)$ by the operators

$$\sigma(J_1) = D(u_2, u_3), \quad \sigma(J_2) = D(u_3, u_1), \quad \sigma(J_3) = D(u_1, u_2)$$

that satisfy the same commutation relations, that is, σ extends to a Lie algebra homomorphism of $so(3)$ into $\text{taut}(\mathcal{S}^3)$. Here of course, u_1, u_2, u_3 is

any fixed TCAR basis of \mathcal{S}^3 . Note that the corresponding representation at the group level is the identity map, if elements of $\text{taut}(\mathcal{S}^3)$ are replaced by their matrices with respect to u_1, u_2, u_3 . For example the rotation matrix $R_1 = \exp t\sigma(J_1)$ generated by J_1 is the 3 by 3 matrix obtained from (1) by omitting the first row and column. Note that since $\overline{R_1 a} = R_1 \bar{a}$, $R_1 \in \text{Dinv}(\mathcal{S}^3)$.

We shall now extend σ , to a linear representation of the Lorentz group.

Let $J_1, J_2, J_3, K_1, K_2, K_3$ be the standard infinitesimal generators of the Lorentz group, the J_j corresponding to the rotation group, and the K_j corresponding to the boosts.

The commutation relations between J_1, J_2, J_3 are given by (4). The other commutation relations are (5)–(8).

It is trivial to check that if J_1, J_2, J_3 are any three elements of a complex Lie algebra satisfying (4), and if we set $K_j = iJ_j$ for $j = 1, 2, 3$, then (5)–(8) hold.

We now define a three dimensional linear representation $\tilde{\sigma}$ of the Lorentz Lie algebra into $\text{dinv}(\mathcal{S}^3)$, extending σ , via:

$$\tilde{\sigma}(K_k) = i\sigma(J_k)$$

for $k = 1, 2, 3$.

Proposition 5.1. $\tilde{\sigma}$ is an irreducible representation with range in $\text{dinv}(\mathcal{S}^3)$.

To see that the image of $\tilde{\sigma}$ lies in $\text{dinv}(\mathcal{S}^3)$, note that for example the one parameter group (boost) $B_1(t) = \exp t\tilde{\sigma}(K_1)$ is the same as the 3 by 3 matrix (10). Thus, although $B_1(t)$ cannot be a unitary matrix, it does preserve the determinant of elements of \mathcal{S}^3 . This connection with (10) and the connection of (23) with (9) shows the relevance of our construction with the natural model for the electromagnetic field intensity.

5.2. Spin-1 Representations of the Lorentz Group in $\text{dinv}(\mathcal{S}^4)$

In the previous subsection, we constructed a three-dimensional linear representation σ of the rotation group, and extended it to the full Lorentz group in such a way that it acts by determinant preserving linear maps on \mathcal{S}^3 . We shall now construct a linear irreducible four-dimensional representation of the Lorentz group which also extends σ and show that it is equivalent to the standard representation of the Lorentz group on space-time.

We shall again let $J_1, J_2, J_3, K_1, K_2, K_3$ be the standard infinitesimal generators of the Lorentz Lie algebra. We also let u_0, u_1, u_2, u_3 denote a TCAR basis for \mathcal{S}^4 and define a representation π from the Lorentz Lie algebra to operators on \mathcal{S}^4 . For notation's sake, we shall use D_{jk} to denote $D(u_j, u_k)$, for $j, k \in \{0, 1, 2, 3\}, j \neq k$. Note that here for example $D(u_2, u_3)$ is acting on the space spanned by u_0, u_1, u_2, u_3 whereas in the previous section it was acting on the span of u_1, u_2, u_3 . We define π by

$$\begin{aligned} \pi(J_1) &= D_{23}, & \pi(J_2) &= D_{31}, & \pi(J_3) &= D_{12} \\ \pi(K_1) &= iD_{01}, & \pi(K_2) &= iD_{02}, & \pi(K_3) &= iD_{03} \end{aligned}$$

Proposition 5.2. π is an irreducible representation with range in $\text{dinv}(\mathcal{S}^4)$.

We omit the proof as it is a straightforward calculation. We just note that for example if $T = \exp t\pi(K_1)$, then with respect to the basis $\{u_0, u_1, u_2, u_3\}$,

$$T = \begin{bmatrix} \cosh t & i \sinh t & 0 & 0 \\ -i \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that if $a = \sum_0^3 a_j u_j$, then $\langle Ta | \overline{Ta} \rangle = \langle a | \overline{a} \rangle$.

Note that $\pi(J_k)$ and $\pi(K_k)$ for $k = 1, 2, 3$ are elements of $\text{dinv}(\mathcal{S}^4)$ which is a subset of $A_4(\mathbf{C})$ by Proposition 3.5. But $A_4(\mathbf{C})$ is the complex span of the D_{jk} and as we have just seen D_{jk} and iD_{jk} belong to $\text{dinv}(\mathcal{S}^4)$. Therefore $\text{dinv}(\mathcal{S}^4) = A_4(\mathbf{C})$.

To understand this representation, we will consider its action on the positive cone $\mathcal{A} = \bigcup_{t>0} tF_{u_0}$ generated by the face F_{u_0} of Proposition 4.4. Note that under the embedding $(t, x, y, z) \mapsto tu_0 + ixu_1 + iyu_2 + izu_3$ of space-time into \mathcal{S}^4 the determinant corresponds to the square of the interval.

Proposition 5.3. Let $\mathcal{A} = \bigcup_{t>0} tF_{u_0}$. Then

- $\mathcal{A} = \{su_0 + \alpha w : s \in \mathbf{R}^+, \alpha w = xiu_1 + yiu_2 + ziu_3, x, y, z \in \mathbf{R}, x^2 + y^2 + z^2 \leq s^2, 0 \leq \alpha \leq 1\}$.
- π is an extension of the usual representation of the Lorentz group on the space-time.

Proof. From Proposition 4.4 we may use the description of F_{u_0} given in (32). If $u_0 + \alpha w \in F_{u_0}$, then $w = \sum_1^3 a_j u_j$ where $a_j = x_j + i y_j \in \mathbb{C}$ satisfies $\sum_1^3 |a_j|^2 = \langle w | w \rangle = 1$. Since w and u_0 are maximal tripotents, $\det w = \text{ard } w$ and $\det u_0 = \text{ard } u_0$, so that for some λ of modulus 1,

$$\left(\sum a_j^2 \right) \lambda = \det w = -\text{ard } w = -\lambda$$

Thus $\sum_1^3 a_j^2 = -1$ so that $a_j = i y_j$ is purely imaginary with $\sum_1^3 y_j^2 = 1$, and hence

$$A = \left\{ t u_0 + t \alpha \sum_1^3 i y_j u_j : y_j \in \mathbb{R}, \sum_1^3 y_j^2 = 1, t > 0 \right\}$$

from which (a) follows.

By taking $s = ct$, where c is the speed of light in vacuum we can identify the cone A with the ‘‘light cone.’’ A direct calculation shows that A is invariant under the representation π . For example with $T = \exp t\pi(K_1)$,

$$T(s, ix, iy, iz) = (\cosh ts + \sinh tx, -i \sinh ts + i \cosh tx, iy, iz)$$

Since this representation preserves the determinant of any element in \mathcal{S}^4 and since for the elements of the cone A , the determinant coincides with the space-time interval, (b) follows. □

5.3. Spin- $\frac{1}{2}$ Representation in $\text{dinv}(\mathcal{S}^4)$

The elements D_{jk} from $\text{dinv}(\mathcal{S}^4)$, defined in the previous section, form a spin grid (as described in Subsec. 2.2) generating the spin factor of dimension 6. The sharp operation on the grid (see Example 2.2) of this factor is thus given by

$$\begin{aligned} D_{01}^\# &= D_{23}, & D_{02}^\# &= D_{31}, & D_{03}^\# &= D_{12} \\ D_{23}^\# &= D_{01}, & D_{31}^\# &= D_{02}, & D_{12}^\# &= D_{03} \end{aligned}$$

and is extended to be conjugate linear on the whole factor.

For the representation π , constructed in the previous section, $\pi(J_k)$ and $\pi(K_k)$ are minimal tripotents of $\text{dinv}(\mathcal{S}^4)$. We now use the conjugation $\#$ to construct, from π , two representations π^+ and π^- of the Lorentz group

using maximal tripotents. The representation π^+ is going to be the self-adjoint part of π with respect to \sharp on J_1, J_2, J_3 and the skew-adjoint part of π with respect to \sharp on K_1, K_2, K_3 . That is,

$$\pi^+(J_k) = \frac{1}{2} (\pi(J_k) + \pi(J_k)^\sharp), \quad \pi^+(K_k) = \frac{1}{2} (\pi(K_k) - \pi(K_k)^\sharp)$$

Note that $\pi^+(K_k) = i\pi^+(J_k)$ for $k = 1, 2, 3$.

For example, the matrices of $\pi^+(J_1)$, $\pi^+(J_2)$ and $\pi^+(J_3)$ in the basis u_0, u_1, u_2, u_3 are extensions of the anti-symmetric matrices J_1, J_2, J_3 to a multiple of 4 by 4 orthogonal matrices as follows:

$$\pi^+(J_1) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & & & \\ 0 & & J_1 & \\ 0 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\pi^+(J_2) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ -1 & & J_2 & \\ 0 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\pi^+(J_3) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & J_3 & \\ -1 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Note that the constant $\frac{1}{2}$ is necessary in order that π^+ will be a representation of $so(3)$. Since $\pi^+(K_k) = i\pi^+(J_k)$, π^+ is a representation of $sl(2, \mathbb{C})$.

Notice also that $\pi^+(J_1)$, $\pi^+(J_2)$, $\pi^+(J_3)$ each have 2 distinct eigenvalues, and using the fact that this number is $2j+1$ where j is called the spin value, for our representation we have $j = 1/2$, implying that this is a spin- $\frac{1}{2}$ representation. This is confirmed in the proposition that follows, whose proof is an elementary calculation.

Proposition 5.4. π^+ and π^- (defined below) are representations of $sl(2, \mathbb{C})$. The representation π^+ of $so(3)$ into $\text{dinv}(\mathcal{S}^4)$ induces a spin- $\frac{1}{2}$ representation of $SO(3)$ in $\text{Dinv}(\mathcal{S}^4)$. For example, the matrix of the operator $R_3(t) = \exp t\pi^+(J_3)$ with respect to the TCAR basis $\{u_0, u_1, u_2, u_3\}$ is

$$R_3(t) = \exp t\pi^+(J_3) = \begin{bmatrix} \cos \frac{t}{2} & 0 & 0 & \sin \frac{t}{2} \\ 0 & \cos \frac{t}{2} & \sin \frac{t}{2} & 0 \\ 0 & -\sin \frac{t}{2} & \cos \frac{t}{2} & 0 \\ -\sin \frac{t}{2} & 0 & 0 & \cos \frac{t}{2} \end{bmatrix}$$

Similarly for $R_1(t) = \exp t\pi^+(J_1)$ and $R_2(t) = \exp t\pi^+(J_2)$.

Let us now define the representation π^- , which is going to be the skew-adjoint part of π with respect to \sharp on J_1, J_2, J_3 and the self-adjoint part of π with respect to \sharp on K_1, K_2, K_3 . Thus,

$$\pi^-(J_k) = \frac{1}{2}(\pi(J_k) - \pi(J_k)^\sharp), \quad \pi^-(K_k) = \frac{1}{2}(\pi(K_k) + \pi(K_k)^\sharp)$$

and again $\pi^-(K_k) = i\pi^-(J_k)$ for any $k = 1, 2, 3$.

The matrices of $\pi^-(J_1), \pi^-(J_2)$ and $\pi^-(J_3)$ in the basis u_0, u_1, u_2, u_3 are extensions of the anti-symmetric matrices J_1, J_2, J_3 to a multiple of 4 by 4 orthogonal matrices as follows:

$$\pi^-(J_1) = \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & & & \\ 0 & & J_1 & \\ 0 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\pi^-(J_2) = \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & & & \\ 1 & & J_2 & \\ 0 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\pi^-(J_3) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & & & \\ 0 & & J_3 & \\ 1 & & & \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The above construction can also be done by using the *Hodge operator*, also called the star operator. Borrowing the definition from the theory of differential forms, we define

$$*D_{jk} = \epsilon_{jklm} g^{jj} g^{kk} D_{lm}$$

where $\{j, k, l, m\} = \{0, 1, 2, 3\}$, g^{pq} is the Lorentz metric: $g^{00} = 1$, $g^{kk} = -1$ for $k = 1, 2, 3$, and $g^{pq} = 0$ if $p \neq q$, and ϵ_{jklm} is the signature of the permutation $(j, k, l, m) \mapsto (0, 1, 2, 3)$. Specifically,

$$\begin{aligned} *D_{01} &= D_{23}, & *D_{02} &= D_{31}, & *D_{03} &= D_{12} \\ *D_{23} &= -D_{01}, & *D_{31} &= -D_{02}, & *D_{12} &= -D_{03} \end{aligned}$$

The representation π^+ is then the skew-adjoint part of π with respect to the Hodge operator, and the representation π^- is the self-adjoint part of π with respect to the Hodge operator.

Proposition 5.5. The representation π^+ is a direct sum of two copies of the standard $\text{spin-}\frac{1}{2}$ representation in terms of the Pauli spin matrices σ_k , $k = 1, 2, 3$.

Proof. Since $\pi^+(K_k) = i\pi^+(J_k)$, it suffices to consider the restriction of π^+ to $\mathfrak{so}(3)$. A simple calculation shows that a complex matrix which commutes with all $\pi^+(J_k)$ has the form

$$\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} = aI_4 + 2b\pi^-(J_1) + 2c\pi^-(J_2) + 2d\pi^-(J_3)$$

The eigenspaces of $\pi^-(J_2)$ and hence two invariant subspaces of the representation π^+ are

$$M = \text{sp}_{\mathbb{C}}\{u_0 + iu_2, u_1 + iu_3\}$$

and its orthogonal complement

$$M^\perp = \text{sp}_{\mathbb{C}}\{u_0 - iu_2, u_1 - iu_3\}$$

By properties of the TCAR basis, the matrix of $\pi^+(J_1)|_M$ with respect to the basis $\{u_0 + iu_2, u_1 + iu_3\}$ is

$$\sigma_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{33}$$

Similarly, the matrices of $\pi^+(J_2)|_M$ and $\pi^+(J_3)|_M$ with respect to the basis $\{u_0 + iu_2, u_1 + iu_3\}$ are

$$\sigma_3 = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{34}$$

and

$$\sigma_1 = \frac{1}{2} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{35}$$

Moreover, the matrices of $\pi^+(J_k)|_{M^\perp}$ with respect to the basis $\{u_0 - iu_2, u_1 - iu_3\}$ turn out to be identical to (33)–(35).

Hence, for $k = 1, 2, 3$,

$$\pi^+(J_k) = \frac{1}{2} \begin{bmatrix} \sigma_{k+1} & 0 \\ 0 & \sigma_{k+1} \end{bmatrix}$$

where of course by σ_4 we mean σ_1 . □

One can obtain other invariant subspaces by considering the eigenspaces of $\pi^-(J_1)$ and $\pi^-(J_3)$. The restriction of π^+ to these subspaces also leads to the Pauli spin matrices, but possibly in a different order and with possible sign changes. By rotating the basis in these subspaces one can obtain the Pauli matrices in the standard order and with the appropriate sign.

For comparison and the sake of completeness, we also analyze the representation π^- . The situation is entirely symmetric in the following sense. As remarked above, the commutant of $\{\pi^+(J_k) : k = 1, 2, 3\}$ is $\text{sp}_{\mathbb{C}}[\{\pi^-(J_k) : k = 1, 2, 3\} \cup \{I\}]$, which, when restricted to real scalars, is a four dimensional associative algebra isomorphic to the quaternions. The commutant of $\{\pi^-(J_k) : k = 1, 2, 3\}$ is therefore $\text{sp}_{\mathbb{C}}[\{\pi^+(J_k) : k = 1, 2, 3\} \cup \{I\}]$, which is also (after restriction to real scalars) isomorphic to the quaternions. Thus we say that the Lie algebra $\text{taut}(\mathcal{S}^4)$ consists of two copies of the quaternions, “glued together” at (i times) the identity.

The representation π^- exhibits all of the characteristics of the representation π^+ ; it is a direct sum of two copies of the $\text{spin-}\frac{1}{2}$ two-dimensional representation given by the Pauli spin matrices. Once again, these matrices occur not always in the natural order and with some change of sign, both of which can be corrected by a change of basis. We omit the details of the calculation.

5.4. Action of the $\text{Spin-}\frac{1}{2}$ Representation in $\text{div}(\mathcal{S}^4)$

Let us now lift the representations π^+ and π^- from acting on \mathcal{S}^4 to an action on $\text{div}(\mathcal{S}^4) = \mathcal{S}^6$ that correspond to the momentum and angular momentum operators on the states represented by \mathcal{S}^4 .

We recall first that, from Example 2.2, a spin grid for $\text{div}(\mathcal{S}^4) = \mathcal{S}^6$ is given by

$$v_1 = D_{01}, \quad \tilde{v} = D_{23}; \quad v_2 = D_{02}, \quad \tilde{v}_2 = D_{31}; \quad v_3 = D_{03}, \quad \tilde{v}_3 = D_{12}$$

So it follows from (16) that a TCAR basis for the spin factor \mathcal{S}^6 is given by

$$\begin{array}{ccc} D_{01} + D_{23}, & D_{02} + D_{31}, & D_{03} + D_{12} \\ i(D_{01} - D_{23}), & i(D_{02} - D_{31}), & i(D_{03} - D_{12}) \end{array}$$

Note that this has the same complex span as

$$\{\pi^+(J_k), \pi^-(K_k) : k = 1, 2, 3\}$$

We shall choose

$$\pi^+(J_1), \quad \pi^+(J_2), \quad \pi^+(J_3), \quad \pi^+(K_1), \quad \pi^-(K_2), \quad \pi^-(K_3) \quad (36)$$

as a basis for $\text{div}(\mathcal{S}^4)$. For any element T in the group $\text{Dinv}(\mathcal{S}^4)$, define an operator \tilde{T} on $\text{Dinv}(\mathcal{S}^4)$ by $\tilde{T}A = TAT^{-1}$ for any $A \in \text{Dinv}(\mathcal{S}^4)$. By taking $T = R_k(t) = \exp(t\pi^+(J_k))$ or $T = B_k(t) = \exp(t\pi^+(K_k))$ we obtain the Jordan theoretic connection between the two types of fundamental particles in nature.

Theorem 4. With respect to the basis (36) of $\text{div}(\mathcal{S}^4)$,

$$\tilde{R}_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}, \quad \tilde{B}_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh t & i \sinh t & 0 \\ 0 & -i \sinh t & \cosh t & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$\tilde{R}_2(t) = \begin{bmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & 1 & 0 & 0 \\ \sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}, \quad \tilde{B}_2(t) = \begin{bmatrix} \cosh t & 0 & -i \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ -i \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

$$\tilde{R}_3(t) = \begin{bmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}, \quad \tilde{B}_3(t) = \begin{bmatrix} \cosh t & i \sinh t & 0 & 0 \\ -i \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_3 \end{bmatrix}$$

Thus our $\text{spin-}\frac{1}{2}$, representation π^+ of $SO(3)$ and the Lorentz group on \mathcal{S}^4 induces a spin-1 representation of $SO(3)$ and the Lorentz group on $\text{Dinv}(\mathcal{S}^4)$.

5.5. Concluding Remarks

We have seen that the action of π^+ and π^- on \mathcal{S}^4 is similar to the action of the Lorentz group on the Dirac bispinors. So, it may be that \mathcal{S}^4 is equivalent or is a replacement for the bispinors representing the relativistic state of an electron. Moreover, using the polar decomposition from Sec. 4.2 of any element of \mathcal{S}^4 , this state could be decomposed into a maximal tripotent and a positive operator acting on it. The maximal tripotent could be identified with the spin of the particle, while the positive operator with the representation of the energy and momentum of the particle. Our representations act properly on each part of this decomposition, mainly as a spin-1/2 on the spin part and as a spin one action on the four-vector representing energy-momentum.

Another interesting observation is that adding an additional symmetry with respect to the Hodge operator (or the sharp operation) to the representation π that is an extension to \mathcal{S}^4 of the regular Lorentz group representation on four-vectors leads to the representation on the bispinors. It will be interesting to find the physical meaning of this observation.

Since \mathcal{S}^3 is imbedded into \mathcal{S}^4 and the representations π^+ and π^- on \mathcal{S}^4 preserve the determinant, from results of Sec. 5.1, these representations act properly on the intensities of the electromagnetic field.

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