Laws and Initial Conditions

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I discuss two case studies from classical electrodynamics challenging the distinction between laws that delineate physically possible words and initial conditions. First, for many reasonable initial conditions there exist no global solutions to the Maxwell-Lorentz equations for continuous charge distributions. Second, in deriving an equation of motion for a charged point particle one needs to invoke an asymptotic condition that seems to express a physically contingent fact even though it is mathematically necessary for the derivation.

1. Introduction. There is a widespread conception of the respective roles of so-called "laws" and initial conditions in scientific theorizing according to which the laws of a theory delineate the worlds that, according to the theory, are physically possible. Particular possible worlds are then picked out by particular contingent initial conditions. Thus, on the standard picture there is a sharp distinction between laws and initial conditions: The former govern the interrelation between states of the world at one time and another; these interrelations are a matter of physical necessity. In the case of the deterministic laws of classical mechanics or electrodynamics, the laws determine how a given state must evolve. By contrast, initial conditions are a purely contingent matter and are not dictated by the theory itself.

In the possible worlds allowed by the theory's laws, the laws hold everywhere and everywhen. In this sense the laws are "global." Of course, most of the equations referred to as "laws" in the sciences do not express universal truths in the actual world and at best govern phenomena in a limited domain. On the standard picture, this is due to a mismatch between the possible worlds allowed by the theory and the real world; there is only an approximate fit between certain regions of some of the possible worlds allowed by the theory and certain regions of the real world, where these latter regions constitute the domain of the theory.

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In this paper I will present certain aspects of one of the central theories of contemporary physics, classical electrodynamics, which suggest that the standard picture may be inadequate: laws need not be global (in the sense of permitting of global solutions) and there are principles that are not easily classified as either laws or initial conditions.

Two preliminary remarks are in order. First, I summarized the standard picture in ways in which it is often presented, but there is one obvious way in which the picture is inadequate as it stands. For there are many examples of equations in physics which appear to be perfectly good instances of physical "laws," but which impose constraints on the state of the world at a given time. In fact, we need not look any further than the laws of classical electrodynamics, the Maxwell equations, for an example of this. That these equations impose not only dynamical constraints on how the state of an electromagnetic system changes follows from the order of the equations. The electromagnetic field has six degrees of freedom at each point (the components of the electric and magnetic field vectors), while the equations provide us with eight independent conditions as constraints: the two equations for the divergence of the electric and magnetic fields, which provide one constraint each, and the two vector equations (Faraday's and Ampere's laws), which are equivalent to six equations relating individual components of the fields to components of the curl of the field.

Several years ago in a paper at the PSA, Professor Sklar argued against the idea that "the initial state of the world at a given time is purely contingent" (Sklar 1991, 553). But if that was all there was at issue, then the case against the standard picture could be made quite easily. Rather, what is at stake is whether the role of the so-called "laws" of a theory is to delineate the range of what is physically possible, independently of whether the lawful constraints are diachronous or synchronous constraints, and whether the role of the laws sharply contrasts with that of merely contingent initial conditions, which can be specified freely within the bounds set by any synchronous lawful constraints.

Second, the type of mathematical problem usually used to motivate the standard picture is that of an initial-value or Cauchy problem. It is assumed that the laws take the form of ordinary partial differential equations which can be solved by specifying the state of the world on an appropriate initial-value surface—the Cauchy data. Phenomena in classical electrodynamics are not usually, however, represented in terms of a pure initial-value problem. Instead the electromagnetic field associated with a collection of charges is usually given in terms of a retarded-field problem, where the field at each point P is determined by the states of charged particles at the points of intersection of their trajectories with the backward light cone with vertex at P. This means that the total field is determined by

the field at the initial time t in conjunction with *the entire trajectories* (from t onward) of all the charged particles present.

One might suspect that the reasons for choosing a retarded-field representation with trajectories given rather than one in terms of a pure initial-value problem are entirely pragmatic, since it is easier simply to assume the trajectories of the charges than to try to determine the trajectories through an initial-value problem as well. But that is not the main reason. Rather phenomena involving charged particles and electromagnetic fields are not represented in terms of pure initial-value problems, because there does not appear to be a fully satisfactory and consistent theory governing the mutual interaction of charges and fields. Generally, the effect of external fields on the motion of a charge is taken to be given by the Lorentz force law, which is treated as a Newtonian force law. However, for finitely charged discrete particles the Lorentz force law is strictly inconsistent with the Maxwell equations and the principle of energy conservation. I will discuss a rival force law below, but the standard response to this problem (when we are interested in the dependence of the field on the motion of charged particles) is to assume that the trajectories of all charges are somehow given and to ignore the question as to how the field affects the motion of a charge. That is, particle trajectories together with the initial value of the field are treated as contingent inputs into the Maxwell equations, which then determine the value of the field everywhere.

The internal inconsistency of the laws or basic equations of classical electrodynamics seriously calls into question the standard view of laws as delineating coherent classes of physically possible worlds. I discuss this issue in Frisch 2005. Here I want to focus on some other aspects of classical electrodynamics here and show that even arguably consistent versions of classical electromagnetism do not fit the standard account of laws and initial conditions.

2.   Continuum Electrodynamics. Unlike a theory of discrete finitely charged particles, classical electrodynamics of continuous charge distributions or *charged fluids* is consistent. (The difference from the particle theory is that self-interactions, which are responsible for the inconsistency in the particle theory are infinitesimal for infinitesimal fluid "particles.") For that reason some would like to think of the continuum theory as the fundamental or basic theory of electromagnetism. But even this seemingly so well-behaved theory spells trouble for the standard view.

For a charged fluid one can formulate a pure Cauchy problem for the full system of the Maxwell-Lorentz equations—that is, for the Maxwell equations, governing the dependence of the fields on the charge distribution, and the Lorentz law, governing the motion of the charge distri-

bution in external fields (Parrott 1987, sec. 5.1–5.3). More specifically, one can show that initial conditions consisting of the initial fields $\mathbf{E}(t = 0, \mathbf{x})$, $\mathbf{B}(t = 0, \mathbf{x})$ and the velocity of the fluid $\mathbf{v}(t = 0, \mathbf{x})$ determine a unique solution to the Maxwell-Lorentz equations on an open subset of R^4, if one assumes that $\mathbf{B}_0 = \mathbf{B}(t = 0, \mathbf{x})$ satisfies $\nabla \cdot \mathbf{B}_0 = 0$. (The charge density ρ need not be introduced separately, but is defined via Coulomb's law $\nabla \cdot \mathbf{E} = 4\pi\gamma\rho$, where $\gamma(\mathbf{v})$ is the factor familiar from relativity theory.) Unlike the Maxwell equations alone, the Maxwell-Lorentz equations are only quasi-linear, but nevertheless the theory seems to fit the standard view rather well. The initial conditions can be freely chosen, with one constraint, and the laws then determine how a "world" with given initial state has to evolve. In other words, the Maxwell-Lorentz equations seem to delineate the class of electrodynamically possible worlds, where different worlds are picked out by different contingent initial conditions.

The problem, however, is that standard existence and uniqueness proofs only ensure that there are *local* solutions to the Maxwell-Lorentz equations, defined on some open set $t_1 < t < t_2$, $r_1 < r < r_2$. And there is in fact a large class of seemingly physically reasonable initial conditions for which *global* solutions to the equations can easily be shown not to exist, namely spherically symmetric charge distributions with zero initial velocity for which the fluid density thins out sufficiently rapidly as $r \to \infty$.

In this case each fluid particle will experience an acceleration proportional to Q/r^2, where Q is the total charge inside the sphere with radius r (centered at the origin) and is constant on the worldline of the particle. Since the initial velocity is zero, fluid particles will move outwards for $t > 0$ with ever increasing velocity and $r(\infty) = \infty$. Q/r functions as the potential energy of a fluid particle at distance r from the origin. If a particle has a higher potential energy at $t = 0$, then it will end up having a higher terminal velocity $v(\infty)$. Now consider two fluid particles 1 and 2 with $r_1(0) < r_2(0)$, such that the inner particle has a higher potential energy; that is, we assume that $Q_1/r_1 > Q_2/r_2$. This will be the case whenever the charge density thins out sufficiently rapidly with r, so that the difference in the total charges contained in the spheres with radii r_1 and r_2 is more than made up for by the increase in radius. Intuitively, the inner particle will have greater potential energy if the total charge contained in the sphere will increase only slightly as the radius of the sphere increases appreciably.

Since particle 1 starts inside the sphere of particle 2, but its terminal velocity $v_1(\infty)$ is larger than that of particle 2, there must be a time t^* at which $r_1(t^*) = r_2(t^*)$. That is, if we take two particles with same angular coordinates, the worldlines of the particles intersect, which is impossible. For, first, this contradicts the assumption that the potential energy of particle 1 is larger than that of 2 or that Q is constant on the worldline

of a charge (which is a consequence of the Maxwell equations). And, second, if two worldlines of infinitesimal fluid particles were to intersect, the velocity of the fluid would not be well defined at that point, since two different velocity vectors would be associated with one and the same point.

The upshot of the discussion is that there are no global solutions to the Maxwell-Lorentz equations for the kind of case we discussed; the solutions cannot be extended beyond some time $t^* > 0$, and (since the equations are time symmetric) backward in time beyond a time $-t^* < 0$. What are we to conclude from this? Given how the problem arises, it seems plausible that the problem is not limited to the specific case discussed and that intersecting worldlines are a typical feature characterizing many more types of initial conditions than the ones we discussed. One possible response might be to argue that the case is just one further example that shows that initial conditions cannot be chosen as freely as we might initially have thought. This is a point familiar from discussions of closed causal loops in general relativity (see Sklar 1991). In that context it is often argued that the impossibility of engineering the death of one's own grandfather, for example, does not show that causal loops are impossible, but only that the relevant initial conditions could not possibly obtain and, hence, that there is an added constraint of self-consistency on possible solutions to the field equations. Applied to our case, one might argue that the upshot of my discussion ought to be that what appear to be physically reasonable initial conditions are shown to be physically impossible after all. But in the current context this response does not seem all that plausible. Unlike in the case of closed causal loops, it is not clear how the breakdown of the solution in the future could have any effects, as it were, on an experimenter's effort to set up a spherically symmetric charge distribution that thins out relatively rapidly, for example. How could the breakdown of the solutions in the future affect our ability to prepare a state that is electromagnetically possible locally?

Parrott suggests that the possibility of intersecting worldlines calls into doubt whether the retardation condition is consistent with the Maxwell-Lorentz equations, since, as he says, retarded solutions must be defined on the entire backward light cones of each field point (Parrott 1987, 187). The worry seems to be this. According to the retarded-field representation the field at each point depends on what happens on the entire backward light cone erected at that point. So if there is a finite time $-t^* < 0$ past which the solution cannot be extended, then the retarded solution is not well defined. But the problem seems to lie more directly with the non-existence of global solutions. For if we set up a *local* initial-value problem for a finite region of spacetime, the retarded solution needs to be defined only within that region, since the contribution of charges in the past of the region will be encoded in the incoming field. Thus, if we rest content

with only local solutions of the Maxwell-Lorentz equations, there is no problem with the retardation condition.

For a wide class of intuitively physically reasonable initial conditions the Maxwell-Lorentz equations have no global solutions. That is, there are no electrodynamically possible worlds in which any of these initial conditions hold. This does not mean, however, that continuum electro-dynamics is practically useless. The theory is useful since it allows us to construct local models of the phenomena, even in situations where initial conditions such as the ones discussed above are appropriate.

3. The Asymptotic Condition in Particle Electrodynamics. The previous example challenged the view that the laws of a theory need to delineate possible worlds, or global models, in order to be used successfully. The next case I want to discuss is meant to call into doubt that the division between physically necessary laws and contingent boundary conditions is exhaustive. In particle electrodynamics the traditional Lorentz force law is inconsistent with the Maxwell equations, since it ignores any effect that the radiation field of a charged particle should have on the motion of the charge. The *Lorentz-Dirac equation* is an alternative equation of motion that includes a radiation reaction term (see Parrott 1987 and Rohrlich 1990 for detailed discussions of the Lorentz-Dirac theory). This equation is unusual in that it is a third-order differential equation of motion. Thus, to solve the equation one needs to specify three initial conditions, such as the initial acceleration in addition to initial position and velocity (and not only the customary two conditions as in the Newtonian case). One odd feature of the Lorentz-Dirac equation is that it allows for so-called *runaway solutions* according to which a charge can accelerate even in the absence of any external forces and then continue to accelerate at an ever increasing rate. These solutions are usually excluded by imposing as one of the "initial" conditions the condition that the acceleration of the charge should vanish at future infinity. Imposing this condition then allows us to reduce the order of the equation and derive a second-order integro-differential equation of motion, which can be solved by assuming standard initial conditions.

What I am interested in here is how we should think about the first condition—that of zero final acceleration. The case is unusual in that the equation is being solved by assuming a mixture of initial and final con-ditions—that is, the final acceleration in conjunction with initial position and velocity—but aside from that one might think that the standard account should apply: the initial conditions are contingent sets of con-ditions that pick out specific possible worlds from the set of worlds defined by the Lorentz-Dirac equation. Yet a first indication that this cannot be the whole story is given by the reasons typically offered for choosing

$a(\infty) = 0$. The condition is chosen in order to avoid the runaway solutions, which are assumed to be unphysical. That is, the vanishing acceleration at future infinity is not understood as one among many physically possible values for the acceleration, but as the *only* possible value; other values do not result in physically possible worlds.

In what sense is a nonzero terminal acceleration physically impossible? The laws of the theory from which the Lorentz-Dirac equation is derived are the Maxwell equations and the principle of energy-momentum conservation. Clearly, a nonzero acceleration does not conflict with the Maxwell equations and arguably is even consistent with energy conservation (although this is a delicate issue to decide). Rather, one reason for excluding nonzero accelerations seems to be that they conflict with some intuitive assessment of what a physically reasonable solution should look like. Thus, one might want to characterize the situation as follows: Since the Lorentz-Dirac equation permits unphysical solutions it is incomplete. Adding the asymptotic condition $a(\infty) = 0$ completes the theory. But this suggests that the condition is more than a mere de facto condition, despite the fact that it looks like a standard contingent "initial" condition.

But there is a deeper reason for demanding that the acceleration at infinity vanish; without assuming the condition the Lorentz-Dirac equation could not be derived. The equation is derived by considering the energy balance for a tube surrounding a portion of the world line of a point charge as the radius of the tube is shrunk to zero. A problem with this derivation is that the field associated with the charge itself is infinite at the location of the charge and the energy integrals over the "caps" of the tube diverge. There appears to be no procedure available for dealing with this infinity in general. In the special case, however, where both the initial and final velocities of the particle are constant, there is a plausible way to proceed. In this case the difference between the integrals over the caps looks formally like a difference between particle four-momenta, $m_f u_\text{out} - m_f u_\text{in}$, where m_f is the diverging integral that is identified as a "field mass" of the particle and u_out and u_in are the particle's final and initial four-velocities. The infinity can then be "swept under the rug" by combining the field mass with the so-called "bare" mass of the particle, assuming that the total empirically observable mass is finite: $m_\text{total} = m_f + m_\text{bare}$. This procedure is known as *mass renormalization*.[1]

The assumption that the initial and final acceleration vanish is generally motivated by assuming that one is dealing with a scattering process, and

1. The equation of mass-energy conservation should read (integral over sides of the tube) + (integral over caps) + $m_\text{bare}u_\text{out} - m_\text{bare}u_\text{in} = 0$. The mass renormalization consists in combining the integral over the caps with the momenta associated with the bare mass, to yield (integral over sides of the tube) + $m_\text{ren}u_\text{out} - m_\text{ren}u_\text{in} = 0$.

looks at the initial or final acceleration at past and future infinity, respectively. That is, one assumes that the charged particle comes in from spatial infinity, interacts with a localized external field, and then flies off again towards spatial infinity. It then seems plausible that far enough away from any field the acceleration ought to vanish, and that therefore the scattered particle will be "asymptotically free"—that is, the particle will have asymptotically constant velocity. The important point here is that the mass renormalization cannot be performed, and hence the Lorentz-Dirac equation cannot be derived, unless one assumes that the acceleration vanishes asymptotically, because otherwise the diverging integrals over the caps of the tube do not have the right mathematical form. That is, the runaway solutions do not merely conflict with what might be an intuitively obvious physical constraint, but violate a condition necessary for deriving the equation.

What, then, is the status of the condition that particles are asymptotically free in the theory? One might think that it has the status of a purely contingent "initial" condition. Similarly to the case of position and velocity, zero initial (or final) acceleration is just one of infinitely many possible values for the acceleration. But in the possible worlds framework, a contingent fact is one that holds in some physically possible worlds but not at all. That is, there ought to be some worlds (in which the laws of the theory hold) that are scattering worlds, and others that are not. Yet since it seems to be impossible to derive an equation of motion for a charged particle from the Maxwell equation without assuming that the particle's acceleration vanishes at infinity, there are no well-defined Maxwellian worlds with charged particles in which the condition does not hold.

Now this suggests that a vanishing acceleration at infinity is physically necessary. But this also does not seem like the right thing to say. Why, for example, one might ask, should a spatially infinite world with a background field that is constant throughout space be physically impossible? Or why shouldn't a world be possible in which two charges are permanently attached to each other through a spring (so that they never move infinitely far away from each other)? In both cases there will be a field force acting on a charged particle at infinity. One reason for thinking that such worlds should be physically impossible, on the standard account, would be if these worlds violated the Maxwell equations (or perhaps energy-momentum conservation). But this is not obviously so. Rather than these worlds violating the Maxwell equations, we simply would not know *how to apply* the Maxwell equations consistently to these worlds, since we would not know what to do with the infinite self-fields associated with the charged particles. Moreover, if we thought that such fields were physically impossible, we would be forced to accept the counterintuitive

conclusion that constant background fields are physically possible in charge-free worlds but not in worlds which also contain a charged particle.

Physicists themselves do not seem to think that a vanishing acceleration at infinity is physically necessary, for some of the standard applications of the Lorentz-Dirac equation concern models in which the condition is not satisfied. Examples include a charge undergoing uniform acceleration (Rohrlich 1990, sec. 6.11) or a charge on a circular orbit (Rohrlich 1990, sec. 6.15). Thus, even though the asymptotic condition is necessary for deriving the Lorentz-Dirac equation, it is not treated as an assumption that all the systems to which the equation can be applied need to satisfy. Curiously enough, then, there are models governed by the Lorentz-Dirac equation which violate the condition of vanishing acceleration. Of course, according to the standard picture this should be impossible, for these models do not satisfy one of the conditions that need to be satisfied for the equation to be applicable. As far as the condition's modal status is concerned, however, since there are models in which the asymptotic condition is not satisfied, the condition is not physically necessary.

I want to suggest that the assumption is best thought of as a condition that is *mathematically* necessary, in a certain sense. It is an assumption without which the Maxwell equations could not be consistently applied to systems of charged particles. As such, the assumption tells us something about the mathematical modeling of the behavior of charged particles, while the connection of the condition to actual systems that may be modeled with its help is rather murky. Take the case of a charged particle in a central field. The relevant idealized mathematical model that can be derived from the Lorentz-Dirac equation does not satisfy the asymptotic condition. Yet Rohrlich argues that *actual* charged particles in a central field, unlike the model, do satisfy the condition, since no actual system has an imposed force doing work over an infinite period of time (Rohrlich 1990, 136). Or consider a particle undergoing uniform acceleration. In his treatment Rohrlich (1990, sec. 6.11), *first* solves the equation of motion for this case, where the solution violates the asymptotic condition, and *then* argues that due to the asymptotic condition uniform acceleration is possible only for a finite time interval. In both these examples the models violate a condition that needs to be made in deriving them; but also in both cases the asymptotic condition is not ignored completely, but is invoked in discussing the model-world relation and is here treated as a contingent fact.

The asymptotic condition is mathematically necessary, I want to submit, in a manner similar to that in which certain continuity or smoothness assumptions are necessary in certain derivations. For example, in deriving the Lorentz-Dirac equation one also needs to assume that all relevant functions are analytic—that is, infinitely differentiable. This is not, how-

ever, treated as an assumption about the world; for the derivation one needs to assume neither that it is contingent nor that it is *physically* necessary that actual fields be infinitely differentiable. And as in the case of the asymptotic condition, the analyticity requirement need not even be satisfied by all the models of the theory in which physicists are interested. Thus, one stock toy model of the theory is a charge subject to a step function field force. The analyticity assumption is a mathematical modeling assumption that may or may not receive a physical interpretation. What may be surprising is that an assumption concerning "initial conditions" can play a similar role. There really seems to be no room for this kind of condition in the standard picture. The distinction between contingent initial conditions and physically necessary laws appears to be exhaustive: Either all possible worlds allowed by classical point electrodynamics satisfy the asymptotic condition—then the condition is physically necessary, according to the theory—or there are some electromagnetically possible worlds that do not satisfy the condition and some that do—then the condition is contingent. But the status of the asymptotic condition appears to fit neither. In order to escape this dilemma, I want to suggest that we should give up the view that models of the phenomena in physics are stand-ins for physically possible worlds.

Another way to make this point is this. For the standard picture to be applicable, there needs to be a unique answer to the question what the laws of the theory are. But the electrodynamics of point charges is treated ambiguously in that respect. If the laws of the theory are the Maxwell equations and the principle of energy conservation (and the Lorentz-Dirac equation has only derivative status) then all mathematical models of that theory have to satisfy the asymptotic condition. By contrast, if the Lorentz-Dirac equation is treated as a law in its own right and we ignore how we initially arrived at the law, then the theory can have models that violate the asymptotic condition. But I think that the standard picture is mistaken in demanding a unique answer to this question. Thus, I want to propose a different picture in its stead.

4. Conclusion. We have seen that even perfectly "respectable" laws, such as the Maxwell-Lorentz equations in continuum electrodynamics, need not have global solutions and that some conditions used in constructing mathematical models of electromagnetic phenomena do not seem to fit the contingent-necessary dichotomy very well. I think we can make sense of both of these features of scientific theorizing, if instead of thinking of laws as delineating the class of possible worlds allowed by a theory, we think of the laws as tools for model-building. On this conception, the primary aim of physical theorizing is to enable us to build mathematical models of the phenomena in which we are interested. This means, for

one, that a theory need not provide us with global models to be successful. Most, if not all, electromagnetic phenomena we are interested in occur in finite regions of spacetime. Models of such phenomena can be adequate even if they cannot be extended to all of spacetime. But it also calls into question what, if any, modal force should be associated with the so-called "laws" of a theory. On the conception I want to advocate here, the assumptions we make in constructing mathematical models of the phenomena range from specific assumptions that do not carry over to other contexts to more general, more robust assumptions that generalize to a wide variety of contexts. The latter type of assumptions often earns the honorific "law," but as I see it there is no sharp distinction between the latter and the former. Whether all or just some models satisfy the asymptotic condition teaches us something about the modeling of electromagnetic phenomena, but not about metaphysics. Moreover, not all assumptions are motivated on purely empirical grounds. Rather, some assumptions, like the asymptotic condition, are adopted for mathematical reasons. And such assumptions, as we have seen, need not even consistently apply to all models; there is no general prohibition against "inconsistent modeling." Of course one needs to be careful in constructing models that violate such a condition, but I am not sure one can say more than the platitude that (echoing Duhem) physicists have to make use of their good sense in such applications.

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