FUSION AND LARGE CARDINAL PRESERVATION

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ABSTRACT. In this paper we introduce some fusion properties of forcing notions which guarantee that an iteration with supports of size $\leq \kappa$ not only does not collapse κ^+ but also preserves the strength of κ (after a suitable preparatory forcing). This provides a general theory covering the known cases of tree iterations which preserve large cardinals (cf. [4, 5, 6, 8, 10, 12]).

1. INTRODUCTION

An important technique in large cardinal set theory is that of extending an elementary embedding $j : M \to N$ of inner models to an elementary embedding $j^* : M[G] \to N[G^*]$ of generic extensions of them. For example, this is the historically first method to get a measurable cardinal κ with $2^\kappa > \kappa^+$. For obtaining such a model (via a reverse Easton iteration of forcings adding α^{++} many Cohen subsets to every inaccessible cardinal $\alpha \leq \kappa$) assuming a certain degree of strength of κ, a preliminary version of the generic G^* must be constructed (possibly in a further generic extension of $M[G]$) and then modified to provide the required G^*. As a complementary technique to the above-mentioned proof of Woodin, Friedman and Thompson suggested in [9] to use perfect trees, using *fusion* as a substitute for distributivity. This allowed them to provide, among other results, a new and easier proof of Woodin's theorem. Combining a result of Kanamori [15] about the application of iterated perfect tree forcing at κ (denoted by Sacks(κ) in the sequel) to Aronszajn trees on κ^{++} and the possibility of extending elementary embeddings, Dobrinen and Friedman [4] determined the exact consistency strength of the tree property at the double successor of a measurable. In addition, this way of getting the tree property is robust

enough: after a suitable collapse of the measurable to ω_ω the tree property holds at $\omega_{\omega+2}$, see [5]. Tree forcings could also be applied to questions involving the number of normal measures, see [8]. The use of the generalized Sacks forcing is also illustrated in [6, 7, 13]. In particular, in [13] a generalized Sacks forcing is used in the context of extender-based Prikry forcing. Yet another example of the use of fusion for extending elementary embeddings is given in [10], where a suitably defined uncountable version of the Miller forcing was applied to the cofinality of the symmetric group.

These results suggest that one should look for some general properties of a poset which would guarantee that iterations with supports of size $\leq \kappa$ do not collapse κ^+ and preserve the strength of κ modulo a suitable preparatory forcing. As was indicated in the papers mentioned above, such iterations might be of importance for the theory of cardinal characteristics at uncountable cardinals, for questions involving the number of normal measures, for the tree property, etc.

In Section 2 we isolate the properties of Miller(κ) needed for the preservation of κ^+ and lifting elementary embeddings in [10] and introduce the notion of good κ-fusion fulfilling the requirements from the previous paragraph. This notion seems to cover all known examples of large cardinal preservation with tree forcings. In addition, it *essentially* includes all κ^+-closed posets, see Example 2.2.

If we forget about the preservation of the strength, the remaining part of this problem (i.e., the preservation of κ^+ by iterations with supports of size $\leq \kappa$) has attracted the attention of Eisworth, Roslanowski, Shelah, and maybe others. As a result many properties guaranteeing this have been found. The idea behind most of them is to generalize properness to the uncountable setting, see [3, 17, 18, 19, 20].

In Section 3 we show that a suitable modification of some properties introduced in [17] will actually guarantee the preservation of the strength by κ-support iterations. This modification does not seem to cover variants of the Sacks forcing considered in [8], although it covers the "standard" Sacks poset Sacks(κ). On the other hand, this allows us to treat singular-splitting-Sacks forcing introduced in [8] (see also [12, 2.1]) and not covered by good κ-fusion. In addition, here we can have a normal filter on κ as a parameter which gives additional elbow room for applications to cardinal characteristics at κ, see Section 4.

In Section 5 we suggest a possible way to unify the main results of Sections 3 and 2 as well as state some open questions.

2. Generalizing the poset Miller(κ): good fusion

The purpose of this section is to introduce a property P of a poset which guarantees that: a) An iteration with supports of size κ of posets with the property P does not collapse κ^+; b) If κ is strong in V then it remains so in forcing extensions by posets described in the previous item; and c) Miller(κ) and various kinds of Sacks(κ) have the property P. An example of such a property P is given in Definition 2.1, and Theorems 2.9, 2.15, 2.22 are the main results of this section.

Throughout the section κ is assumed to be strongly inaccessible.

Definition 2.1. 1. By a *complete κ-tree* we mean a nonempty subtree of $\kappa^{<\kappa}$ which is closed under unions of its nodes (provided that this union is in $\kappa^{<\kappa}$) and has no maximal branches of length $< \kappa$. For a subtree T of $\kappa^{<\kappa}$ and $s \in T \cap \kappa^\alpha$ we denote by $\mathrm{succ}_T(s)$ the set of immediate successors of s, i.e. $\{s'(\alpha) : s' \in T \cap \kappa^{\alpha+1}, s' \restriction \alpha = s\}$. If $s \in T$ then $(T)_s$ is, by the definition, the tree $\{s' \in T : s' \text{ is comparable with } s\}$.

For a tree $T \subset \kappa^{<\kappa}$ we denote by $S(T)$ the set of all splitting nodes of T, i.e., the set $\{s \in T : |\mathrm{succ}_T(s)| > 1\}$. A complete κ-tree T will be called *perfect*, if the set $S(T)$ is nonempty, has no maximal elements, and is closed under unions of increasing sequences of its elements of length less than κ.

2. A forcing $\mathbb{P}$ has *good κ-fusion* iff there exist a map T from $\mathbb{P}$ to the set of all perfect κ-trees, a restriction function R assigning to each pair (p, s) such that $p \in \mathbb{P}$ and $s \in S(T(p))$ an element $R(p, s) \in \mathbb{P}$, often denoted by $(p)_s$,[1] and $h \in \kappa^\kappa$ satisfying the following properties:

(1) *(Basics).* For $p \in \mathbb{P}$ let $S(p)$ denote $S(T(p))$. Then
- $(p)_s \leq p$ for all $s \in S(p)$;
- If $q \leq p$ then $T(q) \subset T(p)$ (and hence $S(q) \subset S(p)$) and $(q)_s \leq (p)_s$ for $s \in S(q)$;
- If $t \supset s$ in $S(p)$ then $(p)_t \leq (p)_s$ and $(p)_s \neq (p)_t$ for any distinct $s, t \in S(p)$;
- If $s \in S(p)$, then $s \in S((p)_s)$ and $((p)_s)_s = (p)_s$;
- If s_0, s_1 are incomparable elements of $S(p)$ for some $p \in \mathbb{P}$, then $(p)_{s_0}$ and $(p)_{s_1}$ are incompatible;
- $\mathbb{P}$ is κ-closed and if $\langle p_\xi : \xi < \alpha \rangle$ is a decreasing sequence for some $\alpha < \kappa$, then there exists a lower bound p of this sequence such that $S(p) = \bigcap_{\xi < \alpha} S(p_\xi)$. For every decreasing sequence

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[1] In most of our proofs we use the notation $(p)_s$ and the function R is often passed over in silence.

$\langle p_\xi : \xi < \alpha \rangle$ of elements of $\mathbb{P}$ we shall fix such a lower bound and denote it by $\bigwedge_{\xi < \alpha} p_\xi$.

If $s \in T(p)$, then $(p)_s$ is, by definition, equal to $(p)_{s_1}$, where s_1 is the minimal extension of s to an element of $S(p)$.

(2) *(Fusion).* For every condition $p \in \mathbb{P}$ we set $S_\alpha(p) = S(p) \cap \alpha^{\leq \alpha}$, and for every $s \in S(p)$ we use the following notation: $\deg_p(s) = o.t.\left(\{t : t \subsetneqq s, t \in S(p)\}\right)$, $S_\alpha^*(p) = \{s \in S_\alpha(p) : \deg_p(s) = \alpha\}$. The notation $q \leq_\alpha p$ means that $q \leq p$ and $S_\alpha(q) = S_\alpha(p)$.

With the above notation we are in a position to formulate the property of $\mathbb{P}$ we are interested in: If $s \in S_\alpha^*(p)$, $s^\smallfrown \xi \in T(p)$, and $r \leq (p)_{s^\smallfrown \xi}$, then there exists $q \leq_\alpha p$ such that $s^\smallfrown \xi \in T(q)$ and $(q)_{s^\smallfrown \xi} \leq r$. [2]

(3) *(Closure).* Any sequence $\langle p_i : i < \kappa \rangle$ which satisfies $i < j \to p_j \leq_i p_i$ and $p_i = \bigwedge_{j < i} p_j$ for limit i has a lower bound p with the property $p \leq_i p_i$ for each $i < \kappa$.

(4) *(The Lifting Condition).* For each condition p there is a club $C(p)$ consisting of limit ordinals α such that whenever $s \in S_\alpha^*(p)$, then $s^\smallfrown h(\alpha) \in T(p)$.   □

Example 2.2. 1. Let $p \subset \kappa^{<\kappa}$. For $s \in p$ we denote by $C(p, s)$ (or simply by $C(s)$ if p is clear from the context) the set $\{\alpha \in \kappa : s^\smallfrown \alpha \in p\}$.

Following [10] we denote by Miller(κ) the following forcing. A condition is a subset p of $\kappa^{<\kappa}$ such that

(i) $s \in p, t \subset s \longrightarrow t \in p$.

(ii) Each $s \in p$ is increasing and has a proper extension in p.

(iii) For every $\alpha < \kappa$ limit, $s \in \kappa^\alpha$, if $s \restriction \beta \in p$ for arbitrary large $\beta < \alpha$, then $s \in p$.

(iv) For every $s \in p$ there is $t \in p$ with $s \subset t$ which splits in p (i.e., $C(p, t)$ has more than one element). Moreover, if t_0, t_1 split in p and $t_0 \subset t_1$, then $C(p, t_1) \subset C(p, t_0)$.

(v) If $s \in p$ splits in p, then the set $C(p, s)$ is club.

(vi) If α is a limit ordinal, $s \in \kappa^\alpha$, and $s \restriction \beta$ splits in p for arbitrary large $\beta < \alpha$, then s splits in p and $C(p, s)$ is the intersection of $C(p, s \restriction \beta)$ for all β such that $s \restriction \beta$ splits in p.

Miller(κ) is ordered by declaring p to be stronger than q (and write $p \leq q$) iff $p \subset q$.

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[2]Such a condition q will be called a *strengthening of p preserving the αth level, with r above $s^\smallfrown \xi$.* For our proofs we shall need this property only for $\xi = h(\alpha)$.

A direct verification shows that $\mathrm{Miller}(\kappa)$ has good κ-fusion witnessed by the function $h_{\mathrm{Miller}} : \alpha \mapsto \alpha$ for all $\alpha < \kappa$.

2. Let us fix a sequence $\vec{A} = \langle A_\alpha : \alpha < \kappa \rangle$ such that $A_\alpha \in [\kappa]^{<\kappa}$ for all α. Let $\mathcal{T}$ be the set of all functions t which satisfy the following conditions.

(i) There exists α such that the domain of t equals α.

(ii) For all $\beta \in \mathrm{dom}(t)$, $t(\beta) \in A_\beta$.

Following [10] we denote by $\mathrm{Sacks}(\vec{A})$ the forcing whose conditions are subsets T of $\mathcal{T}$ such that:

(iii) $s \in T$, $t \subset s \to t \in T$.

(iv) Each t has a proper extension in T.

(v) If $t \in \mathcal{T}$ and the set of such β that $t \upharpoonright \beta \in T$ is unbounded in $\mathrm{dom}(t)$, then $t \in T$.

(vi) There exists a club $C(T)$ such that the set $\mathrm{succ}_T(t)$ of immediate successors of an element $t \in T$ with domain α coincides with $\{t\hat{\ }a : a \in A_\alpha\}$ provided $\alpha \in C(T)$, and $|\mathrm{succ}_T(t)| = 1$ otherwise.

Extension is defined by $S \leq T$ iff S is a subset of T.

A direct verification shows that $\mathrm{Sacks}(\vec{A})$ has good κ-fusion witnessed by any function $h \in \prod_{\alpha<\kappa} A_\alpha$.

For certain sequences $\vec{A}$ the poset $\mathrm{Sacks}(\vec{A})$ has been considered in, e.g., [4, 8, 15].

3. More generally, suppose that $\mathbb{P}$ consists of perfect κ-trees ordered by inclusion and has the following properties:

(i) For every $s \in \kappa^{<\kappa}$ there exists a κ-complete filter $\mathcal{F}_s$ containing no singletons[3] and such that if $p \in \mathbb{P}$ and $s \in S(p)$, then $\mathrm{succ}_p(s) \in \mathcal{F}_s$;

(ii) If $s \in S(p)$, $s\hat{\ }\xi \in p$, and $r \leq (p)_{s\hat{\ }\xi}$, then there exists $q \leq p$ such that $s\hat{\ }\xi \in T(q)$, $(q)_{s\hat{\ }\xi} \leq r$, and all elements of $S(p)$ which are incompatible with $s\hat{\ }\xi$ belong to $S(q)$;
$\mathbb{P}$ is closed under intersections of decreasing sequences of its elements of length $< \kappa$;

(iii) For any sequence $\langle p_i : i < \kappa \rangle$ which satisfies $i < j \to S(p_j) \cap i^{\leq i} = S(p_i) \cap i^{\leq i}$ and $p_i = \bigcap_{j<i} p_j$ for limit i, the intersection $\bigcap_{j<\kappa} p_j$ is an element of $\mathbb{P}$ (observe that this intersection is a perfect κ-tree fulfilling the first item);

(iv) There exists $h \in \kappa^\kappa$ such that for each condition p there is a club $C(p)$ consisting of limit ordinals α such that whenever $s \in S(p) \cap \alpha^\alpha$ is a union of an increasing sequence of elements of $S(p)$ of length α, we have $s\hat{\ }h(\alpha) \in p$.

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[3]$\mathcal{F}_s$ does not have to be free, i.e., we do not require that $\cap \mathcal{F}_s = \emptyset$.

Then a direct verification shows that $\mathbb{P}$ has good κ-fusion.

4. Next, we shall give more examples of posets with good κ-fusion which will allow us to apply Theorems 2.9 and 2.15 in cases when some of the iterands are κ^+-closed, although κ^+-closed posets do not in general enjoy good κ-fusion. Let $\mathbb{P}$ be a poset and $\Vdash_{\mathbb{P}}$"$\mathbb{Q}$ has good κ-fusion witnessed by $\underset{\sim}{S}, \underset{\sim}{R}$, and $h \in \kappa^{\kappa} \cap V$". We say that $p \in \mathbb{P}$ *determines* μ, where μ is a $\mathbb{P}$-name for an element of $\mathbb{Q}$, if p decides the value of $\underset{\sim}{T}(\mu)$ and for every increasing finite sequence $\langle s_0, \ldots, s_n \rangle$ of elements of $\kappa^{<\kappa}$ there exists $A_{\langle s_0, \ldots, s_n \rangle} \in V$, $A_{\langle s_0, \ldots, s_n \rangle} \subset \kappa^{<\kappa}$, such that

$$p \Vdash \underset{\sim}{S}\big((\ldots ((\mu)_{s_0})_{s_1} \ldots)_{s_n}\big) = A_{\langle s_0, \ldots, s_n \rangle}$$

provided that p forces that $(\ldots ((\mu)_{s_0})_{s_1} \ldots)_{s_n}$ is well-defined. Now suppose that $\mathbb{P}$ is κ^+-closed. Then every condition in $\mathbb{P} * \mathbb{Q}$ can obviously be strengthened to a condition $\langle p, \mu \rangle$ such that p determines μ, and the (dense) set of such tuples $\langle p, \mu \rangle$ will be denoted by O.

We claim that O has good κ-fusion. Indeed, for every $\langle p, \mu \rangle \in O$ let us denote by $S\langle p, \mu \rangle$ the set $\underset{\sim}{S}(\mu) \in V$, as determined by p, and for every $s \in S\langle p, \mu \rangle$ let $R(\langle p, \mu \rangle, s) = \langle p, \mu \rangle_s$ be the condition[4] $\langle p, (\mu)_s \rangle$. From the above it follows that $\langle p, (\mu)_s \rangle \in O$. It suffices to show that S, R, and h witness for the good κ-fusion of O. This is rather routine and we shall check only Definition 2.1(2). Suppose that $\langle p, \mu \rangle \in O$, $s \in S^*_\alpha \langle p, \mu \rangle$, $s\hat{\ }\xi \in T\langle p, \mu \rangle$ (here $T\langle p, \mu \rangle$ denotes the subtree of $\kappa^{<\kappa}$ whose splitting nodes are exactly the elements of $S\langle p, \mu \rangle$), and $\langle p(s), \mu(s) \rangle \leq \langle p, \mu \rangle_{s\hat{\ }\xi}$. Since $\Vdash_{\mathbb{P}}$"$\mathbb{Q}$ has good κ-fusion", $p(s)$ forces that there exists ν such that $\nu \in \mathbb{Q}$, $\nu \leq_\alpha \mu$, $s\hat{\ }\xi \in T(\nu)$, and $(\nu)_{s\hat{\ }\xi} \leq \mu(s)$. Therefore there exists $p_1 \leq p(s)$ and a $\mathbb{P}$-name ν such that

$$p_1 \Vdash (\nu \in \underset{\sim}{\mathbb{Q}}) \wedge (\nu \leq_\alpha \mu) \wedge s\hat{\ }\xi \in T(\nu) \wedge (\nu)_{s\hat{\ }\xi} \leq \mu(s).$$

Now let $p_2 \leq p_1$ be such that p_2 determines ν. Then $\langle p_2, \nu \rangle \in O$, $\langle p_2, \nu \rangle \leq_\alpha \langle p, \mu \rangle$, $s\hat{\ }\xi \in T\langle p_2, \nu \rangle$, and $\langle p_2, \nu \rangle_{s\hat{\ }\xi} \leq \langle p(s), \mu(s) \rangle$.

A similar but easier argument shows that a product of a κ^+-closed poset and a poset with good κ-fusion also has good κ-fusion.

It follows from the above that there is no upper bound on the size of posets with good κ-fusion, and hence this notion encompasses not only posets consisting of subtrees of $\kappa^{<\kappa}$. □

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[4]Here we identify conditions $\langle p, \mu \rangle$ and $\langle p, \nu \rangle$ in $\mathbb{P} * \mathbb{Q}$ such that $p \Vdash \mu = \nu$. As a result we have $(\langle p, \mu \rangle_s)_s = \langle p, ((\mu)_s)_s \rangle = \langle p, (\mu)_s \rangle = \langle p, \mu \rangle_s$ for all $\langle p, \mu \rangle \in O$.

Below we collect some straightforward properties of posets with good κ-fusion. We shall often use them without mention. The proof of Theorem 2.15 will resemble that of the main result of [10], and the claims below will allow us to generalize the argument from the Miller case to posets with good κ-fusion by simply making sense out of the steps of the proof from [10] in our context.

Claim 2.3. *Suppose that a poset $\mathbb{P}$ has good κ-fusion witnessed by S, R, and h. Let $p \in \mathbb{P}$ and $s \in T(p)$. Then the statements below hold.*

  (i) *If $t \in S((p)_s)$, then t extends s;*
  (ii) *If $q \leq_\alpha p$, then $S_\alpha^*(q) = S_\alpha^*(p)$;*
  (iii) *If $p \in \mathbb{P}$, D is open dense on $\mathbb{P}$ and $\alpha < \kappa$, then there is $q \leq_\alpha p$ such that $(q)_{s\hat{}\xi}$ belongs to D for all $s \in S_\alpha^*(p)$ and $\xi < \kappa$ such that $s\hat{}\xi \in T(q)$;*
  (iv) *For every decreasing sequence $\langle p_\xi : \xi < \beta \rangle$ of elements of $\mathbb{P}$ of length $\beta < \kappa$, $s \in \bigcap_{\xi < \beta} S(p_\xi)$, and stationary set $A \subset \kappa$ there exists $\alpha \in A$ and $t \in \bigcap_{\xi < \beta} S_\alpha^*(p_\xi)$ such that $s \subset t$; and*
  (v) *If $s = \bigcup_{\xi < \beta} s_\xi$ with $s_\xi \in S_{\mu_\xi}^*(p)$, $s_\xi \subsetneqq s_\eta$ for all $\xi < \eta < \beta$, then $s \in S_\mu^*(p)$, where $\mu = \sup_{\xi < \beta} \mu_\xi$.*

Proof. Items $(ii), (iii)$, and (v) follow directly from corresponding items of Definition 2.1. For instance, (iii) is a consequence of Definition 2.1(2,3).

(i). Suppose, contrary to our claim, that $s \not\subset t$. Notice that $t \in S(p)$ because $(p)_s \leq p$. Then two cases are possible.

a) $t \subsetneqq s$. In this case Definition 2.1 yields $((p)_s)_t \leq (p)_s = ((p)_s)_s$ and $((p)_s)_s < ((p)_s)_t$ (because $t \subsetneqq s$) simultaneously, a contradiction.

b) The elements t and s of $\kappa^{<\kappa}$ are incomparable. Then $(p)_s$ and $(p)_t$ are incompatible. On the other hand, $((p)_s)_t \leq (p)_s$ and $((p)_s)_t \leq (p)_t$, a contradiction.

(iv). Let $p = \bigwedge_{\xi < \beta} p_\xi$. By induction on $\eta < \kappa$ construct a strictly increasing sequence $\langle s_\eta : \eta < \kappa \rangle$ of elements of $S(p)$ such that $s_\eta = \bigcup_{\xi < \eta} s_\xi$ for all limit η. Let C be the club consisting of limit ordinals η such that $s_\eta \in \eta^\eta$ and $\eta = \deg_p(s_\eta)$. Fix $\alpha \in C \cap A$. It follows from the above that α and $t = s_\alpha$ are as required. $\square$

The minimal element of $S(p)$, where p is a condition in a poset $\mathbb{P}$ with good κ-fusion, will be denoted by $\mathrm{stem}(p)$.

Claim 2.4. *If G is a $\mathbb{P}$-generic filter, then $x_G := \bigcup_{p \in G} \mathrm{stem}(p) \in \kappa^\kappa$. Therefore $x_G \restriction \alpha \in T(p)$ and $(p)_{x_G \restriction \alpha} \in G$ for every $p \in G$ and $\alpha < \kappa$.*

Proof. By Claim 2.3(i) for every $\alpha < \kappa$ the set of those conditions $p \in \mathbb{P}$ such that $\text{stem}(p) \in \kappa^\beta$ for some $\beta > \alpha$ is dense in $\mathbb{P}$. So it suffices to prove that for every $p_0, p_1 \in G$ the sequences $\text{stem}(p_0)$ and $\text{stem}(p_1)$ are compatible. Since G is a filter, there exists $p_2 \in G$ such that $p_2 \leq p_0, p_1$, and hence $S(p_2) \subset S(p_0) \cap S(p_1)$. Therefore $\text{stem}(p_2) \supset \text{stem}(p_0)$ and $\text{stem}(p_2) \supset \text{stem}(p_1)$, which proves the first statement of our claim.

Now let us fix $p, q \in G$ and $\alpha < \kappa$. Let $r \leq p, q$ be such an element of G that $\text{stem}(r)$ has length at least α. Then $x_G \restriction \alpha \subset \text{stem}(r) \in S(r) \subset S(p) \cap S(q)$, consequently $(r)_{\text{stem}(r)} \leq (p)_{\text{stem}(r)}, (q)_{\text{stem}(r)}$, and finally $(r)_{\text{stem}(r)} \leq (p)_{x_G \restriction \alpha}, (q)_{x_G \restriction \alpha}$. It follows from the above that $(p)_{x_G \restriction \alpha}$ is compatible with every element of G and hence belongs to G. $\square$

Since the set $\{p \in \mathbb{P} : \text{stem}(p) \not\subset x\}$ is dense in $\mathbb{P}$ for every $x \in \kappa^\kappa \cap V$, the sequence x_G defined in Claim 2.4 is not in V.

Throughout this section κ is an inaccessible cardinal and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\zeta : \xi \leq \gamma, \zeta < \gamma \rangle$ stands for a κ-support iteration such that $\Vdash_{\mathbb{P}_\xi}$ "$\underset{\sim}{\mathbb{Q}}_\xi$ has good κ-fusion witnessed by $\underset{\sim}{S}^\xi, \underset{\sim}{R}_\xi, \check{h}_\xi$" for all $\xi < \gamma$ (i.e., $h_\xi \in \kappa^\kappa \cap V$). We shall write $S(q)$ instead of $\underset{\sim}{S}^\xi(q)$ provided that it is clear from the context that q is a name for a condition in $\underset{\sim}{\mathbb{Q}}_\xi$.

Definition 2.5. Suppose that $\alpha \in \kappa$, $F \in [\gamma]^{<\kappa}$, and $q, p \in \mathbb{P}_\gamma$. $q \leq_{F,\alpha} p$ means that $q \leq p$ and $q \restriction \xi \Vdash q(\xi) \leq_\alpha p(\xi)$ for all $\xi \in F$.

If $\alpha < \kappa$ and $\langle p_\xi : \xi < \alpha \rangle$ is a decreasing sequence of elements of $\mathbb{P}_\gamma$, then the condition $p := \bigwedge_{\xi < \alpha} p_\xi$ is defined[5] as follows: $p \restriction i \Vdash_i p(i) = \bigwedge_{\xi < \alpha} p_\xi(i)$.

A sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ is a *generalized fusion sequence (for $\bar{\mathbb{Q}}$)*, iff

(i) $|F_\alpha| < \kappa$ for all $\alpha \in \kappa$.
(ii) $F_\alpha \supset F_\beta$ for all $\beta \leq \alpha < \kappa$.
(iii) $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ for all α.
(iv) If δ is limit, then $F_\delta = \bigcup_{\beta < \alpha} F_\beta$ and $p_\delta = \bigwedge_{\alpha < \delta} p_\alpha$.
(v) $\bigcup \{F_\alpha : \alpha \in \kappa\} = \bigcup \{\text{supp}(p_\alpha) : \alpha < \kappa\}$. $\square$

The easy but technical proof of the following lemma is left to the reader.

Lemma 2.6. *Let $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ be a generalized fusion sequence for $\bar{\mathbb{Q}}$. Then there exists $q \in \mathbb{P}_\gamma$ such that $q \leq_{F_\alpha,\alpha} p_\alpha$ for all $\alpha \in \kappa$.*

In what follows for every generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ for $\bar{\mathbb{Q}}$ we shall fix such q as in Lemma 2.6 and denote it by $\bigwedge_{\alpha < \kappa} p_\alpha$.

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[5]If $\alpha \geq \kappa$ then $\bigwedge_{\xi < \alpha} p_\xi$ is not always defined.

Definition 2.7. Suppose that $p \in \mathbb{P}_\gamma$, $F \subset \mathrm{supp}(p)$ with $|F| < \kappa$, and $\sigma : F \to \kappa^{<\kappa}$. Then $p|\sigma$ is a function with the same domain as p such that $(p|\sigma)(\xi)$ equals $(p(\xi))_\emptyset$ if $\xi \in \mathrm{supp}(p) \setminus F$ and $(p(\xi))_{\sigma(\xi)}$ otherwise. □

It is clear that $p|\sigma \in \mathbb{P}_\gamma$ if and only if for every $\xi \in F$ we have $(p|\sigma) \upharpoonright \xi \Vdash_\xi \sigma(\xi) \in T(p(\xi))$. If $p|\sigma \in \mathbb{P}_\gamma$, then we say that σ *lies on* p.

Observation 2.8. *Let $\bar{\mathbb{Q}}$ be as above.*

(i) *If $\sigma : F \to \kappa^{<\kappa}$ lies on p, then σ lies on q for every $q \leq p|\sigma$;*

(ii) *If σ lies on p, π lies on q, $\mathrm{dom}(\pi) \supset \mathrm{dom}(\sigma)$, $\sigma(\xi) \subset \pi(\xi)$ for all $\xi \in \mathrm{dom}(\sigma)$, and $p|\sigma \leq q|\pi$, then π lies on p and $p|\sigma = p|\pi = (p|\sigma)|\pi$.*

Proof. The first item can be easily proved by induction on $\xi < \gamma$ using Claim 2.3(i).

Let us prove the second item. Since $p|\sigma \leq q|\pi$, the first item implies that π lies on $p|\sigma$. Let us show by induction on $\xi < \gamma$ that $\pi \upharpoonright \xi$ lies on $p \upharpoonright \xi$ and $(p|\sigma) \upharpoonright \xi = (p|\pi) \upharpoonright \xi$. For $\xi = 1$ this is obvious. Suppose that this is true for all $\xi < \eta$. If η is limit, then the statement above is also true for η. So it suffices to consider the case $\eta = \xi + 1$. If $\xi \notin \mathrm{dom}(\pi)$ then there is nothing to prove. So suppose that $\xi \in \mathrm{dom}(\pi)$. Since $p|\sigma \leq q|\pi$, we have $(p|\sigma) \upharpoonright \xi \Vdash_\xi (p|\sigma)(\xi) \leq (q(\xi))_{\pi(\xi)}$, and consequently $(p|\sigma) \upharpoonright \xi \Vdash_\xi \mathrm{stem}((p|\sigma)(\xi)) \supset \pi(\xi)$. Therefore by Definition 2.1(1), Claim 2.3(i), and Definition 2.7 we have that

$$(p|\sigma) \upharpoonright \xi \Vdash_\xi ((p|\sigma)(\xi))_{\pi(\xi)} = (p(\xi))_{\pi(\xi)} = (p|\pi)(\xi) \ \wedge$$
$$\wedge \ ((p|\sigma)(\xi))_{\pi(\xi)} = ((p|\sigma)(\xi))_\emptyset = (p|\sigma)(\xi)$$

which means that $(p|\sigma) \upharpoonright \xi + 1 = (p|\pi) \upharpoonright \xi + 1$ and thus completes our inductive proof of the fact that $p|\sigma = p|\pi$. Finally, $p|\sigma \geq (p|\sigma)|\pi \geq (p|\pi)|\pi = p|\pi$, which together with $p|\sigma = p|\pi$ implies $p|\sigma = p|\pi = (p|\sigma)|\pi$. □

Theorem 2.9. *The forcing $\mathbb{P}_\gamma$ defined just before Definition 2.5 preserves cardinals $\leq \kappa^+$.*

Suppose that $2^\kappa = \kappa^+$ in V, $\gamma = \kappa^{++}$, and $\Vdash_{\mathbb{P}_\xi} |\mathbb{Q}_\xi| \leq \kappa^+$ for all $\xi < \gamma$. Then $\mathbb{P}_\gamma$ has the κ^{++}-chain condition.

Similar results were discussed in [4, 10, 15] for the Sacks and Miller forcings. Nevertheless, we give complete proofs here. Our exposition closely follows [10].

The first part of Theorem 2.9 follows from the lemma below.

Lemma 2.10. (1) *Assume that $p \in \mathbb{P}_\gamma$ and $p \Vdash z \in V$. Then for every $F \in [\gamma]^{<\kappa}$ and $\alpha_0 \in \kappa$ there exists $q \leq_{F,\alpha_0} p$ and $x \in V$ with $|x| \leq \kappa$ such that $q \Vdash z \in x$.*

(2) *Assume that $p \in \mathbb{P}_\gamma$ and $p \Vdash$ "$\underset{\sim}{z} \in V$ and $|\underset{\sim}{z}| \leq \kappa$". Then for every $F \in [\gamma]^{<\kappa}$ and $\alpha_0 \in \kappa$ there exists $q \leq_{F,\alpha_0} p$ and $x \in V$ with $|x| \leq \kappa$ such that $q \Vdash \underset{\sim}{z} \subset x$.*

Proof. It is well-known how to obtain the second item from the first one, see, e.g., [15, Theorem 2.3].

In order to prove the first item we shall inductively construct a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$ with $(p_\beta, F_\beta) = (p, F)$ for all $\beta \leq \alpha_0$, and $x \in V$ of size $|x| \leq \kappa$ such that $q = \bigwedge_{\alpha \in \kappa} p_\alpha$ and x are as required. The routine description of how to construct the F_α's is omitted. The limit step of the construction is obvious, so we concentrate on the successor case.

Let us enumerate as $\{\sigma_{\alpha,i} : i \in \eta_\alpha\}$ all ground model functions $\sigma : F_\alpha \to \kappa^{\alpha+1}$ which lie on some $r \leq p_\alpha$ so that $r = r|\sigma$, $r \restriction \xi \Vdash \sigma(\xi) \restriction \alpha \in S_\alpha^*(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, and $\sigma(\xi)(\alpha) = h_\xi(\alpha)$ for all $\xi \in F_\alpha$. (Here $\eta_\alpha < \kappa$ is a cardinal.) We shall construct a sequence $\langle p_{\alpha,i} : i \in \eta_\alpha \rangle$ as follows. Set $p_{\alpha,-1} = p_\alpha$ and suppose that we have already constructed a decreasing sequence $\langle p_{\alpha,j} : j < i \rangle$ such that $p_{\alpha,j} \leq_{F_\alpha,\alpha} p_{\alpha,k}$ for all $k \leq j < i$. If i is limit, we set $p_{\alpha,i} = \bigwedge_{j<i} p_{\alpha,j}$. Suppose that $i = j + 1$. If there is no $r \leq p_{\alpha,j}$ such that $r = r|\sigma_{\alpha,j}$ and $r \restriction \xi \Vdash \sigma_{\alpha,j}(\xi) \restriction \alpha \in S_\alpha^*(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, we set $p_{\alpha,i} = p_{\alpha,j}$ and $x_{\alpha,j} = \emptyset$. And if there is such r, let $r_{\alpha,j} \leq r$ and $x_{\alpha,j} \in V$ be such that $r_{\alpha,j} \Vdash \underset{\sim}{z} = x_{\alpha,j}$. Now, using the Maximal Principle we define $p_{\alpha,j+1}$ to be the amalgamation of $p_{\alpha,j}$ and $r_{\alpha,j}$ as in the proof of [15, Theorem 2.2]. More precisely,

(a) $\text{supp}(p_{\alpha,j+1}) = \text{supp}(r_{\alpha,j})$.

(b) If $\xi \in F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$r_{\alpha,j} \restriction \xi$ forces $p_{\alpha,j+1}(\xi)$ to be the strengthening of $p_{\alpha,j}(\xi)$ preserving the αth level, with $r_{\alpha,j}(\xi)$ above $\sigma_{\alpha,j}(\xi)$,

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

(c) If $\xi \notin F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$$r_{\alpha,j} \restriction \xi \Vdash p_{\alpha,j+1}(\xi) = r_{\alpha,j}(\xi),$$

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

Now we let $p_{\alpha+1} = \bigwedge_{i \in \eta_\alpha} p_{\alpha,i}$. It follows that $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$. This completes our construction of $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$. Set $x = \{x_{\alpha,i} : \alpha \in \kappa, i \in \eta_\alpha\}$.

We continue the proof of Lemma 2.10 with the following two auxiliary statements.

Claim 2.11. *Suppose that $r \leq q$, where q is a condition constructed above. Then there exists a sequence $\langle r_\alpha : \alpha \in \kappa \rangle$ of elements of $\mathbb{P}_\gamma$ with $r_0 = r$, a sequence $\langle \sigma_\alpha : F_\alpha \to \kappa^{<\kappa} | \alpha \in \kappa \rangle$, and sequences $\langle \mu_{\alpha,\xi} : \alpha \in \kappa, \xi \in F_\alpha \rangle$ of ordinals less than κ such that*

   (i) *If $\beta < \alpha$, then $r_\alpha \leq r_\beta$.*
   (ii) *If $\xi \in F_\alpha$, then $\mathrm{dom}(\sigma_\alpha(\xi)) = \mu_{\alpha,\xi} + 1$ and $\sigma_\alpha(\mu_{\alpha,\xi}) = h_\xi(\mu_{\alpha,\xi})$.*
   (iii) *If $\beta < \alpha$, then $\sigma_\beta(\xi) \subsetneq \sigma_\alpha(\xi)$ for all $\xi \in F_\beta$.*
   (iv) *For every $\xi \in F_{\alpha+1}$ we have*
      $r_{\alpha+1} \restriction \xi \Vdash$ *"$\mu_{\alpha+1,\xi} \in \bigcap_{\beta \leq \alpha} C(r_\beta(\xi))$, $r_{\alpha+1}(\xi) = (r_{\alpha+1}(\xi))_{\sigma_{\alpha+1}(\xi)}$,*
      and $\sigma_{\alpha+1}(\xi) \restriction \mu_{\alpha+1,\xi} \in \big[\bigcap_{\beta \leq \alpha} S^*_{\mu_{\alpha+1,\xi}}(r_\beta(\xi)) \big] \cap S^*_{\mu_{\alpha+1,\xi}}(q(\xi))$".
   (v) *If δ is limit, then*
      − *$\mu_{\delta,\xi} = \sup_{\alpha < \delta} \mu_{\alpha,\xi}$ for all $\xi \in F_\delta$;*
      − *$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} = \bigcup_{\alpha < \delta} \sigma_\alpha(\xi)$ for all $\xi \in F_\delta$*
      (we assume that $\sigma_\alpha(\xi) = \emptyset$ for all $\xi \notin F_\alpha$);
      − *$r_\delta \restriction \xi \Vdash$ "$\sigma_\delta(\xi) \in T(r_\delta(\xi))$ and*
      $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in \big[\bigcap_{\beta < \delta} S^*_{\mu_{\delta,\xi}}(r_\beta(\xi)) \big] \cap S(r_\delta(\xi)) \cap S^*_{\mu_{\delta,\xi}}(q(\xi))$" for
      all $\xi \in F_\delta$.

Proof. The construction proceeds by induction. For limit δ we simply set $\sigma_\delta(\xi)$ and $\mu_{\delta,\xi}$ to be as required in (ii,v) and $r_\delta = \bigwedge_{\alpha < \delta} r_\alpha$. Thus $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in \big[\bigcap_{\beta < \delta} S^*_{\mu_{\delta,\xi}}(r_\beta(\xi)) \big] \cap S^*_{\mu_{\delta,\xi}}(q(\xi))$ by Claim 2.3(v). Since $\Vdash_{\mathbb{P}_\xi} S(r_\delta(\xi)) = \bigcap_{\alpha < \delta} S(r_\alpha(\xi))$, it follows from the above that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in S(r_\delta(\xi))$. It remains to show that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \in T(r_\delta(\xi))$.

From the conditions (i)-(v) and the inductive assumption for every $\alpha < \beta < \delta$ we have that $r_\delta \restriction \xi \Vdash \mu_{\beta,\xi} \in C(r_\alpha(\xi))$, and hence $r_\delta \restriction \xi \Vdash \mu_{\delta,\xi} \in C(r_\alpha(\xi))$. Therefore $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \in T(r_\alpha(\xi))$ by the definition of $C(r_\alpha(\xi))$ and $\sigma_\delta(\xi)(\mu_{\delta,\xi}) = h_\xi(\mu_{\delta,\xi})$. Let us fix any $\mathbb{P}_\xi$-generic filter H containing $r_\delta \restriction \xi$ and in $V[H]$ find an increasing sequence $\langle s_\alpha : \alpha < \delta \rangle$ of elements of $\kappa^{<\kappa}$ such that $\sigma_\delta(\xi) \leq s_\alpha \in S(r_\alpha(\xi))$ for all $\alpha < \delta$. Then $s = \bigcup_{\zeta < \delta} s_\zeta \in S(r_\alpha(\xi))$ for every $\alpha < \delta$, and hence $s \in S(r_\delta(\xi)) = \bigcap_{\alpha < \delta} S(r_\alpha(\xi))$, and finally $\sigma_\delta(\xi) \in T(r_\delta(\xi))$ because $\sigma_\delta(\xi) \leq s$. Since H was chosen arbitrary, we conclude that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \in T(r_\delta(\xi))$ which completes the limit case of our proof.

At successor step $\alpha + 1$ consider the increasing enumeration $\langle \xi_i : i < \eta \rangle$ of $F_{\alpha+1}$ and find a decreasing sequence $\langle u_i : i < \eta \rangle$ of elements of $\mathbb{P}_\gamma$ as follows: Set $u_i = \bigwedge_{j < i} u_j$ for limit i. Now given u_i, using Claim 2.3(iv) and Definition 2.1(4) find $v \leq u_i \restriction \xi_i$ and $\pi \in \kappa^{\mu+1}$ for some $\mu \in \kappa$ such that the

following conditions are satisfied provided that $\xi_i \in F_{\alpha+1}$:

$$\pi \supset \sigma_\alpha(\xi_i), \ \pi(\mu) = h_{\xi_i}(\mu), \ \text{and}$$

$$v \Vdash_{\xi_i} \text{``}\mu \in \bigcap_{\beta \leq \alpha} C(r_\beta(\xi_i)) \ \wedge \ \pi \upharpoonright \mu \in \big[\bigcap_{\beta \leq \alpha} S_\mu^*(r_\beta(\xi_i))\big] \cap S_\mu^*(q(\xi_i))\text{''}.$$

Then we set

$$u_{i+1} = v ^\frown (r_\alpha(\xi_i))_\pi ^\frown r_\alpha \upharpoonright (\gamma \setminus (\xi_i + 1)),$$

$\sigma_{\alpha+1}(\xi_i) = \pi$. ($\mu_{\alpha+1,\xi_i}$ automatically becomes equal to μ.) With u_i's thus defined, we set $r_{\alpha+1} = \bigwedge_{i < \eta} u_i$. This completes the inductive construction, hence the proof of the claim. $\qquad\square$

The following claim is obvious.

Claim 2.12. *There exists a club $C \subset \kappa$ such that $\mu_{\alpha,\xi} = \alpha$ for every $\alpha \in C$ and $\xi \in F_\alpha$. Consequently, $r_\alpha \upharpoonright \xi \Vdash \sigma_\alpha(\xi) \upharpoonright \alpha \in S_\alpha^*(q(\xi))$ for every such $\alpha \in C$ and $\xi \in F_\alpha$.*

We are in a position now to finish the proof of Lemma 2.10. Let C be such as in Claim 2.12 and $\alpha \in C$. Then $\sigma_\alpha = \sigma_{\alpha,i}$ for some $i < \eta$ by Claim 2.3(*iii*) (see the construction of $p_{\alpha+1}$ at the beginning of the proof of Lemma 2.10). Since $r_{\alpha+1} \leq q \leq p_{\alpha,i}$, Claim 2.11(*iv*) implies that for every $\xi \in F_{\alpha+1} \supset F_\alpha$ we have $r_{\alpha+1} \upharpoonright \xi \Vdash r_{\alpha+1}(\xi) = (r_{\alpha+1}(\xi))_{\sigma_\alpha(\xi)}$. (Indeed, by Claim 2.3(*i*) and equality $r_{\alpha+1}(\xi) = (r_{\alpha+1}(\xi))_{\sigma_{\alpha+1}(\xi)}$, for every $t \in S(r_{\alpha+1}(\xi))$ we have $t \supset \sigma_{\alpha+1}(\xi)$, and hence $r_{\alpha+1}(\xi) = (r_{\alpha+1}(\xi))_{\sigma_{\alpha+1}(\xi)} = (r_{\alpha+1}(\xi))_{\sigma_\alpha(\xi)}$.) Therefore the construction of $p_{\alpha,i+1}$ is nontrivial. Since $r_{\alpha+1} \leq q \leq p_{\alpha,i+1}$, $r_{\alpha+1} = r_{\alpha+1}|\sigma_\alpha \leq p_{\alpha,i+1}|\sigma_{\alpha,i} \leq r_{\alpha,i}$, and hence $r_{\alpha+1} \Vdash \underset{\sim}{z} = x_{\alpha,i}$. Therefore for every $r \leq q$ there exists $r' \leq r$ such that $r' \Vdash \underset{\sim}{z} \in x$, which finishes our proof. $\qquad\square$

Let χ be a regular cardinal much bigger than κ. Following [3] we define an elementary submodel N of $H(\chi)$ to be *relevant*, if $|N| = \kappa$, $N^{<\kappa} \subset N$, and N can be written as a union $\bigcup_{\alpha < \kappa} N_\alpha$, where $\langle N_\alpha : \alpha < \kappa \rangle$ is a continuous $\in$-increasing chain of elementary submodels of $H(\chi)$ such that $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$ and $|N_\alpha| < \kappa$ for all $\alpha < \kappa$.

A poset $\mathbb{P}$ is *κ-proper*, if for every relevant model N containing $\{\mathbb{P}, \ldots\}$ and $p \in \mathbb{P} \cap N$ there exists $q \leq p$ which is $(N, \mathbb{P})$-generic, i.e. for every dense subset D of $\mathbb{P}$ which is an element of N, the intersection $D \cap N$ is pre-dense below q. In contrast to the properness, the κ-properness is not preserved by κ-support iterations, see [22, App. 3.6(2)]. However, it is easy to check that if $\mathbb{P}$ is κ-proper and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is κ-proper", then $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is κ-proper as well.

Lemma 2.13. *The forcing $\mathbb{P}_\gamma$ defined just before Definition 2.5 is κ-proper.*

Proof. Let $N \supset \{\mathbb{P}_\gamma, \ldots\}$ and $p \in \mathbb{P}_\gamma \cap N$. Let $\langle D'_\alpha : \alpha < \kappa \rangle$ be the enumeration of all open dense subsets of $\mathbb{P}_\gamma$ which are elements of N and set $D_\alpha = \bigcap_{\beta \leq \alpha} D'_\beta$. Now, repeating the proof of Lemma 2.10 with the additional requirement[6] $r_{\alpha,j} \in D_\alpha$ and in such a way that the p_α's are all in N we arrive at some $q \leq p$. We claim that q is $(N, \mathbb{P}_\gamma)$-generic. Indeed, given any $r \leq q$ and $\beta < \kappa$, there exists a decreasing sequence $\langle r_\alpha : \alpha < \kappa \rangle$ of elements of $\mathbb{P}_\gamma$ below r (namely the sequence given by Claim 2.11) such that there are $\alpha \geq \beta$ and i with the property $r_{\alpha+1} \leq r_{\alpha,i}$ (see the paragraph after Claim 2.12). Since $r_{\alpha,i} \in D_\alpha$ by the construction of q, we have $r_{\alpha+1} \in D_\alpha \subset D'_\beta$, which completes our proof. $\square$

The second part of Theorem 2.9 is a direct consequence of Lemma 2.13 and the following theorem.

Theorem 2.14. [3, Proposition 3.1.]. *Assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ in V, and let $\langle \mathbb{R}_i, \mathbb{S}_i : i < \lambda^{++} \rangle$ be a λ-support iteration such that $\mathbb{R}_i$ is λ-proper and $\Vdash_{\mathbb{R}_i} |\underset{\sim}{\mathbb{S}}_i| \leq \lambda^+$ for all i. Then $\mathbb{R}_{\lambda^{++}}$ has the λ^{++}-c.c.*

Assume GCH in V and let κ be an inaccessible limit of inaccessible cardinals in V. We define in V a preparatory forcing $\mathbb{R}_\kappa$ as follows. Let $\mathbb{R}_0$ be the trivial forcing. For $i < \kappa$ let $\underset{\sim}{\mathbb{S}}_i$ be an $\mathbb{R}_i$-name for the *sum* of all $< \rho_i$-closed posets whose underlying set is a subset of $H(\rho_i^{++})^{V^{\mathbb{R}_i}}$, where ρ_i is the ith inaccessible cardinal below κ. In other words, let $\underset{\sim}{\mathbb{S}}_i$ be an $\mathbb{R}_i$-name for the poset $\{\langle \mathbb{S}, s \rangle : \mathbb{S}$ is a $< \rho_i$-directed closed posets whose underlying set is a subset of $H(\rho_i^{++})^{V^{\mathbb{R}_i}}$ and $s \in \mathbb{S}\} \cup \{1\}$, ordered with 1 above everything else and $\langle \mathbb{S}, s \rangle \leq \langle \mathbb{S}', s' \rangle$ when $\mathbb{S} = \mathbb{S}'$ and $s \leq s'$. Let $\mathbb{R}_\kappa$ be the iteration $\langle \mathbb{R}_\xi, \underset{\sim}{\mathbb{S}}_\xi : \xi \leq \kappa \rangle$ with Easton support. The poset $\mathbb{R}_\kappa$ is a subset of $H(\kappa)$ and it is well-known [1, Corollary 2.4] that $\mathbb{R}_\kappa$ has κ-c.c.

Theorem 2.15. *Suppose GCH holds and $j : V \to M$ is an ultrapower embedding via a (κ, κ^{++})-extender in V such that $H(\kappa^{++})$ of V is contained in M. Also let $\mathbb{R}_\kappa$ be the "preparatory" forcing defined above and in $V^{\mathbb{R}_\kappa}$ let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ be an iteration of forcings with good κ-fusion as above, where $\gamma \leq \kappa^{++}$. Then j can be extended in $V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma}$ to an elementary embedding $j^{**} : V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma} \to M^{j(\mathbb{R}_\kappa * \mathbb{P}_\gamma)}$ so that the $H(\kappa^{++})$ of $V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma}$ is contained in $M^{j(\mathbb{R}_\kappa * \mathbb{P}_\gamma)}$. In particular, κ remains measurable in $V^{\mathbb{R}_\kappa * \mathbb{P}_\gamma}$.*

Proof. The poset $j(\mathbb{R}_\kappa)$ is an iteration $\langle \bar{\mathbb{R}}_\xi, \bar{\mathbb{S}}_\xi : \xi < j(\kappa) \rangle$ of length $j(\kappa)$ in M with Easton support. It is clear that $\bar{\mathbb{R}}_\xi = \mathbb{R}_\xi$ and $\bar{\mathbb{S}}_\xi = \underset{\sim}{\mathbb{S}}_\xi$ for all $\xi \leq \kappa$.

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[6]Notice that we could take $r_{\alpha,j}$ to be *any* strengthening of a certain condition r in that proof.

In addition, $\mathbb{S}_\kappa$ is an $\mathbb{R}_\kappa$-name for the sum of all $< \kappa = \rho_\kappa$-closed posets whose underlying set is a subset of $H(\kappa^{++})^{V^{\mathbb{R}_\kappa}}$.

Let G be an $\mathbb{R}_\kappa$-generic filter over V. Since $\mathbb{R}_\kappa$ has κ-c.c. and M and V have the same $H(\kappa^{++})$, so do $M[G]$ and $V[G]$ (see the proof of Claim 2.16 below for a slightly more involved argument), and hence (an isomorphic copy of) $\mathbb{P}_\gamma$ is among the summands in $\bar{\mathbb{S}}_\kappa$. Let g be a $\mathbb{P}_\gamma$-generic over $V[G]$. Then the set $\{\langle \mathbb{P}_\gamma, p \rangle : p \in g\} \cup \{1^G_{\underset{\sim}{\mathbb{S}}_\kappa}\} \in V[G]$ is a $\bar{\mathbb{S}}^G_\kappa$-generic over $M[G]$, and we shall identify it with g.

The following claim is analogous to [4, Lemma 4.4] and to [10, Claim 5.2]. Its proof is given for the sake of completeness.

Claim 2.16. *$M[G * g]$ and $V[G * g]$ have the same $H(\kappa^{++})$. Moreover, $M[G*g]$ (resp. $M[G]$) is closed under κ sequences in $V[G*g]$ (resp. $V[G]$).*

Proof. To show that $H(\kappa^{++})$ of $M[G * g]$ and $V[G * g]$ coincide it is enough to prove that these models have the same subsets of κ^+. Let us fix $X \in \mathcal{P}(\kappa^+) \cap V[G * g]$ with a $\mathbb{R}_\kappa * \mathbb{P}_\gamma$-name $\underset{\sim}{X}$ and for every $\alpha \in \kappa^+$ fix an antichain $A_\alpha \in V$ in $\mathbb{R}_\kappa * \mathbb{P}_\gamma$ such that each element of A_α forces $\alpha \in \underset{\sim}{X}$ and A_α is maximal with this property. By Theorem 2.9 each A_α has size at most κ^+. Combining this with the fact that $\mathbb{R}_\kappa * \mathbb{P}_\gamma \subset H(\kappa^{++})$ we conclude that the whole sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ belongs to $H(\kappa^{++})$ and hence is an element of M. Thus $X = \underset{\sim}{X}^{G*g} = \{\alpha \in \kappa^+ : G * g \cap A_\alpha \neq \emptyset\} \in M[G * g]$, which completes the proof of the first part of the claim.

Regarding the second part of the claim, we shall prove only the $G*g$ case. The other case is analogous. Let us fix $X \subset M[G * g]$ with a $\mathbb{R}_\kappa * \mathbb{P}_\gamma$-name $\underset{\sim}{X}$ such that $V[G * g] \vDash |X| \leq \kappa$. Since $\mathbb{R}_\kappa * \mathbb{P}_\gamma$ is κ-proper as an iteration of two κ-proper posets, there exists $Y \in V$ such that Y has size $\leq \kappa$ in V and $X \subset Y$. For every $y \in Y$ let D_y be the set of those conditions in $\mathbb{R}_\gamma * \mathbb{P}_\gamma$ which determine whether $y \in \underset{\sim}{X}$ or not. Then D_y is an open dense subset of $\mathbb{R}_\kappa * \mathbb{P}_\gamma$. Given a condition u in $\mathbb{R}_\kappa * \mathbb{P}_\gamma$, the κ-properness of $\mathbb{R}_\kappa * \mathbb{P}_\gamma$ yields a condition $w \leq u$ such that for every $y \in Y$ there exists $A_y \in [D_y]^\kappa$ which is predense below w. Therefore $G * g$ contains such a condition w. By the choice of j we have that $M^\kappa \subset M$, and hence $\langle A_y : y \in Y \rangle \in M$, and hence $X = \{y \in Y : \text{some of the (equivalently, all) elements of } A_y \cap G * g \text{ forces } y \in \underset{\sim}{X}\}$ belongs to $M[G * g]$. □

Claim 2.16 allows us to find a $j(\mathbb{R}_\kappa) \restriction (\kappa, j(\kappa))$-generic filter $H \in V[G*g]$ over $M[G * g]$, see the proof of [4, Th. 4.1] for more details. Thus $j[G] = G \subset G * g * H$, and hence j lifts to an embedding $j^* : V[G] \to M[G * g * H]$ definable in $V[G * g]$, see [2, Prop. 9.1]. Let M^* denote $M[G * g * H]$.

Next, we shall extend j^* to an elementary embedding $j^{**} : V[G * g] \to$ $M^*[\hat{g}]$ for some $j^*(\mathbb{P}_\gamma)$-generic filter $\hat{g}$. By [2, Prop. 9.1] it is enough to find a $j^*(\mathbb{P}_\gamma)$-generic $\hat{g} \in V[G * g]$ over M^* for which $j^*[g] \subset \hat{g}$. For this we shall need auxiliary Definition 2.17 and Claims 2.18 and 2.19, which we present in full generality below.

Definition 2.17. Let ρ be a strongly inaccessible cardinal and $\bar{\mathbb{U}} = \langle \mathbb{T}_\xi, \mathbb{U}_\xi :$ $\xi < \nu \rangle$ be a ρ-support iteration such that $\mathbb{U}_\xi$ is forced to have good ρ-fusion witnessed by $\underline{S}^\xi$, $\underline{R}_\xi$, and h_ξ. Let $q \in \mathbb{T}_\nu$, $i \in \rho$, and $F \in [\nu]^{<\rho}$. We say that a function $\sigma : F \to \rho^{i+1}$ *lies (F, i)-potentially on q*, if $\sigma(\xi)(i) = h_\xi(i)$ for all $\xi \in F$ and σ lies on some $r \leq q$ such that $r \upharpoonright \xi \Vdash \sigma(\xi) \upharpoonright i \in S_i^*(q(\xi))$ for all $\xi \in F$.

Claim 2.18. *Let ρ, $\bar{\mathbb{U}}$, q, i be such as in Definition 2.17 and $q^* \in \mathbb{T}_\nu$ be stronger than q. If $\sigma : T \to \rho^{i+1}$ lies (T, i)-potentially on q^*, then $\sigma \upharpoonright F$ lies (F, i)-potentially on q for every $F \subset T$.*

Proof. Let $r \in \mathbb{T}_\nu$ be a condition witnessing that σ lies (T, i)-potentially on q^* and such that $r|\sigma = r$. We claim that r also witnesses that $\sigma \upharpoonright F$ lies (F, i)-potentially on q. For this we have to show that $r \upharpoonright \xi \Vdash \sigma(\xi) \upharpoonright i \in S_i^*(q(\xi))$ for all $\xi \in F$. It follows from the hypotheses that $r \upharpoonright \xi \Vdash \sigma(\xi) \upharpoonright i \in S_i^*(q^*(\xi))$ for all $\xi \in T \supset F$ and $r \upharpoonright \xi \Vdash q^*(\xi) \leq q(\xi)$. Let us fix some $\xi \in F$. By Definition 2.1 we know that $r \upharpoonright \xi \Vdash$ "$\sigma(\xi) \upharpoonright i \in S_i(q^*(\xi))$ and $\deg_{q^*(\xi)}(\sigma(\xi) \upharpoonright i) = i$". Since $r \upharpoonright \xi \Vdash S(q^*(\xi)) \subset S(q(\xi))$, we have that $r \upharpoonright \xi \Vdash$ "$\sigma(\xi) \upharpoonright i \in S(q(\xi))$ and $\deg_{q(\xi)}(\sigma(\xi) \upharpoonright i) \geq i$". Since $\sigma(\xi) \upharpoonright i \in i^i$, we conclude that $r \upharpoonright \xi \Vdash \deg_{q(\xi)}(\sigma(\xi) \upharpoonright i) \leq i$, and hence $r \upharpoonright \xi \Vdash \deg_{q(\xi)}(\sigma(\xi) \upharpoonright i) = i$. This means that $r \upharpoonright \xi \Vdash \sigma(\xi) \upharpoonright i \in S_i^*(q(\xi))$ and thus completes our proof. $\square$

Claim 2.19. *Suppose that ρ and $\bar{\mathbb{U}}$ are such as in Definition 2.17, $\langle I_\alpha : \alpha < \rho \rangle$ is an increasing sequence of elements of $[\nu]^{<\rho}$, $p \in \mathbb{T}_\nu$, and $\langle D_\alpha : \alpha < \rho \rangle$ is a sequence of open dense subsets of $\mathbb{T}_\nu$. Then there exists a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \rho \rangle$ for $\mathbb{T}_\nu$ such that $I_\alpha \subset F_\alpha$ for all $\alpha < \rho$, $p_0 = p$, and if $\sigma : F_\alpha \to \rho^{\alpha+1}$ lies (F_α, α)-potentially on $p_{\alpha+1}$ for some limit α, then σ lies on $p_{\alpha+1}$ and $p_{\alpha+1}|\sigma \in D_\alpha$.*

Proof. The proof is contained in that of Lemma 2.10. Indeed, take $r_{\alpha,j} \in D_\alpha$ in the construction of a fusion sequence from the proof of Lemma 2.10 (the part before Claim 2.11) instead of demanding that $r_{\alpha,j}$ decides $\underline{z}$ as a ground model object. The resulting fusion sequence is easily seen to be as required. $\square$

Under assumptions made in Definition 2.17, the set of all those generalized fusion sequences $\langle (u_\alpha, T_\alpha) : \alpha \in \rho \rangle$ such that

for every limit $\alpha \in \rho$ and $\sigma : T_\alpha \to \rho^{\alpha+1}$ which lies (T_α, α)-potentially on $u_{\alpha+1}$, σ lies on $u_{\alpha+1}$,

will be denoted by $\mathbb{D}(\mathbb{T}_\nu)$. Claim 2.19 implies that we can construct a generalized fusion sequence in $\mathbb{D}(\mathbb{T}_\nu)$ with arbitrary u_0 and with the T_α's growing as quickly as we wish.

Let us come back to our main task, namely to the construction of a $j^*(\mathbb{P}_\gamma)$-generic filter $\hat{g} \in V[G * g]$ over M^* for which $j^*[g] \subset \hat{g}$. Using Claim 2.4, for every $\xi < \gamma$ we denote by $x(\xi) \in \kappa^\kappa \cap V[G * g \upharpoonright (\xi + 1)]$ the (unique!) branch through all trees of the form $T(q)$, where $q \in g(\xi)$, and set $a_\xi = j^*(h_\xi)(\kappa)$. For every $\bar{I} \in M^*$ such that $\bar{I} \subset j[\gamma]$ and $|\bar{I}| = \kappa$ we define $\sigma_{\bar{I}} : \bar{I} \to \kappa^{\kappa+1}$ by letting $\sigma_{\bar{I}}(j(\xi)) = x(\xi)\hat{\ }a_\xi$ for all $j(\xi) \in \bar{I}$.

Claim 2.20. *Suppose that $\bar{I} \in M^*$ such that $\bar{I} \subset j[\gamma]$ and $|\bar{I}| = \kappa$. If $q \in g$, then $\sigma_{\bar{I}}$ lies $(\bar{I}, \kappa)$-potentially on $\bar{q} := j^*(q)$.*

Proof. Let us write $\bar{I}$ in the form $\bar{I} = \bigcup_{\alpha < \kappa} j[I_\alpha]$ such that $I_\alpha \subset I_\beta$ for all $\alpha < \beta$, $I_\delta = \bigcup_{\alpha < \delta} I_\alpha$ for limit δ, and $|I_\alpha| < \kappa$ for all $\alpha < \kappa$. By the definition of $x(\xi)$ and Claim 2.4 we can construct a decreasing sequence $\langle r_\alpha : \alpha < \kappa \rangle$ of elements of g below q and a sequence $\langle \pi_\alpha : \alpha < \kappa \rangle$ such that

(i) $\pi_\alpha : I_\alpha \to \kappa^{<\kappa}$ and $x(\xi) \supset \pi_\beta(\xi) \supsetneqq \pi_\alpha(\xi)$ for all $\xi \in I_\alpha$ and $\beta > \alpha$;

(ii) $r_\alpha = r_\alpha | \pi_\alpha$ and $r_\beta \upharpoonright \xi \Vdash \pi_\beta(\xi) \in S(r_\alpha(\xi))$ for all $\beta > \alpha$ and $\xi \in I_\alpha$.

Then, letting $\bar{\pi}_\alpha = j^*(\pi_\alpha) : j^*[I_\alpha] \to \kappa^{<\kappa}$ and $\bar{r}_\alpha = j^*(r_\alpha)$, the elementarity of j^* yields

(iii) $\sigma_{\bar{I}}(j(\xi)) \supset \bar{\pi}_\beta(j(\xi)) \supsetneqq \bar{\pi}_\alpha(j(\xi))$ for all $\xi \in I_\alpha$ and $\beta > \alpha$;

(iv) $\bar{r}_\alpha = \bar{r}_\alpha | \bar{\pi}_\alpha$ and $\bar{r}_\beta \upharpoonright j(\xi) \Vdash \bar{\pi}_\beta(j(\xi)) \in S(\bar{r}_\alpha(j(\xi)))$ for all $\beta > \alpha$ and $\xi \in I_\alpha$.

Set $\bar{r}_\kappa = \bigwedge_{\alpha < \kappa} \bar{r}_\alpha$ and $\bar{\pi}_\kappa : \bar{I} \to \kappa^\kappa$, $\bar{\pi}_\kappa : j(\xi) \mapsto x(\xi)$. It follows from the above that $\bar{\pi}_\kappa = \bigcup_{\alpha < \kappa} \bar{\pi}_\alpha$ and hence $\bar{\pi}_\kappa$ lies on $\bar{r}_\kappa$. Let us fix $\alpha < \kappa$ and $\xi \in I_\alpha$. Condition (iv) implies that $\bar{r}_\kappa \upharpoonright j(\xi)$ forces that $x(\xi) = \bigcup_{\alpha < \beta < \kappa} \bar{\pi}_\beta(j(\xi)) \in S(\bar{r}_\alpha(j(\xi)))$, and hence it forces that $x(\xi) \in S_\kappa^*(\bar{r}_\alpha(j(\xi)))$ and $\kappa \in C(\bar{r}_\alpha(j(\xi)))$. Applying Claim 2.18 we conclude that $\bar{r}_\kappa \upharpoonright j(\xi) \Vdash x(\xi) \in S_\kappa^*(\bar{q}(j(\xi)))$. Also, $\bar{r}_\kappa \upharpoonright j(\xi) \Vdash \sigma_{\bar{I}}(j(\xi)) = x(\xi)\hat{\ }a_\xi \in T(\bar{r}_\alpha(j(\xi)))$, which yields $\bar{r}_\kappa \upharpoonright j(\xi) \Vdash \sigma_{\bar{I}}(j(\xi)) \in T(\bar{r}_\kappa(j(\xi)))$ (remember that $\alpha < \kappa$ was chosen arbitrary). Thus $\bar{r}_\kappa$ witnesses $\sigma_{\bar{I}}$ lying $(\bar{I}, \kappa)$-potentially on $\bar{q}$. $\square$

Let us denote by $\hat{g}$ the upwards closure of the set

$$\tilde{g} := \{ \bar{u}_{\kappa+1} | \sigma_{\bar{T}_\kappa} : \langle (u_\alpha, T_\alpha) : \alpha \in \kappa \rangle \in \mathbb{D}(\mathbb{P}_\gamma) \text{ and } \bigwedge_{\alpha < \kappa} u_\alpha \in g \} \subset j^*(\mathbb{P}_\gamma),$$

where $\langle (\bar{u}_{\bar{\alpha}}, \bar{T}_{\bar{\alpha}}) : \bar{\alpha} \in j(\kappa) \rangle = j^*(\langle (u_\alpha, T_\alpha) : \alpha \in \kappa \rangle)$. We shall prove that $\hat{g}$ is a $j^*(\mathbb{P}_\gamma)$-generic filter over M^* and $j^*[g] \subset \hat{g}$.

Given any $\langle (u_\alpha, T_\alpha) : \alpha \in \kappa \rangle \in \mathbb{D}(\mathbb{P}_\gamma)$ such that $\bigwedge_{\alpha < \kappa} u_\alpha \in g$, let us notice that $\sigma_{\bar{T}_\kappa}$ lies $(\bar{T}_\kappa, \kappa)$-potentially on $\bigwedge_{\bar{\alpha} < j(\kappa)} \bar{u}_{\bar{\alpha}} = j^*(\bigwedge_{\alpha < \kappa} u_\alpha)$ by Claim 2.20, and hence $\sigma_{\bar{T}_\kappa}$ also lies $(\bar{T}_\kappa, \kappa)$-potentially on $\bar{u}_{\kappa+1} \geq \bigwedge_{\bar{\alpha} < j(\kappa)} \bar{u}_{\bar{\alpha}}$ by Claim 2.18. Therefore $\sigma_{\bar{T}_\kappa}$ lies on $\bar{u}_{\kappa+1}$ by the definition of $\mathbb{D}(\mathbb{P}_\gamma)$, which shows that the definition of $\tilde{g}$ above makes sense.

Claim 2.21. *$\hat{g}$ is a filter containing $j^*[g]$.*

Proof. Let us fix $p \in g$ and any two elements $\langle (u_\alpha, T_\alpha) : \alpha \in \kappa \rangle$ and $\langle (u'_\alpha, T'_\alpha) : \alpha \in \kappa \rangle$ of $\mathbb{D}(\mathbb{P}_\gamma)$ such that $\bigwedge_{\alpha < \kappa} u_\alpha, \bigwedge_{\alpha < \kappa} u'_\alpha \in g$, and find $q \in g$ below all of the conditions $\{\bigwedge_{\alpha < \kappa} u_\alpha, \bigwedge_{\alpha < \kappa} u'_\alpha, p\}$. By Claim 2.19 the set W of all those conditions w such that $w = \bigwedge_{\alpha < \kappa} w_\alpha$ for some $\langle (w_\alpha, L_\alpha) : \alpha \in \kappa \rangle \in \mathbb{D}(\mathbb{P}_\gamma)$ with $T_\alpha, T'_\alpha \subset L_\alpha$ for all $\alpha < \kappa$ and $w_0 \leq q$, is dense below q. Therefore there exists $w \in g \cap W$. Thus $\tilde{g} \ni \bar{w}_{\kappa+1}|\sigma_{\bar{L}_\kappa} \leq \bar{u}_{\kappa+1}|\sigma_{\bar{T}_\kappa}, \bar{u}'_{\kappa+1}|\sigma_{\bar{T}'_\kappa}$, which implies that $\hat{g}$ is a filter. In addition, $w_0 \leq q$, and hence $\bar{w}_{\kappa+1}|\sigma_{\bar{L}_\kappa} \leq j^*(w_0) \leq j^*(q) \leq j^*(p)$, consequently $j^*(p) \in \hat{g}$, which yields $j^*[g] \subset \hat{g}$. $\square$

In light of Claim 2.21 we are left with the task to show that $\tilde{g}$ meets all open dense subsets of $j^*(\mathbb{P}_\gamma)$ which are elements of M^*. Let us fix such an open dense $\bar{D} \subset j^*(\mathbb{P}_\gamma)$ and write $\bar{D}$ as $j^*(f)(\bar{a})$, where f has domain $H(\kappa)^V$, $f \in V[G]$, and $\bar{a} \in H(\kappa^{++})^V$. There is no loss of generality to assume that $f(a)$ is open dense in $\mathbb{P}_\gamma$ for all $a \in H(\kappa)^V$. Let us enumerate $H(\kappa)^V$ as $\langle a_k : k \in \kappa \rangle$ and set $D'_k = \bigcap_{k' \leq k} f(a_{k'})$ and $\vec{D}' = \langle D'_k : k \in \kappa \rangle$. As a result we have that for every $a \in H(\kappa)^V$ there exists $k < \kappa$ such that $\vec{D}'(k) \subset f(a)$. The elementarity of j^* yields $\bar{k} < j^*(\kappa)$ such that $j^*(\vec{D}')(\bar{k}) \subset j^*(f)(\bar{a}) = \bar{D}$. The extender nature of j allows to find a strictly increasing sequence $\vec{\beta} = \langle \beta_k : k < \kappa \rangle$ of ordinals below κ such that $j^*(\vec{\beta})(\kappa) > \bar{k}$, see the beginning of the proof [9, Lemma 4] for details. Set $D_k = D'_{\beta_k}$ and $\vec{D} = \langle D_k : k < \kappa \rangle$. Then $\vec{D}$ has the property $j^*(\vec{D})(\kappa) \subset j^*(\vec{D}')(\bar{k}) \subset \bar{D}$.

Given any $u \in \mathbb{P}_\gamma$, using Claim 2.19 we can construct a fusion sequence $\langle (u_k, T_k) : k \in \kappa \rangle$ with $u_0 = u$ satisfying the following condition:

If $\sigma : T_k \to \kappa^{k+1}$ lies (T_k, u_k)-potentially on u_{k+1}, then σ lies on u_{k+1} and $u_{k+1}|\sigma \in D_k$. In particular, $\langle (u_k, T_k) : k \in \kappa \rangle \in \mathbb{D}(\mathbb{P}_\gamma)$.

Let $\langle \bar{T}_{\bar{k}} : \bar{k} \in j(\kappa) \rangle$ and $\langle \bar{u}_{\bar{k}} : \bar{k} \in j(\kappa) \rangle$ be the results of applying j^* to $\langle T_k : k \in \kappa \rangle$ and $\langle u_k : k \in \kappa \rangle$ respectively, $v = \bigwedge_{k < \kappa} u_k$, and $\bar{v} = j^*(v) = \bigwedge_{\bar{k} < j(\kappa)} \bar{u}_{\bar{k}}$. By elementarity of j^*, for every $\bar{\sigma} : \bar{T}_\kappa \to j(\kappa)^{\kappa+1}$ lying (T_κ, κ)-potentially on $\bar{u}_{\kappa+1}$, $\bar{\sigma}$ lies on $\bar{u}_{\kappa+1}$ and $\bar{u}_{\kappa+1}|\bar{\sigma} \in j^*(\vec{D})(\kappa) \subset \bar{D}$.

Since u was chosen arbitrarily, we can assume that $v \in g$. Claim 2.20 implies that $\sigma_{\bar{T}_\kappa}$ lies $(\bar{T}_\kappa, \kappa)$-potentially on $j^*(v)$, and hence it lies $(\bar{T}_\kappa, \kappa)$-potentially on $\bar{u}_{\kappa+1}$ by Claim 2.18. It follows from the above that $\sigma_{\bar{T}_\kappa}$ lies on $\bar{u}_{\kappa+1}$ and $\tilde{g} \ni \bar{u}_{\kappa+1} | \sigma_{\bar{T}_\kappa} \in j^*(\vec{D})(\kappa) \subset \bar{D}$, which finishes our proof of Theorem 2.15. $\square$

If we use preparation relative to a fast function (see [11]) instead of the poset $\mathbb{R}_\kappa$, we can prove the following theorem by almost literal repetition of the proof of Theorem 2.15.

Theorem 2.22. *Suppose GCH holds, θ is a regular cardinal, and $j : V \to M$ is an ultrapower embedding via a (κ, θ)-extender in V such that $H(\theta)$ of V is contained in M. Then there exists a κ-c.c. poset $\mathbb{R}$ of size κ such that in $V^{\mathbb{R}}$, for every iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ with supports of size $\leq \kappa$ of forcings with good κ-fusion, where $\gamma \leq \theta$ and $\Vdash_{\mathbb{R}*\mathbb{P}_\xi} |\mathbb{Q}_\xi| < \max\{\kappa^{++}, \theta\}$, j can be extended in $V^{\mathbb{R}*\mathbb{P}_\gamma}$ to an elementary embedding $j^{**} : V^{\mathbb{R}*\mathbb{P}_\gamma} \to M^{j(\mathbb{R}*\mathbb{P}_\gamma)}$ so that the $H(\theta)$ of $V^{\mathbb{R}*\mathbb{P}_\gamma}$ is contained in $M^{j(\mathbb{R}*\mathbb{P}_\gamma)}$. In particular, κ remains θ-strong in $V^{\mathbb{R}*\mathbb{P}_\gamma}$.*

Although the iterations considered in Theorem 2.22 have θ-c.c., they could collapse all cardinals between κ^+ and θ.

Corollary 2.23. *Suppose GCH holds and λ is a strong cardinal. Then there exists a λ-c.c. poset $\mathbb{R}$ of size λ such that λ remains strong in $V^{\mathbb{R}*\mathbb{P}_\gamma}$ for every iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ with supports of size $\leq \kappa$ of forcings with good κ-fusion.*

3. Reasonably bounded forcing notions and extending elementary embeddings

Here we introduce a strengthening of the reasonable B-boundedness property from [17] suitable for extending elementary embeddings. Throughout the section λ stands for a strongly inaccessible cardinal and $\bar{\mu}$ denotes a nondecreasing sequence $\langle \mu_\alpha : \alpha < \lambda \rangle$ of regular cardinals $\leq \lambda$ such that $|\prod_{\xi<\alpha} f(\xi)| < \mu_\alpha$ for every $f : \alpha \to \mu_\alpha$. For example, the sequences $\langle |2^\alpha|^+ : \alpha < \lambda \rangle$ and $\langle \lambda : \alpha < \lambda \rangle$ are as above. Whenever there is no need to consider a particular $\mathbb{Q}$-generic filter for some poset $\mathbb{Q}$, the forcing extension of V by $\mathbb{Q}$ will be denoted by $V^{\mathbb{Q}}$. Let $\mathcal{U}$ be a family of unbounded subsets of λ which is closed under diagonal intersections. For a poset $\mathbb{Q}$ we denote by $\mathcal{U}^{\mathbb{Q}}$ the closure of $\mathcal{U}$ under diagonal intersections in $V^{\mathbb{Q}}$. It is easy to check that if $\mathbb{Q}$ is $< \lambda$-strategically closed then $\mathcal{U}^{\mathbb{Q}}$ consists of unbounded

subsets of λ, see, e.g., [20]. We shall also denote by $\mathcal{D}_\lambda$ the collection of all clubs of λ.

Definition 3.1. Let $\mathbb{Q}$ be a forcing notion.

For a condition $p \in \mathbb{Q}$ we define a game $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ between two players, Generic and Antigeneric, as follows. A play of $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ lasts λ steps and results in a sequence

$$\left\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \right\rangle$$

constructed by the players. The αth round is played as follows:

(1) First, Generic chooses a non-empty set I_α of cardinality $< \mu_\alpha$ and a collection $\langle p^\alpha_t : t \in I_\alpha \rangle$ of pairwise incompatible elements of $\mathbb{Q}$ such that

    (a) for any $J \subset \alpha$, $t \in I_\alpha$, and $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} I_\xi$, if there exists a lower bound for the set $\{q^\xi_{t_\xi} : \xi \in J\} \cup \{p^\alpha_t\}$, then p^α_t is such a lower bound (i.e., for any $\beta < \alpha$ and $t' \in I_\beta$ either the conditions $q^\beta_{t'}$, q^α_t are incompatible or else $q^\beta_{t'} \geq q^\alpha_t$);

    (b) if $\alpha < \lambda$ is limit, then for any cofinal subset J of α and a sequence $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} I_\xi$, the set $\{t \in I_\alpha : \forall\, \xi \in J\ (p^\alpha_t \leq q^\xi_{t_\xi})\}$ has size at most $|\alpha|$.

(2) Antigeneric answers by picking a collection $\langle q^\alpha_t : t \in I_\alpha \rangle$ such that $q^\alpha_t \leq p^\alpha_t$ for all $t \in I_\alpha$.

Generic wins this play $\left\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \right\rangle$ if there exists $p^* \leq p$ such that

$$p^* \Vdash \{\alpha < \lambda : \exists\, t \in I_\alpha\ (q^\alpha_t \in \Gamma_{\mathbb{Q}})\} \in \mathcal{U}^{\mathbb{Q}},$$

where $\Gamma_{\mathbb{Q}}$ is the canonical name for the $\mathbb{Q}$-generic filter.

We say that $\mathbb{Q}$ is *reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$* if $\mathbb{Q}$ is $< \lambda$ strategically closed and Generic has a winning strategy in $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ for all $p \in \mathbb{Q}$.

If $\mathcal{U} = \{\lambda\}$, then forcing notions which are reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ will be called *reasonably A_e-bounding over $\bar{\mu}$*.

$\square$

Remark 3.2. If we remove items $(a), (b)$ (or just item (b)) from Definition 3.1(1), we get the definition of the game $\mathfrak{D}^{rc\mathbf{B}}_{\mathcal{U},\bar{\mu}}(p, \mathbb{Q})$ and of *reasonably B-bounding over $\mathcal{U}, \bar{\mu}$* forcing notions introduced in [17]. If $\mathcal{U} = \{\lambda\}$, then forcing notions which are reasonably B-bounding over $\mathcal{U}, \bar{\mu}$ are called in [17] *reasonably A-bounding over $\bar{\mu}$*.

The subscript "e" in the notation $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}$ comes from *extending elementary embeddings*.

$\square$

Example 3.3. 1. We shall illustrate the reasonably A_e-boundedness over $\bar{\mu}$ by proving that the poset $\mathrm{Sacks}(\lambda)$ has this property for $\mu_\alpha = \lambda$, see [15, Def. 1.1] for the definition of $\mathrm{Sacks}(\lambda)$.

For a condition $p \in \mathrm{Sacks}(\lambda)$ and $\alpha < \lambda$ let us denote by $\mathrm{Split}(p)$ the set of all splitting nodes of p and by $\mathrm{Split}_\alpha(p)$ the set $\{s \in \mathrm{Split}(p) : o.t.(\{t : t \subsetneqq s, t \in \mathrm{Split}(p)\}) \le \alpha\}$. The notation $q \le_\alpha p$ means that $q \le p$ and $\mathrm{Split}_\alpha(q) = \mathrm{Split}_\alpha(p)$[7].

Let us fix $p_0 \in \mathrm{Sacks}(\lambda)$ and suppose that we have already reached αth round of the play $\mathfrak{D}^{\mathbf{B}_e}_{\{\lambda\},\bar{\mu}}(p_0, \mathrm{Sacks}(\lambda))$. Suppose also that the players have constructed a sequence $\langle p_\xi, q_\xi : \xi < \alpha \rangle$ of conditions in $\mathrm{Sacks}(\lambda)$ such that

(i) $p_{\xi+1} \le_{\xi+1} q_\xi \le_\xi p_\xi$ for all $\xi + 1 < \alpha$;
(ii) $I_\xi = \{s\hat{\ }0, s\hat{\ }1 : s \in \mathrm{Split}_\xi(p_\xi)\}$;
(iii) $p_t^\xi = (p_\xi)_t$ for all $t \in I_\xi$;
(iv) $q_t^\xi = (q_\xi)_t$ for all $t \in I_\xi$; and
(v) $p_\xi = \bigcap_{\zeta < \xi} q_\zeta$ for all $0 < \xi < \alpha$.

As required in (v) we set $p_\alpha = \bigcap_{\xi < \alpha} q_\xi$ (thus p_α is simply equal to q_β provided that $\alpha = \beta + 1$). Now, Generic is instructed to play $I_\alpha = \{s\hat{\ }0, s\hat{\ }1 : s \in \mathrm{Split}_\alpha(p_\alpha)\}$ and $p_t^\alpha = (p_\alpha)_t$ for all $t \in I_\alpha$. Suppose that Antigeneric replies with a sequence $\langle q_t^\alpha : t \in I_\alpha \rangle$ such that $q_t^\alpha \le p_t^\alpha$ for all $t \in I_\alpha$. Then we set $q_\alpha = \bigcup_{t \in I_\alpha} q_t^\alpha$. This completes the description of a strategy for Generic, and we are left with the task to show that it is winning.

Let us fix a play $\langle I_\xi, \langle p_t^\xi, q_t^\xi : t \in I_\xi \rangle : \xi < \lambda \rangle$ in the game $\mathfrak{D}^{\mathbf{B}_e}_{\{\lambda\},\bar{\mu}}(p_0, \mathrm{Sacks}(\lambda))$ and let $\langle p_\xi, q_\xi : \xi < \lambda \rangle$ be the sequence of conditions in $\mathrm{Sacks}(\lambda)$ constructed aside from this play such that the conditions (i)-(v) are satisfied. It is easy to check that items $(a), (b)$ of Definition 3.1(1) are satisfied. In fact, for any limit $\alpha < \lambda$, cofinal subset J of α, and sequence $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} I_\xi$, the set $\{t \in I_\alpha : \forall \xi \in J \, (p_t^\alpha \le q_{t_\xi}^\xi)\}$ has size 2. Set $p^* = \bigcap_{\xi < \lambda} p_\xi = \bigcap_{\xi < \lambda} q_\xi$ and notice that $p^* \le_\xi q_\xi \le_\xi p_\xi$ for all $\xi < \lambda$, and hence

$$\{(q_\xi)_{s\hat{\ }i} : s \in \mathrm{Split}_\xi(q_\xi), i \in 2\} = \{q_t^\xi : t \in I_\xi\}$$

is predense below p^* for all $\xi < \lambda$. Thus

$$p^* \Vdash \{\alpha < \lambda : \exists \, t \in I_\alpha \, (q_t^\alpha \in \Gamma_{\mathbb{Q}})\} = \lambda,$$

which finishes our proof.

In the same way we can prove that singular-splitting-Sacks forcing is reasonably A_e-bounding, see [12, 2.1] for its definition.

2. It seems that not all natural variants of Sacks forcing are reasonably A_e-bounding. For example, if $\vec{A} = \langle A_\xi : \xi < \lambda \rangle$ is a sequence of sets of

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[7]Notation $q \le_\alpha p$ has a different meaning in [15].

cardinality less than λ and the sequence $\langle |A_\xi| : \xi < \lambda \rangle$ grows fast enough, then the proof presented above does *not* show that the poset $\mathrm{Sacks}(\vec{A})$ (see Example 2.2)[8] is reasonably A_e-bounding, the main obstacle being condition $1(b)$ of Definition 3.1. However, these posets have good λ-fusion introduced in Section 2. □

Definition 3.4. ([17, Definition 2.2].)
Let γ be an ordinal, $0 \in w \subset \gamma$. A *standard $(w, 1)^\gamma$-tree* is a pair $\mathcal{T} = (T, \mathrm{rk})$ such that

- $\mathrm{rk} : T \to w \cup \{\gamma\}$;
- If $t \in T$ and $\mathrm{rk}(t) = \varepsilon$, then t is a sequence $\langle t(\zeta) : \zeta \in w \cap \varepsilon \rangle$;
- $(T, \lhd)$ is a tree with root $\emptyset$, where $\lhd$ is the end-extension relation, such that every chain in T has a $\lhd$-upper bound in T; and
- If $t \in T$, then there is $t' \in T$ such that $t \lhd t'$ and $\mathrm{rk}(t') = \gamma$.

Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle$ be a λ-support iteration, i.e., an iteration with supports of size at most λ. A *standard tree of conditions in $\bar{\mathbb{Q}}$* is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that

- (T, rk) is a standard $(w, 1)^\gamma$-tree for some $w \subset \gamma$;
- $p_t \in \mathbb{P}_{\mathrm{rk}(t)}$ for all $t \in T$; and
- If $s, t \in T$, $s \lhd t$, then $p_s = p_t \restriction \mathrm{rk}(s)$.

A standard tree of conditions in $\bar{\mathbb{Q}}$ will be called *regular*, if p_t and $p_{t'}$ are incompatible for all $t \neq t'$ such that $\mathrm{rk}(t) = \mathrm{rk}(t')$.

The next definition is abstracted from the proof of the main result of [10].

Definition 3.5. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ be a λ-support iteration. We say that the poset $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is $B_e(\bar{\mathbb{Q}})$-*bounding over* $\mathcal{U}, \bar{\mu}$, if it is $< \lambda$-strategically closed and for every $p \in \mathbb{P}_\gamma$ and sequence $\bar{D} = \langle D_\alpha : \alpha < \lambda \rangle$ of open dense subsets of $\mathbb{P}_\gamma$ there are sequences $\bar{\mathcal{T}}$ and $\bar{q}$ such that

(1) $\bar{\mathcal{T}} = \langle \mathcal{T}_\alpha : \alpha < \lambda \rangle$, where each $\mathcal{T}_\alpha = (T_\alpha, \mathrm{rk}_\alpha)$ is a standard $(w_\alpha, 1)^\gamma$-tree of size $< \mu_\alpha$;
(2) $\bar{q} = \langle \bar{q}^\alpha : \alpha < \lambda \rangle$, where $\bar{q}^\alpha = \langle q_t^\alpha : t \in \mathcal{T}_\alpha \rangle$ is a regular standard tree of conditions;
(3) $|w_\alpha| \leq |\alpha|$ for all α;
(4) $\{q_t^\alpha : \mathrm{rk}_\alpha(t) = \gamma\} \subset D_\alpha$;
(5) For any $J \subset \alpha$ and sequence $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} \mathcal{T}_\xi$, if $\mathrm{rk}_\xi(t_\xi) = \gamma$ for all $\xi \in J$, $t \in \mathcal{T}_\alpha$, $\mathrm{rk}_\alpha(t) = \gamma$, and there exists a lower bound for the

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[8]Similar variations on $\mathrm{Sacks}(\kappa)$ were considered in [8].

set $\{q_{t_\xi}^\xi : \xi \in J\} \cup \{q_t^\alpha\}$, then q_t^α is such a lower bound (i.e., for any $\xi < \alpha$, $t' \in \mathcal{T}_\xi$, and $t \in \mathcal{T}_\alpha$ such that $\mathrm{rk}_\xi(t') = \mathrm{rk}_\alpha(t) = \gamma$, either the conditions $q_{t'}^\xi$, q_t^α are incompatible or else $q_{t'}^\xi \geq q_t^\alpha$);

(6) For any limit $\alpha < \lambda$, cofinal subset J of α, and a sequence $\bar{t} = (t_\xi)_{\xi \in J} \in \prod_{\xi \in J} \mathcal{T}_\xi$ such that $\mathrm{rk}_\xi(t_\xi) = \gamma$ for all $\xi \in J$, the set $\mathcal{T}_\alpha(\bar{t})$ consisting of those $t \in \mathcal{T}_\alpha$ which can be extended to $t' \in \mathcal{T}_\alpha$ with $\mathrm{rk}_\alpha(t') = \gamma$ and $q_{t'}^\alpha \leq q_{t_\xi}^\xi$ for all $\xi \in J$, has the following property: *for every two consecutive elements $\xi < \eta$ of w_α and $t \in \mathcal{T}_\alpha(\bar{t})$ with $\mathrm{rk}_\alpha(t) = \xi$, the set $\{t' \in \mathcal{T}_\alpha(\bar{t}) : \mathrm{rk}_\alpha(t') = \eta \text{ and } t \lhd t'\}$ has size at most $|\alpha|$*;

(7) There exists a condition $r \in \mathbb{P}_\gamma$ such that $p \geq r$ and
$$r \Vdash_{\mathbb{P}_\gamma} \{\alpha < \lambda : \exists\, t \in \mathcal{T}_\alpha\, (\mathrm{rk}_\alpha(t) = \gamma \wedge q_t^\alpha \in \Gamma_\mathbb{Q})\} \in \mathcal{U}^\mathbb{Q}. \qquad \square$$

The following theorem is analogous to [17, Th. 3.1].

Theorem 3.6. *Assume that*

(1) λ *is a strongly inaccessible cardinal;*

(2) $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ *is a λ-support iteration such that*

$$\Vdash_{\mathbb{P}_\xi} \mathbb{Q}_\xi \text{ is reasonably } B_e\text{-bounding over } \mathcal{U}, \bar{\mu}.$$

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is $B_e(\bar{\mathbb{Q}})$-bounding over $\mathcal{U}, \bar{\mu}$.

Proof. The proof will be done by "adding an ε" to that of [17, Th. 3.1]. In order to avoid unnecessary repetitions we keep our notations as close to those of [17] as possible and indicate what kind of changes are to be made in the proof of [17, Th. 3.1] in order to get our theorem. We shall also refer to equations from [17].

Let us fix $p \in \mathbb{P}_\gamma$ and a sequence $\bar{D}$ of open dense subsets of $\mathbb{P}_\gamma$. Like in [17, Th. 3.1], Generic constructs a winning strategy $\mathbf{st}$ in the game $\eth_{\mathcal{U},\bar{\mu}}^{rc\mathbf{b}}(p, \mathbb{P}_\gamma)$ with the auxiliary objects mentioned in $(\otimes)_\delta$ so that $\underline{\mathbf{st}}_\xi$ is a $\mathbb{P}_\xi$-name for a winning strategy of Generic in $\eth_{\mathcal{U},\bar{\mu}}^{\mathbf{B}_e}(r_\delta(\xi), \mathbb{Q}_\xi)$, see $(*)_3$. (Note that that a strategy for Generic in $\eth_{\mathcal{U},\bar{\mu}}^{\mathbf{B}_e}(-, -)$ is also a strategy for Generic in $\eth_{\mathcal{U},\bar{\mu}}^{rc\mathbf{B}}(-, -)$.) While constructing its winning strategy in the game $\eth_{\mathcal{U},\bar{\mu}}^{rc\mathbf{b}}(p, \mathbb{P}_\gamma)$, Generic is allowed to choose *any* condition p_ζ^δ stronger than all r_i^ζ, s_i^ζ for $i < \zeta$, see line 6 on page 212 in [17]. We shall additionally require that $p_\zeta^\delta \in D_\delta$, and for any $J \subset \delta$ and a sequence $(t_\xi)_{\xi \in J} \in \prod_{\xi \in J} \mathcal{T}_\xi$, *if* $\mathrm{rk}_\xi(t_\xi) = \gamma$ for all $\xi \in J$, there exists a lower bound for the set $\{q_{*,t_\xi}^\xi : \xi \in J\}$, and every element of the latter set is compatible with p_ζ^δ, *then* p_ζ^δ is actually a lower bound of $\{q_{*,t_\xi}^\xi : \xi \in J\}$. This can be easily achieved using the $< \lambda$ strategical completeness of $\mathbb{P}_\gamma$. Now let r, w_α's, and $\mathcal{T}^\delta$'s be the same as in

the proof of [17, Th. 3.1] and set $\bar{q}^\alpha = \bar{q}^\alpha_*$. We claim that $\bar{\mathcal{T}} = \langle \bar{\mathcal{T}}^\delta : \delta < \lambda \rangle$ and $\bar{q} = \langle \bar{q}^\delta : \delta < \lambda \rangle$ fulfill the conditions in Definition 3.5. Conditions (1)-(3) and (5) are satisfied by the construction. Since

$$q^\delta_{*,t_\zeta} = s^\zeta_{\zeta_\delta} \leq s^\xi_\zeta \leq r^\zeta_\zeta = q^\delta_\zeta \leq p^\delta_\zeta \in D^\delta$$

for all $\delta < \lambda$ and $\zeta < \zeta_\delta$ (see [17, pp. 211-212]), we conclude that the condition (4) of Definition 3.5 is satisfied as well. Condition (7) is just the last but one formula on page 215 of [17]. Thus we are left with the task to prove (6).

Let us fix a limit $\delta < \lambda$, a cofinal subset J of δ, and a sequence $\bar{t}$ such as in Definition 3.5(6), and let $\xi < \eta$ be two consecutive elements of w_δ. If $\mathcal{T}_\delta(\bar{t})$ is empty then there is nothing to prove. So assume that $\mathcal{T}_\delta(\bar{t}) \neq \emptyset$ and fix some $t \in \mathcal{T}_\delta(\bar{t})$ with $\mathrm{rk}(t) = \xi$. By the definition of $\mathcal{T}_\delta(\bar{t})$ we conclude that $q^\delta_t \leq q^\beta_{t_\beta} \restriction \xi$ for all $\beta \in J$. Suppose that $t' \in \mathcal{T}_\delta(\bar{t})$ is such that $t \lhd t'$ and $\mathrm{rk}_\delta(t') = \eta$. Again, $q^\delta_{t'} \leq q^\beta_{t_\beta} \restriction \eta$ for all $\beta \in J$. It follows that $q^\delta_t \Vdash_{\mathbb{P}_\xi} q^\delta_{t'}(\xi) \leq q^{\beta_1}_{t_{\beta_1}}(\xi) \leq q^{\beta_2}_{t_{\beta_2}}(\xi)$ for all $\beta_1 > \beta_2 \in J$. Let $\alpha < \delta$ be the minimal ordinal such that $\xi \in w_{\alpha+1}$. By $(*)_9$ we have that $q^\delta_t \Vdash_{\mathbb{P}_\xi}$ "$\langle \varepsilon_{\beta,\xi}, \bar{p}_{\beta,\xi}, \bar{q}_{\beta,\xi} : \alpha < \beta \leq \delta \rangle$ is an initial segment of a delayed play of $\mathfrak{D}^{\mathbf{B}_e}_{\mathcal{U},\bar{\mu}}(r_\alpha(\xi), \mathbb{Q}_\xi)$ in which Generic uses $\underline{\mathbf{st}}_\xi$". Condition $(*)_{10}$ yields that $q^\delta_t \Vdash_{\mathbb{P}_\xi}$ "$q^\beta_{t_\beta}(\xi)$ is a member of the sequence $\bar{q}_{\beta,\xi}$ for all $\beta \in J \cap (\alpha, \delta)$ and $q^\delta_{t'}(\xi)$ is a member of the sequence $\bar{q}_{\delta,\xi}$". By Definition 3.1(1a) we have that $q^\delta_t \Vdash_{\mathbb{P}_\xi} |\{\varepsilon < \varepsilon_{\delta,\xi} : \bar{q}_{\delta,\xi}(\varepsilon) \leq_{\mathbb{Q}_\xi} q^\beta_{t_\beta}(\xi)$ for all $\beta \in J \cap (\alpha, \delta)\}| \leq |\delta|$. Notice that $q^\delta_t \Vdash_{\mathbb{P}_\xi} \varepsilon_{\delta,\xi} = \varepsilon^t_{\delta,\xi}$ and therefore

$$|\{t'' \in \mathcal{T}_\delta(\bar{t}) : t \lhd t'' \wedge \mathrm{rk}_\delta(t'') = \eta\}| \leq$$

$$\leq |\{\varepsilon < \varepsilon^t_{\delta,\xi} : q^\delta_t \Vdash_{\mathbb{P}_\xi} \bar{q}_{\delta,\xi}(\varepsilon) \leq q^\beta_{t_\beta}(\xi) \text{ for all } \beta \in J \cap (\alpha, \delta)\}|.$$

Let H be a $\mathbb{P}_\xi$-generic filter containing q^δ_t. From the above it follows that

$$\{\varepsilon < \varepsilon^t_{\delta,\xi} : q^\delta_t \Vdash_{\mathbb{P}_\xi} \bar{q}_{\delta,\xi}(\varepsilon) \leq q^\beta_{t_\beta}(\xi) \text{ for all } \beta \in J \cap (\alpha, \delta)\} \subset$$

$$\subset \{\varepsilon < \varepsilon^t_{\delta,\xi} : (\bar{q}_{\delta,\xi}(\varepsilon))^H \leq (q^\beta_{t_\beta}(\xi))^H \text{ (in } \mathbb{Q}^H_\xi) \text{ for all } \beta \in J \cap (\alpha, \delta)\},$$

and the latter set has size at most $|\delta|$ in $V[H]$. However, $\mathbb{P}_\xi$ is $< \lambda$-distributive, and hence we have

$$|\{\varepsilon < \varepsilon^t_{\delta,\xi} : q^\delta_t \Vdash_{\mathbb{P}_\xi} \bar{q}_{\delta,\xi}(\varepsilon) \leq q^\beta_{t_\beta}(\xi) \text{ for all } \beta \in J \cap (\alpha, \delta)\}| \leq |\delta|$$

in V, which implies

$$|\{t'' \in \mathcal{T}_\delta(\bar{t}) : t \lhd t'' \wedge \mathrm{rk}_\delta(t'') = \eta\}| \leq |\delta|$$

and thus completes our proof.   □

We collect below some basic facts about the relationship between $\mathcal{U}$ and $\mathcal{U}^{\mathbb{Q}}$. A poset $\mathbb{Q}$ is called λ^λ-*bounding*, if for every $x \in \lambda^\lambda \cap V^{\mathbb{Q}}$ there exists $y \in \lambda^\lambda \cap V$ such that $x(\alpha) < y(\alpha)$ for all $\alpha < \lambda$.

Observation 3.7. *Let $\mathbb{Q}$ be a $< \lambda$-distributive poset. Then*

(1) *$\mathcal{D}_\lambda^{\mathbb{Q}}$ is the collection of all clubs in $V^{\mathbb{Q}}$, i.e., it equals $\mathcal{D}_\lambda$ interpreted in $V^{\mathbb{Q}}$;*

(2) *If $\mathbb{Q}$ is λ^λ-bounding and C_1 is a club in $V^{\mathbb{Q}}$, then there exists a club $C \in V$ such that $C \subset C_1$; and*

(3) *If $\mathcal{U}$ is a base for a normal filter on λ, and for every set $X \in V^{\mathbb{Q}}$ such that $X \subset V$ and $|X| \leq \lambda$ in $V^{\mathbb{Q}}$ there exists $Y \in V$ such that $|Y| \leq \lambda$ in V and $X \subset Y$, then for every sequence $\langle U_\xi : \xi < \lambda \rangle \in \mathcal{U}^\lambda \cap V^{\mathbb{Q}}$ there exists $W \in \mathcal{U}$ and a club $C \in V^{\mathbb{Q}}$ such that $\Delta_{\xi < \lambda} U_\xi \supset W \cap C$.*

Proof. 1. The first item is straightforward.

2. Let us fix a club $C_1 \in V^{\mathbb{Q}}$ and let $\langle c_\alpha : \alpha < \lambda \rangle$ be the increasing enumeration of C_1. Observe that the set of the limit points of $\Delta_{\alpha < \lambda}(\lambda \setminus (c_\alpha + 1))$ equals to the set $\{\alpha : \alpha$ is limit and $c_\alpha = \alpha\}$ and hence is a subset of C_1. Since $\mathbb{Q}$ is λ^λ-bounding, there exists a function $x \in \lambda^\lambda \cap V$ such that $x(\alpha) > c_\alpha$ for all α. Then

$$C := Lim\big(\Delta_{\alpha < \lambda}(\lambda \setminus (x(\alpha) + 1))\big) \subset Lim\big(\Delta_{\alpha < \lambda}(\lambda \setminus (c_\alpha + 1))\big) \subset C_1$$

and $C \in V$.

3. Let $\mathcal{W} \in [\mathcal{U}]^\lambda \cap V$ be such that $\{U_\xi : \xi < \lambda\} \subset \mathcal{W}$ and $\langle W_\eta : \eta < \lambda \rangle$ be an enumeration of $\mathcal{W}$. Set

$$C = \big\{\alpha \ : \ \{W_\eta : \eta < \alpha\} \cap \{U_\xi : \xi < \lambda\} = \{U_\xi : \xi < \alpha\}\big\},$$

$W = \Delta_{\eta < \lambda} W_\eta$, and notice that C is a club in $V^{\mathbb{Q}}$. We claim that $\Delta_{\xi < \lambda} U_\xi \supset W \cap C$. Indeed, if $\alpha \in W \cap C$, then $\alpha \in \bigcap_{\eta < \alpha} W_\eta \subset \bigcap_{\xi < \alpha} U_\xi$ because $\{U_\xi : \xi < \alpha\} \subset \{W_\eta : \eta < \alpha\}$, and hence $\alpha \in \Delta_{\xi < \lambda} U_\xi$.   □

Following [14, Definition 21.6] we say that a poset $\mathbb{P}$ is $< \kappa$-*directed closed,* if for every subset A of $\mathbb{P}$ of size $< \kappa$, if every finite subset of A has a lower bound *in A* (such subsets A are called *directed*), then A has a lower bound in $\mathbb{P}$. A subset A of $\mathbb{P}$ is *centered,* if any finite subset of A has a lower bound *in $\mathbb{P}$*. If A is a centered subset of $\mathbb{P}$ and every finite subset of A has the greatest lower bound, that by closing A under these bounds we get a directed subfamily of $\mathbb{P}$.

We shall say that all finite subsets of a poset $\mathbb{P}$ have greatest lower bounds if so do all finite subsets of $\mathbb{P}$ which are bounded from below. Using this convention, for posets in which all finite subsets have greatest lower

bounds, being $< \kappa$-directed closed is equivalent to the existence of a lower bound for every centered subset of size $< \kappa$.

Let A, B be subsets of a poset $\mathbb{P}$. We say that A is *predense below* B if every lower bound of B is compatible with some element of A.

The following theorem is the main result of this section.

Theorem 3.8. *Suppose GCH holds and $j : V \to M$ is a (λ, λ^{++})-extender ultrapower[9] such that $H(\lambda^{++})^V = H(\lambda^{++})^M$. Let $\mathcal{U}$ be a normal filter on λ contained in the measure derived from j. Let also $\mathbb{R}_\lambda$ be the poset defined before Theorem 2.15 and in $V^{\mathbb{R}_\lambda}$ let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \lambda^{++} \rangle$ be a λ-support iteration such that $\Vdash_{\mathbb{R} * \mathbb{P}_\xi}$ "$\mathbb{Q}_\xi$ is a $< \lambda$-directed closed reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ poset and of size $\leq \lambda^+$ all of whose finite subsets have a greatest lower bound" for all $\xi < \lambda^{++}$. Then j can be extended to an elementary embedding $j^{**} : V^{\mathbb{R}_\lambda * \mathbb{P}_{\lambda^{++}}} \to M^{j(\mathbb{R}_\lambda * \mathbb{P}_{\lambda^{++}})}$ so that $H(\lambda^{++})$ of $V^{\mathbb{R}_\lambda * \mathbb{P}_{\lambda^{++}}}$ and $H(\lambda^{++})$ of $M^{j(\mathbb{R}_\lambda * \mathbb{P}_{\lambda^{++}})}$ coincide, where $\mathbb{P}_{\lambda^{++}}$ is the direct limit of the sequence $\langle \mathbb{P}_\alpha : \alpha < \lambda^{++} \rangle$. In particular, λ remains measurable in $V^{\mathbb{R}_\lambda * \mathbb{P}_{\lambda^{++}}}$.*

Proof. In the same way as at the beginning of the proof of Theorem 2.15 let us fix a $\mathbb{R}_\lambda$-generic filter G over V, a $\mathbb{P}_{\lambda^{++}}$-generic filter g over $V[G]$, and identify g with an $\bar{\mathbb{S}}_\lambda^G$-generic filter over $M[G]$. Next, let us find a $j(\mathbb{R}_\lambda) \upharpoonright (\lambda, j(\lambda))$-generic filter $H \in V[G * g]$ over $M[G * g]$, and using $j[G] = G \subset G * g * H$ let us lift j to an embedding $j^* : V[G] \to M^* := M[G * g * H]$ definable in $V[G * g]$.

Next, we shall extend j^* to an elementary embedding $j^{**} : V[G * g] \to M^*[h]$ for some $j^*(\mathbb{P}_{\lambda^{++}})$-generic filter h. By [2, Prop. 9.1] it is enough to find a $j^*(\mathbb{P}_{\lambda^{++}})$-generic $h \in V[G * g]$ over M^* for which $j^*[g] \subset h$. Applying Theorem 3.6 we conclude that $\mathbb{P}_{\lambda^{++}} = \lim(\bar{\mathbb{Q}})$ is $B_e(\bar{\mathbb{Q}})$-bounding over $\mathcal{U}, \bar{\mu}$, and hence it fulfills the premises of Observation 3.7(3). For every $p \in \mathbb{P}_{\lambda^{++}}$ and sequence $\bar{D} = \langle D_\alpha : \alpha < \lambda \rangle$ of open dense subsets of $\mathbb{P}_{\lambda^{++}}$ choose $\bar{\mathcal{T}}^{p, \bar{D}} = \langle \mathcal{T}_\alpha^{p, \bar{D}} : \alpha < \lambda \rangle$, $\bar{q}^{p, \bar{D}} = \langle \bar{q}^{\alpha, p, \bar{D}} : \alpha < \lambda \rangle$, where $\bar{q}^{\alpha, p, \bar{D}} = \langle q_t^{\alpha, p, \bar{D}} : t \in \mathcal{T}_\alpha^{p, \bar{D}} \rangle$, and $r = r(p, \bar{D}) \leq p$ as in Definition 3.5.

Let us fix some $p \in \mathbb{P}_{\lambda^{++}}$ and $\bar{D}$ such that $r(p, \bar{D}) \in g$. Set $C_{p, \bar{D}} = \{\alpha < \lambda : g \cap \{q_t^{\alpha, p, \bar{D}} : t \in \mathcal{T}_\alpha^{p, \bar{D}}, \mathrm{rk}_\alpha(t) = \lambda^{++}\} \neq \emptyset\} \in V[G * g]$. Observation 3.7(3) implies that $C_{p, \bar{D}}$ contains an intersection of an element of $\mathcal{U}$ and a club. More precisely, strengthening $r(p, \bar{D})$, if necessary, we may assume that there exists a $\mathbb{P}_{\lambda^{++}}$-name $\sigma_{p, \bar{D}} \in V[G]$ and $U_{p, \bar{D}} \in \mathcal{U}$ such that $r(p, \bar{D}) \Vdash$ "$\sigma_{p, \bar{D}}$ is a club and $\sigma_{p, \bar{D}} \cap \check{U}_{p, \bar{D}} \subset \check{C}_{p, \bar{D}}$". In $V[G * g]$, let $\langle \beta_\zeta^{p, \bar{D}} : \zeta < \lambda \rangle$ be

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[9]I.e., $M = \{j(f)(a) : f \in V, f : H(\lambda) \to V, \text{ and } a \in H(\lambda^{++})\}$.

the increasing enumeration of $\sigma_{p,\bar{D}}^g$. Let us also fix a decreasing sequence $\langle x^{\alpha,p,\bar{D}} \rangle$ of elements of g such that $x^{\alpha,p,\bar{D}}$ decides the first α elements of $\sigma_{p,\bar{D}}$. Item (2) of Definition 3.5 yields $|g \cap \{q_t^{\alpha,p,\bar{D}} : t \in \mathcal{T}_\alpha^{p,\bar{D}}, \mathrm{rk}_\alpha(t) = \lambda^{++}\}| = 1$ for all $\alpha \in C_{p,\bar{D}}$, and we denote the corresponding element of $\mathcal{T}_\alpha^{p,\bar{D}}$ by $t_\alpha^{p,\bar{D}}$. Let $\langle q_*^{\alpha,p,\bar{D}} : \alpha < \lambda \rangle$ be a decreasing sequence of elements of g such that $q_*^{\alpha,p,\bar{D}} \leq x^{\alpha,p,\bar{D}}$ for all $\alpha < \lambda$ and $q_*^{\alpha,p,\bar{D}} \leq q_{t_\alpha^{p,\bar{D}}}^{\alpha,p,\bar{D}}$ for all $\alpha \in C_{p,\bar{D}}$. From now on we shall omit p and $\bar{D}$ in the indices if they are clear from the context. For example, we shall write $q_*^\alpha, q_{t_\alpha}^\alpha, x^\alpha$, and U instead of $q_*^{\alpha,p,\bar{D}}, q_{t_\alpha^{p,\bar{D}}}^{\alpha,p,\bar{D}}, x^{\alpha,p,\bar{D}}$, and $U_{p,\bar{D}}$, respectively.

Set $\tilde{\mathcal{T}} = j^*(\bar{\mathcal{T}})(\lambda)$ and $\tilde{q} = j^*(\bar{q})(\lambda) = \langle \tilde{q}_t : t \in \tilde{\mathcal{T}} \rangle$. By elementarity of j^*, $\tilde{\mathcal{T}}$ is a regular standard $(w_\lambda, 1)^{j(\lambda^{++})}$-tree for some $w_\lambda \subset j(\lambda^{++})$ of size $|w_\lambda| \leq \lambda$. Let us also denote by $\tilde{\tilde{\mathcal{T}}}$ the collection of those $t \in \tilde{\mathcal{T}}$ which can be extended to a $t' \in \tilde{\mathcal{T}}$ such that $\mathrm{rk}_\lambda(t') = j(\lambda^{++})$ and $\tilde{q}_{t'} \leq j^*(q_{t_\alpha}^\alpha)$ for all $\alpha \in C$. By elementarity of j^* and item (6) of Definition 3.5 we have that $\tilde{\tilde{\mathcal{T}}}$ has the following property: for every two consecutive elements $\xi < \eta$ of w_λ and $t \in \tilde{\tilde{\mathcal{T}}}$ with $\mathrm{rk}_\lambda(t) = \xi$, the set $\{t' \in \tilde{\tilde{\mathcal{T}}} : \mathrm{rk}_\lambda(t') = \eta$ and $t \lhd t'\}$ has size at most λ. Let us denote by $\downarrow (p, \bar{D})$ the set of all lower bounds in $j^*(\mathbb{P}_{\lambda^{++}})$ of the collection $\{j^*(r)\} \cup \{j^*(q_*^\alpha) : \alpha < \lambda\} \subset j^*[g]$. Observe that $j^*(\mathbb{P}_{\lambda^{++}})$ is $< j(\lambda)$-directed closed in M^*, and hence $\downarrow (p, \bar{D})$ is non-empty.

Claim 3.9. $\{\tilde{q}_t : t \in \tilde{\tilde{\mathcal{T}}}, \mathrm{rk}_\lambda(t) = j(\lambda^{++})\}$ *is predense below* $\downarrow (p, \bar{D})$.

Proof. Let us fix $\tilde{r} \in \downarrow (p, \bar{D})$. Since $\tilde{r} \leq j^*(r)$, $\tilde{r} \Vdash_{j(\mathbb{R}_\lambda) * j(\mathbb{P}_{\lambda^{++}})}$ "the set Π of those $\zeta < j(\lambda)$ such that $\Gamma_{j^*(\mathbb{P}_{\lambda^{++}})} \cap \{j^*(\bar{q})_t^\zeta : t \in j^*(\bar{\mathcal{T}})_\zeta, \mathrm{rk}_\zeta(t) = j(\lambda^{++})\} \neq \emptyset$ contains the intersection of $j^*(U)$ and of a club $j^*(\sigma)$". Since $\tilde{r} \leq j^*(x^\alpha)$ for all $\alpha < \lambda$, we conclude that $\tilde{r} \Vdash_{j(\mathbb{R}_\lambda) * j(\mathbb{P}_{\lambda^{++}})}$ "β_ζ is the ζth element of $j^*(\sigma)$ for all $\zeta < \lambda$". Thus $\tilde{r}$ forces that $j^*(\sigma)$ is unbounded below λ, and hence it forces $\lambda \in j^*(\sigma)$. Recall that $\mathcal{U}$ is a subset of the measure derived from j^*, and hence $\lambda \in j^*(U)$, and consequently $\tilde{r}$ forces $\lambda \in \Pi$. From the above it follows that the set

$$\{j^*(\bar{q})_t^\lambda : t \in j^*(\bar{\mathcal{T}})_\lambda, \mathrm{rk}_\lambda(t) = j(\lambda^{++})\} = \{\tilde{q}_t : t \in \tilde{\mathcal{T}}, \mathrm{rk}_\lambda(t) = j(\lambda^{++})\}$$

is predense below $\tilde{r}$. The regularity of $\tilde{\mathcal{T}}$, the fact that $\tilde{r}$ is below $j^*(q_{t_\alpha}^\alpha)$ for all $\alpha < \lambda$, and Definition 3.5(5) imply that all elements of $\{\tilde{q}_t : t \in \tilde{\mathcal{T}} \setminus \tilde{\tilde{\mathcal{T}}}, \mathrm{rk}(t) = j(\lambda^{++})\}$ are incompatible with $\tilde{r}$, and hence $\{\tilde{q}_t : t \in \tilde{\tilde{\mathcal{T}}}, \mathrm{rk}(t) = j(\lambda^{++})\}$ is predense below $\tilde{r}$. $\square$

Now we start working with different pairs $(p, \bar{D})$, and hence it is useful for understanding the rest of the proof to keep in mind our previous convention

regarding the simplification of notations: $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^{p,\bar{D}}$, $\tilde{\tilde{\mathcal{T}}} = \tilde{\tilde{\mathcal{T}}}^{p,\bar{D}}$, $\mathrm{rk}_\lambda = \mathrm{rk}_\lambda^{p,\bar{D}}$, and $w_\lambda = w_\lambda^{p,\bar{D}}$.

Let us write the set of all pairs $\langle p, \bar{D} \rangle$ such that $p \in \mathbb{P}_{\lambda^{++}}$, $\bar{D}$ is a λ-sequence of open dense subsets of $\mathbb{P}_{\lambda^{++}}$, and $r(p, \bar{D}) \in g$ in the form $\{\langle p^i, \bar{D}^i \rangle : i < \nu\}$ for some cardinal ν. In the rest of the proof we shall simply use i instead of the tuple $\langle p^i, \bar{D}^i \rangle$ in notations of the objects corresponding to this tuple defined above. For example, we shall write $r(i)$ and $q_{t^i_\alpha}^{\alpha,i}$ instead of $r(p^i, \bar{D}^i)$ and $q_{t^{p^i, \bar{D}^i}_\alpha}^{\alpha, p^i, \bar{D}^i}$, respectively.

Now, for every $\xi < \lambda^+$ we shall construct a sequence $\langle t^{\xi,i} : i < \nu \rangle$ such that

- (i) $t^{\xi,i} \in \tilde{\tilde{\mathcal{T}}}^i$ and $\mathrm{rk}_\lambda^i(t^{\xi,i})$ equals the ξth element of $w_\lambda^i \cup \{j(\lambda^{++})\}$ if ξ does not exceed the order type of $w_\lambda^i \cup \{j(\lambda^{++})\}$, and equals $j(\lambda^{++})$ otherwise;
- (ii) $t^{\xi,i} \lhd t^{\eta,i}$ for all $\xi < \eta$; and
- (iii) The set $\{\tilde{q}_{t^{\xi,i}}^i : \xi < \lambda^+, i < \nu\} \cup \{j^*(r(i)) : i < \nu\} \cup \{j^*(q_*^{\alpha,i}) : \alpha \in \lambda, i < \nu\}$ is centered.

Suppose that for some $0 < \xi < \lambda^+$ and all $\zeta < \xi$ we have already constructed $t^{\zeta,i}$'s so that the items (i)-(ii) above are satisfied and the set

$$\{\tilde{q}_{t^{\zeta,i}}^i : \zeta < \xi, i < \nu\} \cup \{j^*(r(i)) : i < \nu\} \cup \{j^*(q_*^{\alpha,i}) : \alpha \in \lambda, i < \nu\}$$

is centered.

If ξ is limit then we define $t^{\xi,i}$ to be the smallest upper bound for the sequence $\langle t^{\zeta,i} : \zeta < \xi \rangle$ for all $i < \nu$. Items (i)-(ii) are clearly satisfied for all $\zeta \leq \xi$. Thus we are left with the task to check that the set

$$O = \{\tilde{q}_{t^{\zeta,i}}^i : \zeta \leq \xi, i < \nu\} \cup \{j^*(r(i)) : i < \nu\} \cup \{j^*(q_*^{\alpha,i}) : \alpha \in \lambda, i < \nu\}$$

is centered. Suppose to the contrary that O fails to be centered and let $i < \nu$ be the minimal ordinal such that there exists a finite subset E of O with $\max\{k : \tilde{q}_{t^{\xi,k}}^k \in E\} = i$ which has no lower bound. Enlarging E and then eliminating those of its elements which have a lower bound in E, we may assume that

$$E = \{\tilde{q}_{t^{\xi,k}}^k : k \in F \cap i\} \cup \{\tilde{q}_{t^{\xi,i}}^i\} \cup \{j^*(r(i)) : i \in F\} \cup \{j^*(q_*^{\alpha,i}) : i \in F\}$$

for some $\alpha < \lambda$ and F such that $i \in F \in [\nu]^{<\omega}$. The minimality of i implies that

$$(E \setminus \{\tilde{q}_{t^{\xi,i}}^i\}) \cup \{\tilde{q}_{t^{\zeta,i}}^i\} \cup \{j^*(q_*^{\beta,i}) : \beta \in L\}$$

has a lower bound for every $\zeta < \xi$ and $L \in [\lambda]^{<\omega}$. The existence of exact lower bounds and $< j(\lambda)$-directed closeness of $j^*(\mathbb{P}_{\lambda^{++}})$ implies that the set

$$(E \setminus \{\tilde{q}^i_{t\xi,i}\}) \cup \{r^i_\xi\} \cup \{j^*(q^{\beta,i}_*) : \beta \in \lambda\}$$

has a lower bound, say r, where $r^i_\xi = \tilde{q}^i_{t\xi,i} \upharpoonright \sup_{\zeta<\xi} \mathrm{rk}^i_\lambda(t^{\zeta,i})$. (Notice that $\tilde{q}^i_{t\zeta,i} \in j^*(\mathbb{P}_{\lambda^{++}}) \upharpoonright \mathrm{rk}^i_\lambda(t^{\zeta,i})$, but it is not necessary that $\mathrm{rk}^i_\lambda(t^{\xi,i})$ equals $\sup_{\zeta<\xi} \mathrm{rk}^i_\lambda(t^{\zeta,i})$, and hence there is no reason to think that $r^i_\xi = \tilde{q}^i_{t\xi,i}$.)

Claim 3.9 yields that the set $\{\tilde{q}^i_t : t \in \tilde{\tilde{\mathcal{T}}}^i, t^{\xi,i} \lhd t,$ and $\mathrm{rk}^i_\lambda(t)$ equals $j(\lambda^{++})\}$ is predense below the set $\{r^i_\xi, j^*(r(i))\} \cup \{j^*(q^{\alpha,i}_*) : \alpha \in \lambda\}$. Therefore the one-element set $\{\tilde{q}^i_{t\xi,i}\} = \{\tilde{q}^i_t : t \in \tilde{\tilde{\mathcal{T}}}^i, t^{\xi,i} \lhd t,$ and $\mathrm{rk}^i_\lambda(t)$ equals ξth element of $w^i_\lambda \cup \{j(\lambda^{++})\}\}$ is predense below the set $\{r^i_\xi, j^*(r(i))\} \cup \{j^*(q^{\alpha,i}_*) : \alpha \in \lambda\}$, which means that $\tilde{q}^i_{t\xi,i}$ is weaker than any lower bound of the set $\{r^i_\xi, j^*(r(i))\} \cup \{j^*(q^{\alpha,i}_*) : \alpha \in \lambda\}$. In particular $\tilde{q}^i_{t\xi,i} \geq r$, which means that r is a lower bound for E, a contradiction.

Let us now consider the case $\xi = \zeta + 1$. We shall construct $t^{\xi,i}$ by induction on $i < \nu$. Suppose that $t^{\xi,i'}$ have been already constructed for all $i' < i$ such that (i) holds for all $i' < i$ and the following conditions are satisfied:

(iv) $t^{\zeta,i'} \lhd t^{\xi,i'}$ for all $i' < i$; and

(v) The set $\{\tilde{q}^{i'}_{t\xi,i'} : i' < i\} \cup \{\tilde{q}^k_{t\zeta,k} : i \leq k < \nu\} \cup \{j^*(r(k)) : k < \nu\} \cup \{j^*(q^{\alpha,k}_*) : \alpha \in \lambda, k < \nu\}$ is centered.

If $\zeta \geq o.t.(w^i_\lambda \cup \{j(\lambda^{++})\})$, then we set $t^{\xi,i} = t^{\zeta,i}$. So assume that $\zeta < o.t.(w^i_\lambda \cup \{j(\lambda^{++})\})$. To construct $t^{\xi,i}$ observe that Claim 3.9 implies that the set $A = \{\tilde{q}^i_t : t \in \tilde{\tilde{\mathcal{T}}}^i, t^{\zeta,i} \lhd t,$ and $\mathrm{rk}^i_\lambda(t)$ equals to the ξth element of $w^i_\lambda \cup \{j(\lambda^{++})\}\}$ is predense below the set $B^{\zeta,i} := \{\tilde{q}^i_{t\zeta,i}, j^*(r(i))\} \cup \{j^*(q^{\alpha,i}_*) : \alpha \in \lambda\}$. Suppose that there is no $\tilde{q}^i_t \in A$ such that the set

$$\{\tilde{q}^{i'}_{t\xi,i'} : i' < i\} \cup \{\tilde{q}^i_t\} \cup \{\tilde{q}^k_{t\zeta,k} : i < k < \nu\} \cup \{j^*(r(k)) : k < \nu\} \cup$$
$$\cup \{j^*(q^{\alpha,k}_*) : \alpha \in \lambda, k < \nu\}$$

is centered. This means that for every $\tilde{q}^i_t \in A$ there are $I'_t \in [i]^{<\omega}$, $K_t \in [\nu \setminus (i+1)]^{<\omega}$, $K^*_t \in [\nu]^{<\omega}$, and $S_t \in [\lambda \times \nu]^{<\omega}$ such that the union of the set

$$X_t := \{\tilde{q}^{i'}_{t\xi,i'} : i' \in I'_t\} \cup \{\tilde{q}^k_{t\zeta,k} : k \in K_t\} \cup \{j^*(r(k)) : k \in K^*_t\} \cup$$
$$\cup \{j^*(q^{\alpha,k}_*) : (\alpha,k) \in S_t\}$$

with $\{\tilde{q}^i_t\}$ has no lower bound. Set $X = \bigcup_{\tilde{q}^i_t \in A} X_t \cup B^{\zeta,i}$. By Definition 3.5(6) we have that $|A| \leq \lambda$, and hence $|X| \leq \lambda$. Combined with (v) and the directed $< j(\lambda)$-directed closeness of $j(\mathbb{P}_{\lambda^{++}})$ this implies that there exists a lower bound $\tilde{q}$ of X. In particular, $\tilde{q} \leq \tilde{q}^i_{t\zeta,i}$, and hence there exists $\tilde{q}^i_t \in A$

compatible with $\tilde{q}$. But then the set

$$\{\tilde{q}_{t\xi,i'}^{i'} : i' \in I_t'\} \cup \{\tilde{q}_t^i\} \cup \{\tilde{q}_{t\zeta,k}^k : k \in K_t\} \cup \{j^*(r(k)) : k \in K_t^*\} \cup$$
$$\cup \{j^*(q_*^{\alpha,k}) : (\alpha, k) \in S_t\}$$

has a lower bound, a contradiction. From the above it follows that there exists $\tilde{q}_t^i \in A$ such that the set

$$\{\tilde{q}_{t\xi,i'}^{i'} : i' < i\} \cup \{\tilde{q}_t^i\} \cup \{\tilde{q}_{t\zeta,k}^k : i < k < \nu\} \cup \{j^*(r(k)) : k < \nu\} \cup$$
$$\cup \{j^*(q_*^{\alpha,k}) : \alpha \in \lambda, k < \nu\}$$

is centered, and we denote the corresponding t by $t^{\xi,i}$. This finishes our construction of $t^{\xi,i}$'s satisfying conditions (i)-(iii).

For every $i < \nu$ we denote by $\tilde{q}^i$ the condition $\tilde{q}_{t^{o.t.(w_\lambda^i \cup \{j(\lambda^{++})\}),\,i}}^i \in j^*(\mathbb{P}_{\lambda^{++}})$. From the above it follows that

$$\mathrm{rk}_\lambda^i(t^{o.t.(w_\lambda^i \cup \{j(\lambda^{++})\}),\,i}) = j(\lambda^{++}) \text{ and } t^{\xi,i} = t^{o.t.(w_\lambda^i \cup \{j(\lambda^{++})\}),\,i}$$

for all $\xi > o.t.(w_\lambda^i \cup \{j(\lambda^{++})\})$. Condition (iii) implies that the set $\{\tilde{q}^i : i < \nu\} \cup \{j^*(r(i)) : i < \nu\}$ is centered.

Next, let us show that for every open dense subset $\tilde{D}$ of $j^*(\mathbb{P}_{\lambda^{++}})$ which is an element of M^* there exists $i < \nu$ such that $\tilde{q}_{t\xi,i}^i \in \tilde{D}$. In the same way as in the proof of Theorem 2.15 we can find a sequence $\bar{D}' = \langle D_\alpha : \alpha < \lambda \rangle$ of open dense subsets of $\mathbb{P}_{\lambda^{++}}$ such that $j^*(\bar{D}')(\lambda) \subset \tilde{D}$. Notice that the set $\{r(p, \bar{D}') : p \in \mathbb{P}_{\lambda^{++}}\}$ is dense in $\mathbb{P}_{\lambda^{++}}$, and hence there exists $p \in \mathbb{P}_{\lambda^{++}}$ such that $r(p, \bar{D}') \in g$. Let $i < \nu$ be such that $\langle p, \bar{D}' \rangle = \langle p^i, \bar{D}^i \rangle$. Then

$$\tilde{q}^i = \tilde{q}_{t^{o.t.(w_\lambda^i \cup \{j(\lambda^{++})\}),\,i}}^i \in j^*(\bar{D}^i)(\lambda) \subset \tilde{D}$$

by Definition 3.5(4) and item (i).

Thus the set $h := \{\tilde{q} \in j^*(\mathbb{P}_{\lambda^{++}}) : \exists i < \nu \, (\tilde{q}^i \leq \tilde{q})\} \in V[G * g]$ is an upwards closed centered subset of $j^*(\mathbb{P}_{\lambda^{++}})$ meeting each open dense subset of $j^*(\mathbb{P}_{\lambda^{++}})$ which are elements of M^*. Therefore h is $j^*(\mathbb{P}_{\lambda^{++}})$-generic over M^*, see, e.g., [16, Ch.VII, Lemma 7.4]. In addition, $j^*(r(i))$ is compatible with every element of h for every $i < \nu$, and hence $\{j^*(r(i)) : i < \nu\} \subset h$. Let us note that for every sequence $\bar{D} = \langle D_\alpha : \alpha < \lambda \rangle$ of open dense subsets of $\mathbb{P}_{\lambda^{++}}$ the set $O := \{r(p, \bar{D}) : p \in \mathbb{P}_{\lambda^{++}}\}$ is dense in $\mathbb{P}_{\lambda^{++}}$ and $g \cap O \subset \{r(i) : i < \nu\}$. Therefore $j^*[g \cap O] \subset h$, and hence $j^*[g] \subset h$, which finishes our proof. $\square$

If we use preparation relative to a fast function (see [11]) instead of the poset $\mathbb{R}_\lambda$, we can prove the following theorem by almost literal repetition of the proof of Theorem 3.8.

Theorem 3.10. *Suppose GCH holds, θ is a regular cardinal, and $j : V \to M$ is an ultrapower embedding via a (λ, θ)-extender in V such that $H(\theta)$ of V is contained in M. Then there exists a λ-c.c. poset $\mathbb{R}$ of size λ such that for every normal filter $\mathcal{U}$ on λ contained in the measure derived from j, and for every λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle \in V^{\mathbb{R}}$ such that*

(a) *$\gamma \leq \theta$, and*

(b) *$\Vdash_{\mathbb{R} * \mathbb{P}_\xi}$ "$\mathbb{Q}_\xi$ is a $< \lambda$-directed closed reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ poset of size $< \max\{\lambda^{++}, \theta\}$ all of whose finite subsets have a greatest lower bound" for all $\xi < \gamma$,*

the embedding j can be extended to an elementary embedding $j^{**} : V^{\mathbb{R} * \mathbb{P}_\gamma} \to M^{j(\mathbb{R} * \mathbb{P}_\gamma)}$ so that $H(\theta)$ of $V^{\mathbb{R} * \mathbb{P}_\gamma}$ and $H(\theta)$ of $M^{j(\mathbb{R} * \mathbb{P}_\gamma)}$ coincide. In particular, λ remains measurable in $V^{\mathbb{R} * \mathbb{P}_\gamma}$.

Corollary 3.11. *Suppose GCH holds and λ is a strong cardinal. Then there exists a λ-c.c. poset $\mathbb{R}$ of size λ such that for every λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle \in V^{\mathbb{R}}$, if $\Vdash_{\mathbb{R} * \mathbb{P}_\xi}$ "$\mathbb{Q}_\xi$ is a $< \lambda$ directed closed reasonably B_e-bounding over $\mathcal{U}, \bar{\mu}$ all of whose finite subsets have a greatest lower bound" for all $\xi < \gamma$, then λ remains strong in $V^{\mathbb{R} * \mathbb{P}_\gamma}$.*

4. APPLICATIONS

To the best knowledge of the authors, all of the known results stating that a certain degree of strongness is preserved by iterations of tree forcings follow from one of the Theorems 2.15, 2.22, 3.8, or 3.10.

Another application of Theorem 3.8 uses the following poset from [20].

Definition 4.1. 1. A set $E \subset [\lambda]^\lambda$ will be called a *normal collection*, if

- E is closed under diagonal intersections of sequences of length λ of its elements;
- If $E' \in [E]^{<\lambda}$, then $\bigcap E' \in E$;
- If $A \in E$ and $|B| < \lambda$, then $A \setminus B \in E$.

2. Let $\bar{E} = \langle E_\nu : \nu \in \lambda^{<\lambda} \rangle$ be a system of $< \lambda$-complete non-principal filters on λ and let $E \subset [\lambda]^\lambda$ be a normal collection. We define a forcing notion $\mathbb{Q}_E^{\bar{E}}$ as follows.

A condition p in $\mathbb{Q}_E^{\bar{E}}$ is a complete λ-tree $p \subset \lambda^{<\lambda}$ (see Definition 2.1) such that

- for every $\nu \in \lambda^{<\lambda}$, either $|\mathrm{succ}_p(\nu)| = 1$ or $\mathrm{succ}_p(\nu) \in E_\nu$;
- for every $\bar{\nu} \in [p]$ the set $\{\xi < \lambda : \mathrm{succ}_p(\bar{\nu} \restriction \xi) \in E_{\bar{\nu} \restriction \xi}\}$ is in E, where $[p]$ is the set of all maximal branches through p.   □

For a binary relation R on λ and $x, y \in \lambda^\lambda$ we denote by $[x \, R \, y]$ the set $\{\xi < \lambda : x(\xi) R y(\xi)\}$. Using this notation, for a family $\mathcal{U} \subset [\lambda]^\lambda$ the notation $x \leq_{\mathcal{U}} y$ means that $[x \leq y] \in \mathcal{U}$. Let $\bar\mu$ be such as in section 3.

The following lemma is analogous to [20, Prop. 1.12].

Lemma 4.2. *Suppose that $\mathcal{U}$ is a normal filter on λ, $S \in \mathcal{U}$, $\bar{E}$, E are such as in Definition 4.1, and $\lambda \setminus S \in E$. Then*

(1) $\mathbb{Q}_E^{\bar{E}}$ *is reasonably B_e-bounding over $\mathcal{U}, \bar\mu$;*

(2) *If $E = \mathcal{D}_\lambda \upharpoonright (\lambda \setminus S) := \{(\lambda \setminus S) \cap C : C \in \mathcal{D}_\lambda\}$, then $\mathbb{Q}_E^{\bar{E}}$ adds a function $\ell \in \lambda^\lambda$ such that $x \leq_{\tilde{E}} \ell$ for all $x \in \lambda^\lambda \cap V$, where $\tilde{E}$ is the closure of E in $V^{\mathbb{Q}_E^{\bar{E}}}$ under diagonal intersections of sequences of its elements of length λ.*[10]

Proof. 1. Let $p \in \mathbb{Q}_E^{\bar{E}}$. We are going to describe a strategy **st** for Generic in $\mathfrak{D}_{\mathcal{U},\bar\mu}^{\mathbf{B}_e}(p, \mathbb{Q}_E^{\bar{E}})$. In the course of the play, Generic constructs an auxiliary sequence $\langle T_\xi : \xi < \lambda \rangle$ so that if $\langle I_\xi, \langle p_t^\xi, q_t^\xi : t \in I_\xi \rangle : \xi < \lambda \rangle$ is the sequence constructed by two players in the course of the play, then the following conditions are satisfied:

(i) $T_\xi \in \mathbb{Q}_E^{\bar{E}}$ and if $\xi < \zeta < \lambda$ then $p = T_0 \supset T_\xi \supset T_\zeta$ and $T_\zeta \cap \lambda^\xi = T_\xi \cap \lambda^\xi$;

(ii) If $\xi < \lambda$ is limit, then $T_\xi = \bigcap_{\zeta < \xi} T_\zeta$;

(iii) If $\xi \in S$, then

     $- \; I_\xi = T_\xi \cap \xi^\xi$ and $p_t^\xi = (T_\xi)_t$ for $t \in I_\xi$,

     $- \; T_{\xi+1} = \bigcup\{q_t^\xi : t \in I_\xi\} \cup \bigcup\{(T_\xi)_\nu : \nu \in (T_\xi \cap \lambda^\xi) \setminus \xi^\xi\}$;

(iv) If $\xi \notin S$, then $I_\xi = \emptyset$ and $T_{\xi+1} = T_\xi$.

Conditions (i)-(iv) fully describe the strategy **st**. To show that it is a winning strategy of Generic consider a play $\langle I_\xi, \langle p_t^\xi, q_t^\xi : t \in I_\xi \rangle : \xi < \lambda \rangle$ in which Generic uses **st**. Let G be a $\mathbb{Q}_E^{\bar{E}}$-generic containing $p^* = \bigcap_{\xi < \lambda} T_\xi$ and $x_G \in \lambda^\lambda$ be the union of all the stems of elements of G. (The fact that $p^* \in \mathbb{Q}_E^{\bar{E}}$ can be proved in the same way as in [20, Prop. 1.12(2)].) Then $C_G := \{\xi \in \lambda : x_G \upharpoonright \xi \in \xi^\xi\} \in V[G]$ is a club. Let us fix $\xi \in C_G \cap S$. It follows that $x_G \upharpoonright \xi \in T_\xi$, and hence $(T_{\xi+1})_{x_G \upharpoonright \xi} = q_{x_G \upharpoonright \xi}^\xi \in G$. Therefore

$$\{\xi : \exists t \in I_\xi \, (q_t^\xi \in G)\} = \{\xi : \exists t \in I_\xi \, (x_G \in [q_t^\xi])\} \supset S \cap C_G \in \mathcal{U}^{\mathbb{Q}_E^{\bar{E}}},$$

which means that $p^* \Vdash \{\xi : \exists t \in I_\xi \, (q_t^\xi \in \Gamma_{\mathbb{Q}_E^{\bar{E}}})\} \in \mathcal{U}^{\mathbb{Q}_E^{\bar{E}}}$.

The verification of Definition 3.1(1) is straightforward.

2. Let G and x_G be such as before and $x \in \lambda^\lambda \cap V$. We claim that $x \leq_{E^{\mathbb{Q}_E^{\bar{E}}}} x_G$. Let D_x be the subset of $\mathbb{Q}_E^{\bar{E}}$ consisting of those p such that for

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[10]In other words, $\tilde{E} = E^{\mathbb{Q}_E^{\bar{E}}}$.

every splitting node $t \in p \cap \lambda^\xi$ and every extension s of t we have $s(\xi) > x(\xi)$. It is clear that D_x is a dense subset of $\mathbb{Q}_E^{\bar{E}}$, and hence there exists $p_x \in G \cap D_x$, which yields $x_G \in [p_x]$. Set $A = \{\xi \in \lambda : |\mathrm{succ}_{p_x}(x_G \restriction \xi)| > 1\} \in V[G]$. Since $E = \mathcal{D}_\lambda \restriction (\lambda \setminus S)$, we have that $A \subset (\lambda \setminus S)$ and A contains all its limit points which are elements of $\lambda \setminus S$. Indeed, let $\beta \in \lambda \setminus S$ be a limit point of A, $f \in [p_x] \cap V$ be such that $f \restriction \beta = x_G \restriction \beta$, and $C \in \mathcal{D}_\lambda \cap V$ be such that $A_f := \{\xi \in \lambda : |\mathrm{succ}_{p_x}(f \restriction \xi)| > 1\} = (\lambda \setminus S) \cap C$. Since β is a limit point of A, $\beta \in \lambda \setminus S$, and $A \cap \beta = A_f \cap \beta = ((\lambda \setminus S) \cap C) \cap \beta$, we conclude that $\beta \in C$, and hence $\beta \in A_f$, which means that $f \restriction \beta$ splits in p_x. But $f \restriction \beta = x_G \restriction \beta$, hence $x_G \restriction \beta$ splits in p_x, and consequently $\beta \in A$.

It follows from the above that A can be written in the form $C_1 \cap (\lambda \setminus S)$ for some club $C_1 \in V[G]$ and hence $A \in E^{\mathbb{Q}_E^{\bar{E}}}$. It suffices to observe that $x_G(\xi) > x(\xi)$ for all $\xi \in A$ by the definition of D_x. □

A standard argument shows that if $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \gamma \rangle$ is a λ-support iteration such that $\mathbb{P}_\gamma$ is $B_e(\bar{\mathbb{Q}})$-bounding over some $\mathcal{U}, \bar{\mu}$, then $\mathbb{P}_\gamma$ is λ-proper (see the paragraph before Theorem 2.14 for the definition of λ-properness).

Given a filter $\mathcal{F}$ on λ, we denote by $\mathfrak{d}_\mathcal{F}$ the smallest size of a subset of λ^λ which is dominating with respect to $\leq_\mathcal{F}$.

The following corollary is analogous to [20, Cor. 4.5] and [17, Cor. 5.1].

Corollary 4.3. *Suppose GCH holds and $j : V \to M$ is an (λ, λ^{++})-extender ultrapower such that $H(\lambda^{++})^V = H(\lambda^{++})^M$. Let $\mathcal{U}$ be the measure derived from j. Then there exists a λ^{++}-c.c. λ-proper poset $\mathbb{P}$ of size λ^{++} such that*

(1) *j can be extended to an elementary embedding $j^* : V^\mathbb{P} \to M^{j(\mathbb{P})}$;*
(2) *If $\mathcal{W} \in V^\mathbb{P}$ is a normal filter on λ such that $\mathcal{U} \subset \mathcal{W}$, then $\mathfrak{d}_\mathcal{W} = \lambda^+$;*
(3) *If $\mathcal{W} \in V^\mathbb{P}$ is a normal filter on λ such that $\mathcal{U} \not\subset \mathcal{W}$, then $\mathfrak{d}_\mathcal{W} = \lambda^{++}$.*

Proof. Let us write the collection of all elements of $\mathcal{U}$ with stationary complement in the form $\{U_i : i < \lambda^+\}$ and fix a map $\psi : \lambda^{++} \to \lambda^+$ such that $\psi^{-1}(i)$ is unbounded in λ^{++} for all $i \in \lambda^+$. Let also $\mathbb{R}$ be such as in Theorem 3.10. In $V^\mathbb{R}$ consider a λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \mathbb{Q}_\xi : \xi < \lambda^{++} \rangle$ such that $\mathbb{Q}_\xi$ is a $\mathbb{P}_\xi$-name for the poset $\mathbb{Q}_{E^\xi}^{\bar{E}^\xi}$ defined as follows: For $\nu \in \lambda^{<\lambda}$ the filter E_ν^ξ is generated by $\mathcal{D}_\lambda$, and E^ξ is the family $\mathcal{D}_\lambda \restriction (\lambda \setminus U_{\psi(\xi)})$ (both defined in $V^{\mathbb{R} * \mathbb{P}_\xi}$). We claim that $\mathbb{P} = \mathbb{R} * \mathbb{P}_{\lambda^{++}}$ is as required.

First of all, $\mathbb{P}_{\lambda^{++}}$ is $B_e(\bar{\mathbb{Q}})$-bounding over $\mathcal{U}, \bar{\mu}$ by Lemma 4.2 and Theorem 3.6. Therefore j can be lifted to the forcing extension by $\mathbb{P}$ by Theorem 3.8. In addition, it is a direct consequence of the $B_e(\bar{\mathbb{Q}})$-boundedness over $\mathcal{U}, \bar{\mu}$ that $V \cap \lambda^\lambda$ is dominating in $V^\mathbb{P} \cap \lambda^\lambda$ with respect to the preorder

$\leq_{\mathcal{U}^{\mathbb{P}}}$ (recall that $\mathbb{R}$ has λ-c.c.), and hence $\lambda^+ \leq \mathfrak{d}_{\mathcal{W}} \leq \mathfrak{d}_{\mathcal{U}^{\mathbb{P}}} = \lambda^+$ for every normal filter $\mathcal{W} \in V^{\mathbb{P}}$ extending $\mathcal{U}$.

Now suppose that $\mathcal{W} \in V^{\mathbb{P}}$ is a normal filter on λ which does not extend $\mathcal{U}$. Then there exists $i < \lambda^+$ such that $\lambda \setminus U_i \in \mathcal{W}$, and hence by Lemma 4.2 for every $\xi \in \psi^{-1}(i)$ the poset $\mathbb{Q}_\xi$ adds an element of λ^λ which dominates all elements of $\lambda^\lambda \cap V^{\mathbb{R}*\mathbb{P}_\xi}$ with respect to $\leq_{\mathcal{W}}$ (notice that $\mathcal{D}_\lambda \upharpoonright (\lambda \setminus U_i) \subset \mathcal{W}$). This proves item (3). $\qquad\square$

Remark 4.4. We use the notation of Corollary 4.3 here. The first item implies that there exists a normal measure $\mathcal{W} \in V^{\mathbb{P}}$ on λ such that $\mathcal{U} \subset \mathcal{W}$, and hence $\mathfrak{d}_{\mathcal{W}} = \lambda^+$ by item (2). However, we do not know whether in $V^{\mathbb{P}}$ there exists a normal measure on λ which does not contain $\mathcal{U}$. $\qquad\square.$

Corollary 4.5. *Suppose GCH holds and λ is a measurable cardinal. Let $\mathcal{U}$ be a normal measure on λ. Then there exists a λ-proper poset $\mathbb{P}$ of size λ^+ such that*

(1) *$j_{\mathcal{U}}$ can be extended to an elementary embedding $j^* : V^{\mathbb{P}} \to M^{j(\mathbb{P})}$;*
(2) *If $\mathcal{W} \in V^{\mathbb{P}}$ is a normal measure on λ such that $\mathcal{U} \not\subset \mathcal{W}$, then $\lambda^\lambda \cap V$ is bounded in $\lambda^\lambda \cap V^{\mathbb{P}}$ with respect to $\leq_{\mathcal{W}}$. In particular, if $\mathcal{U} \neq \mathcal{U}_1 \in V$ is a normal measure on λ, then $j_{\mathcal{U}_1}$ cannot be lifted to an elementary embedding from $V^{\mathbb{P}}$ to some $M' \supset M$.*

Proof. Let us write the collection of all elements of $\mathcal{U}$ with stationary complement in the form $\{U_i : i < \lambda^+\}$. Let also $\mathbb{R}$ be such as in Theorem 3.10. In $V^{\mathbb{R}}$ consider a λ-support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \lambda^+ \rangle$ such that $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}_i$-name for the poset $\mathbb{Q}_{\bar{E}^i}^{\bar{E}^i}$ defined as follows: For $\nu \in \lambda^{<\lambda}$ the filter E_ν^i is generated by $\mathcal{D}_\lambda$, and E^i is the family $\mathcal{D}_\lambda \upharpoonright (\lambda \setminus U_i)$ (both defined in $V^{\mathbb{R}*\mathbb{P}_i}$). We claim that $\mathbb{P} = \mathbb{R} * \mathbb{P}_{\lambda^+}$ is as required.

First of all, $\mathbb{P}_{\lambda^+}$ is $B_e(\bar{\mathbb{Q}})$-bounding over $\mathcal{U}, \bar\mu$ by Lemma 4.2 and Theorem 3.6, and $|\mathbb{P}| = |\mathbb{P}_{\lambda^+}| = \lambda^+$. Therefore $j_{\mathcal{U}}$ can be lifted to the forcing extension by $\mathbb{P}$ by Theorem 3.10 (in this case $\theta = \lambda^+$).

Now suppose that $\mathcal{W} \in V^{\mathbb{P}}$ is a normal filter on λ which does not extend $\mathcal{U}$. Then there exists $i < \lambda^+$ such that $\lambda \setminus U_i \in \mathcal{W}$, and hence by Lemma 4.2 the poset $\mathbb{Q}_i$ adds an element of λ^λ which dominates all elements of $\lambda^\lambda \cap V^{\mathbb{R}*\mathbb{P}_i}$ with respect to $\leq_{\mathcal{W}}$ (notice that $\mathcal{D}_\lambda \upharpoonright (\lambda \setminus U_i) \subset \mathcal{W}$).

Let $\mathcal{U}_1 \neq \mathcal{U}$ be a normal measure on λ. Suppose, contrary to our claim, that $j_{\mathcal{U}_1}$ can be extended to an elementary embedding $j' : V^{\mathbb{P}} \to M' \supset M$ and denote by $\mathcal{W}$ the measure on λ derived from j'. It follows from the above that there exists a function $x \in \lambda^\lambda \cap V^{\mathbb{P}}$ such that $y \leq_{\mathcal{W}} x$ for all $y \in \lambda^\lambda \cap V$. Therefore $j'(x)(\lambda) > j'(y)(\lambda) = j_{\mathcal{U}_1}(y)(\lambda)$ for all $y \in \lambda^\lambda \cap V$,

and hence $j'(x)(\lambda) \geq j_{\mathcal{U}_1}(\lambda)$. On the other hand, letting $\bar{\lambda}$ be the constant sequence of length λ all of whose entries are λ, we have $j_{\mathcal{U}_1}(\lambda) = j'(\lambda) = j'(\bar{\lambda})(\lambda) > j'(x)(\lambda)$, which leads to a contradiction. $\square$

5. Concluding remarks and open questions

Let κ be an inaccessible cardinal, A be a stationary subset of κ containing all successor ordinals, and $\mathcal{D}$ be any normal filter on κ containing A and all clubs in κ. We refer the reader to [20, § 6] for the definition of reasonably merry over $(A, \mathcal{D})$ forcing notions.

Proposition 5.1. *If $\mathbb{Q}$ has a good κ-fusion, then $\mathbb{Q}$ is reasonably merry over $(A, \mathcal{D})$.*

Proof. Let us fix $p \in \mathbb{Q}$ and suppose that we have reached stage α of the game $\mathfrak{D}^{\mathrm{master}}_{A,\mathcal{D}}(p, \mathbb{Q})$. Suppose also that we have already constructed aside a decreasing fusion sequence $\langle p_\xi : \xi < \alpha \rangle \in \mathbb{Q}^\alpha$. Let $p'_\alpha = \bigwedge_{\xi < \alpha} p_\xi$. If α is limit, then Generic plays

$$I_\alpha = \{s \,\hat{}\, h(\alpha) : s \in S^*_\alpha(p'_\alpha)\} \cap T(p'_\alpha)$$

and the collection $\{p^{\alpha,t} = (p'_\alpha)_t : t \in I_\alpha\}$, where S^*, T, and h are as in Definition 2.1. Suppose that $\{q^{\alpha,t} : t \in I_\alpha\}$ is the reply of the second player. Then using Claim 2.3(iii) we can find a condition p_α such that $p_\alpha \leq_\alpha p'_\alpha$ and $(p_\alpha)_t \leq q^{\alpha,t}$ for all $t \in I_\alpha \cap T(p_\alpha)$.

If α is a successor, then Generic chooses $I_\alpha = \emptyset$.

This finishes our definition of a strategy of Generic in the game $\mathfrak{D}^{\mathrm{master}}_{A,\mathcal{D}}(p, \mathbb{Q})$ which we shall denote by Υ. Now we shall show that Υ is winning. Let us fix a play $\langle I_\alpha, \langle p^{\alpha,s}, q^{\alpha,s} : s \in I_\alpha \rangle : \alpha < \kappa \rangle$ in which Generic uses Υ, and let $\langle p_\alpha : \alpha < \kappa \rangle$ be the fusion sequence of elements of $\mathbb{Q}$ constructed by him aside of the game $\mathfrak{D}^{\mathrm{master}}_{A,\mathcal{D}}(p, \mathbb{Q})$ as described above. Set $q = \bigwedge_{\alpha < \kappa} p_\alpha$.

Now we shall describe a winning strategy st for COM in the game $\mathfrak{D}^{\mathrm{servant}}_{A,\mathcal{D}}(\bar{q}, q, \mathbb{Q})$. Informally speaking, the strategy st instructs COM to mimic the proof of Claim 2.11 for all $\alpha \in A$ (in this case $\gamma = 1$ which simplifies the matter). This way for every $\alpha \in A$ the player COM constructs a condition $r_\alpha \in \mathbb{Q}$, $\sigma_\alpha \in \kappa^{<\kappa}$, and $\mu_\alpha \in \kappa$. More precisely, at stage $\alpha + 1 \in A$ of the game $\mathfrak{D}^{\mathrm{servant}}_{A,\mathcal{D}}(\bar{q}, q, \mathbb{Q})$ the player COM in the same way as in the proof of Claim 2.11 constructs a condition $r_{\alpha+1} \in \mathbb{Q}$, $\sigma_{\alpha+1} \in \kappa^{<\kappa}$, and $\mu_{\alpha+1} \in \kappa$ such that

(i) If $\beta \leq \alpha$, then $r_{\alpha+1} \leq r_\beta$ and $\sigma_\beta \subsetneq \sigma_{\alpha+1}$.
(ii) $\mathrm{dom}(\sigma_{\alpha+1}) = \mu_{\alpha+1} + 1$ and $\sigma_{\alpha+1}(\mu_{\alpha+1}) = h(\mu_{\alpha+1})$.

(iii) $\mu_{\alpha+1} \in \bigcap_{\beta \leq \alpha} C(r_\beta)$, $r_{\alpha+1} = (r_{\alpha+1})_{\sigma_{\alpha+1}}$, and

$$\sigma_{\alpha+1} \restriction \mu_{\alpha+1} \in \left[\bigcap_{\beta \leq \alpha} S^*_{\mu_{\alpha+1}}(r_\beta)\right] \cap S^*_{\mu_{\alpha+1}}(q),$$

and sets $A_{\alpha+1} = C(q) \setminus \mu_{\alpha+1}$.

If $\alpha \in A$ is limit, then the player COM sets $r'_\delta = \bigwedge_{\xi < \alpha} r_\xi$, $\mu_\alpha = \sup_{\xi < \alpha} \mu_\xi$ and $\sigma_\alpha = (\bigcup_{\xi < \alpha} \sigma_\xi) \hat{} h(\mu_\alpha)$. In the same way as in the proof of Claim 2.11 we can show that $\sigma_\alpha \in T(r'_\alpha)$ and

$$\sigma_\alpha \restriction \mu_\alpha \in \left[\bigcap_{\xi < \alpha} S^*_{\mu_\alpha}(r_\xi)\right] \cap S(r'_\alpha) \cap S^*_{\mu_\alpha}(q).$$

This allows COM to choose $r_\alpha = (r'_\alpha)_{\sigma_\alpha}$ and $A_\alpha = C(q) \setminus \mu_\alpha$.

Now suppose that α is a limit ordinal in A and

$$\alpha \in \bigcap_{\xi < \alpha} A_\beta = \bigcap_{\beta \in \alpha \cap A} (C(q) \setminus \mu_\beta) \cap \bigcap_{\beta \in \alpha \setminus A} A_\beta.$$

Then $\mu_{\beta+1} < \alpha$ for all $\beta < \alpha$, and hence $\mu_\alpha = \sup_{\beta < \alpha} \mu_{\beta+1} = \alpha$. Therefore $\sigma_\alpha \restriction \alpha \in S^*_\alpha(q) = S^*_\alpha(p_\alpha)$ and $\alpha \in C(q)$, and consequently

$$\sigma_\alpha = (\sigma_\alpha \restriction \alpha) \hat{} h(\alpha) \in T(q) \subset T(p_\alpha) \subset T(p'_\alpha).$$

It follows form the above that $\sigma_\alpha \in I_\alpha$ and

$$r_\alpha = (r'_\alpha)_{\sigma_\alpha} \leq (q)_{\sigma_\alpha} \leq (p_\alpha)_{\sigma_\alpha} \leq q^{\alpha,\sigma_\alpha},$$

which completes our proof. □

Although Proposition 5.1 combined with [20, Theorem 6.4] imply Theorem 2.9, we have presented a complete proof of Theorem 2.9 because some of its parts are used in the proof of Theorem 2.15, see, e.g., Claim 2.19.

Let $\mathcal{U}$ be a normal filter on κ. By [20, Observation 6.6] every reasonably B-bounding over $\mathcal{U}$ forcing notion is reasonably merry over $(S,\mathcal{U})$ for any $S \in \mathcal{U}$. In light of Proposition 5.1 and Theorems 3.8 and 2.15 it is natural to ask the following

Question 5.2. *Is there an analogue of Theorem 3.8 for (a suitably defined subclass of) posets which are reasonably merry over $(S,\mathcal{U})$?*

In case of a positive answer one could probably get a common generalization of Theorems 3.8 and 2.15.

We also do not know whether Theorem 3.8 is true for all reasonably B-bounding over $\mathcal{U}$ forcing notions provided that $\mathcal{U}$ is a subset of the normal measure derived from the embedding j. Even the following is open.

Question 5.3. *Suppose that κ is a strong cardinal and $\mathbb{Q}$ is $< \kappa$-directed closed reasonably A-bounding poset of size $|\mathbb{Q}| \leq \kappa^+$. Does κ remain measurable in $V^{\mathbb{K}*\mathbb{Q}}$ for some preparatory forcing $\mathbb{K}$?*

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