ON BOREL EQUIVALENCE RELATIONS IN GENERALIZED BAIRE SPACE

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Abstract. We construct two Borel equivalence relations on the generalized Baire space κ^κ, $\kappa^{<\kappa} = \kappa > \omega$, with the property that neither of them is Borel reducible to the other. A small modification of the construction shows that the straightforward generalization of the Glimm-Effros dichotomy fails.

By λ^κ we denote the set of all functions $\kappa \to \lambda$. We define a topology to $(\lambda^\kappa)^n$ by letting the sets $N_{(\eta_1 \dots, \eta_n)} = \{(f_1, \dots, f_n) \in (\lambda^\kappa)^n | \ \eta_i \subseteq f_i \text{ for all } 1 \le i \le n\}$, be the basic open sets, where for some $\alpha < \kappa$ for all $1 \le i \le n$, η_i is a function from α to λ. We write N_η for $N_{(\eta)}$. For $\kappa > \omega$, the spaces κ^κ are called generalized Baire spaces. The study of these spaces started already in [Va] and since then many papers have been written on these, more on the history can be found from [FHK]. Most of the study of these spaces (for $\kappa > \omega$) is done under the assumption that $\kappa^{<\kappa} = \kappa$ and we make this assumption also.

By closing open sets under complementation and unions of size $\le \kappa$, we get the class of Borel sets. A function between these spaces is Borel if the inverse image of every open set is Borel. As in the case $\kappa = \omega$, a Borel function F is continuous on a co-meager set i.e. there are open and dense sets U_i, $i < \kappa$, such that $F \restriction (\bigcap_{i<\kappa} U_i)$ is continuous, see [FHK].

Let $X, Y \in \{\kappa^\kappa, 2^\kappa\}$ and let $E \subseteq X^2$ and $E' \subseteq Y^2$ be equivalence relations. We say that E is Borel reducible to E' and write $E \le_B E'$ if there is Borel function $F \colon X \to Y$ such that for all $f, g \in X$, fEg if and only if $F(f)E'F(g)$. We say that they are Borel bi-reducible if both $E \le_B E'$ and $E' \le_B E$ hold.

In [FHK] these Borel reductions were studied. We were mostly interested in equivalence relations like isomorphism among (codes of) models of some first-order theory but also some general theory was developed. And we were annoyed when we found out that we could not find Borel equivalence relations which are incomparable with respect to Borel reducibility. Let us see why one cannot just take some example from the case $\kappa = \omega$ and carry out a straightforward generalization.

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Arguments like the one in [LV] use machinery available only in the case $\kappa = \omega$. But with some basic results like the Borel incomparability of E_1 and E_0^ω this is not the case. And indeed one can generalize the definitions of these relations in a straightforward way and prove that E_0^κ is not Borel reducible to E_1, see the proof of Lemma 5. Also if one takes the classical proof of the other direction, see Theorem 8.2 in [Hj], one notices that everything in the proof holds also in the case $\kappa > \omega$. However, this does not prove that the result is true for $\kappa > \omega$. In the proof two functions are constructed by induction on $i < \kappa$ and if $\kappa > \omega$, one needs to go over limits and at least without major changes in the construction, this cannot be done (and one can prove this).

In this paper, we will modify the definitions of E_1 and E_0^κ and prove the following theorem:

1 Theorem. *Suppose $\kappa^{<\kappa} = \kappa > \omega$. There are Borel equivalence relations on κ^κ such that neither of them is Borel reducible to the other.*

The rest of this paper gives a proof for this theorem.

Before defining the equivalence relations, we want to point out that if $\lambda < \kappa$ and we define $id_\lambda \subseteq (2^\kappa)^2$ to be the set of pairs (f, g) such that $|\{\alpha < \kappa|\ f(\alpha) \neq g(\alpha)\}| < \lambda$, then id_λ is Borel bi-reducible with the identity, see [FHK].

For $\alpha, \beta < \kappa$, by $\alpha - \beta$ we denote the (unique) ordinal γ such that $\alpha + \gamma = \beta$ or $\beta + \gamma = \alpha$.

2 Definition. *We let E_* be the set of pairs (f, g) of function from κ to κ such that for some $\alpha < \kappa$, $f(\beta) - g(\beta) < \alpha$ for all $\beta < \kappa$.*

Clearly E_* is a Borel equivalence relation on κ^κ.

Let $\gamma \leq \kappa$ and $\pi\colon \kappa \to \gamma \times \kappa$ be one to one and onto. We define a topology on $2^{\gamma\times\kappa}$ so that $f \mapsto g$, $g(\alpha) = f(\pi(\alpha))$, is a homeomorphism from $2^{\gamma\times\kappa}$ onto 2^κ. For $f \in 2^{\gamma\times\kappa}$ and $\alpha < \gamma$, by f_α we mean the function $f_\alpha(x) = f(\alpha, x)$.

3 Definition.

(I) We let E_0' be the set of pairs $(f, g) \in (2^\kappa)^2$ such that $\{\alpha < \kappa|\ f(\alpha) \neq g(\alpha)\}$ is a finite union of intervals bounded in κ (i.e. a finite union of sets of the form $[\gamma, \delta)$, $\gamma < \delta < \kappa$).

(II) We let E^* be the set of pairs $(f, g) \in (2^{\kappa\times\kappa})^2$ such that for all $\alpha < \kappa$, $f_\alpha E_0' g_\alpha$.

Clearly both E_0' and E^* are Borel and it is easy to see that they are also equivalence relations. Also it is easy to see that E^* is Borel bi-reducible with a Borel equivalence relation on κ^κ (also E_* is Borel bi-reducible with a Borel equivalence relation on 2^κ). So to prove Theorem 1, it is enough to prove Lemmas 4 and 5 below.

4 Lemma. $E_* \not\leq_B E^*$.

Proof. For a contradiction, suppose that $F\colon \kappa^\kappa \to 2^{\kappa\times\kappa}$ is a Borel reduction of E_* to E^*. As mentioned above, there are dense and open subsets U_i, $i<\kappa$, of κ^κ such that F is continuous on $U=\bigcap_{i<\kappa} U_i$.

By induction on $i<\kappa$ we construct ordinals $\alpha_i,\beta_i<\kappa$ and functions $f^0_i, f^1_i\colon \alpha_i\to\kappa$ and $g^0_i, g^1_i\colon \beta_i\times\beta_i\to 2$ so that

(i) for $i<j$, $\alpha_i<\alpha_j$, $\beta_i<\beta_j$, $f^0_i\subseteq f^0_j$, $f^1_i\subseteq f^1_j$, $g^0_i\subseteq g^0_j$ and $g^1_i\subseteq g^1_j$,

(ii) $N_{f^0_i}\cup N_{f^1_i}\subseteq U_j$ for all $j<i$,

(iii) $F(N_{f^0_i}\cap U)\subseteq N_{g^0_i}$ and $F(N_{f^1_i}\cap U)\subseteq N_{g^1_i}$,

(iv) for some $\gamma<\alpha_{i+1}$, $f^0_{i+1}(\gamma)-f^1_{i+1}(\gamma)\geq i$,

(v) for all $i<\kappa$ and $\gamma<\beta_i$, there is $\beta_i\leq\delta<\beta_{i+1}$ such that $g^0_{i+1}(\gamma,\delta)=g^1_{i+1}(\gamma,\delta)$.

Notice that if we can construct these so that (i)-(v) hold, then $f^0=\cup_{i<\kappa}f^0_i$ and $f^1=\cup_{i<\kappa}f^1_i$ belong to U, $F(f^0)=g^0=\cup_{i<\kappa}g^0_i$, $F(f^1)=g^1=\cup_{i<\kappa}g^1_i$ and f^0 and f^1 are not in the relation E_*. Also it is not hard to see (see below) that for all $i<\kappa$ there are $h^0\supseteq g^0_i$ and $h^1\supseteq g^1_i$ such that they are in the relation E^*.

For $i=0$, we let $\alpha_i=\beta_i=0$ (and $f^0_i=f^1_i=g^0_i=g^1_i=\emptyset$) and at limits we take unions. Clearly these are as required.

So suppose we have constructed these for i and we construct then for $i+1$. For all $j<\kappa$ we construct first $h^0_j, h^1_j\in\kappa^{\gamma_j}$, $\gamma_j<\kappa$, as follows: $h^0_0=f^0_i$ and $h^1_0=f^1_i$ and at limits we take unions. For $j=2k+1$, we choose first $h^1_j\supseteq h^1_{2k}$ so that $N_{h^1_j}\subseteq U_k$ and h^1_j properly extends h^1_{2k} and then we choose $h^0_j\supseteq h^0_{2k}$ so that $dom(h^0_j)=dom(h^1_j)$ and for all $\gamma\in dom(h^0_j)-dom(h^0_{2k})$, $h^0_j(\gamma)=h^1_j(\gamma)+i$. For $j=2k+2$ we do the reverse i.e. $N_{h^0_j}\subseteq U_k$ and for all $\gamma\in dom(h^1_j)-dom(h^1_{2k+1})$, $h^1_j(\gamma)=h^0_j(\gamma)+i$. Then $h^0=\cup_{j<\kappa}h^0_j$ and $h^1=\cup_{j<\kappa}h^1_j$ belong to U and they are E_*-equivalent. Then $F(h^0)$ and $F(h^1)$ are E^*-equivalent and since at stage i the elements satisfy (iii), $F(h^0)\supseteq g^0_i$ and $F(h^1)\supseteq g^1_i$. And so by choosing β_{i+1} large enough and letting $g^0_{i+1}=F(h^0)\restriction(\beta_{i+1}\times\beta_{i+1})$ and $g^1_{i+1}=F(h^1)\restriction(\beta_{i+1}\times\beta_{i+1})$ the requirement (v) and relevant parts of (i) are satisfied. Since F is continuous on U and $h^0,h^1\in U$, by choosing α_{i+1} large enough and letting $f^0_{i+1}=h^0\restriction\alpha_{i+1}$ and $f^1_{i+1}=h^1\restriction\alpha_{i+1}$, the rest of the requirements can be satisfied.

So now we have f^0 and f^1 and since they are not E_*-equivalent, $g^0=\cup_{i<\kappa}g^0_i=F(f^0)$ and $g^1=\cup_{i<\kappa}g^1_i=F(f^1)$ are not E^*-equivalent. Let $\alpha<\kappa$ witness this i.e. $(g^0)_\alpha$ and $(g^1)_\alpha$ are not E'_0-equivalent. By (v) from the construction of g^0 and g^1, it is not possible that $(g^0)_\alpha(\gamma)\neq(g^1)_\alpha(\gamma)$ for all large enough γ. Thus there must exist an increasing sequence $(\gamma_i)_{i<\omega}$ of ordinals $<\kappa$ such that $(g^0)_\alpha(\gamma_i)=(g^1)_\alpha(\gamma_i)$ iff i is odd.

Now choose $i^*<\kappa$ so that $\beta_{i^*}>\alpha\cup\bigcup_{i<\omega}\gamma_i$. Then there are no h^0 and h^1 extending $g^0_{i^*}$ and $g^1_{i^*}$, respectively, so that h^0 and h^1 are E^*-equivalent. As pointed out above, this is a contradiction. □

5 Lemma. $E^*\not\leq_B E_*$.

Proof. For a contradiction, suppose $F\colon 2^{\kappa\times\kappa}\to\kappa^\kappa$ is a Borel reduction of E^* to E_*. As above, there are open and dense subsets U_i, $i<\kappa$, of $2^{\kappa\times\kappa}$ such that on $U=\cap_{i<\kappa}U_i$, F is continuous.

By induction on $i<\kappa$ we construct ordinals $\alpha_i,\beta_i<\kappa$ and functions $f^0_i,f^1_i\colon \alpha_i\times\alpha_i\to 2$ and $g^0_i,g^1_i\colon\beta_i\to\kappa$ so that

(i) for $i<j$, $\alpha_i<\alpha_j$, $\beta_i<\beta_j$, $f^0_i\subseteq f^0_j$, $f^1_i\subseteq f^1_j$, $g^0_i\subseteq g^0_j$, $g^1_i\subseteq g^1_j$ and $\alpha_0=\beta_0=0$,

(ii) $N_{f^0_i}\cup N_{f^1_i}\subseteq U_j$ for all $j<i$,

(iii) $F(N_{f^0_i}\cap U)\subseteq N_{g^0_i}$ and $F(N_{f^1_i}\cap U)\subseteq N_{g^1_i}$,

(iv) for some $\gamma<\alpha_{i+1}$, $g^0_{i+1}(\gamma)-g^1_{i+1}(\gamma)\geq i$,

(v) For all $\alpha_i\leq\alpha<\alpha_{i+1}\leq\alpha_j$ the following hold:

(a) for all $\gamma<\alpha_{i+1}$, $(f^0_j)_\alpha(\gamma)\neq(f^1_j)_\alpha(\gamma)$

(b) for all $\alpha_{i+1}\leq\gamma<\alpha_j$, $(f^0_j)_\alpha(\gamma)=(f^1_j)_\alpha(\gamma)$.

If we can construct these so that (i)-(v) hold, we have a contradiction: By (v), $f^0=\bigcup_{i<\kappa}f^0_i$ and $f^1=\bigcup_{i<\kappa}f^1_i$ are E^*-equivalent. By (ii) and (iii), $F(f^0)=g^0=\bigcup_{i<\kappa}g^0_i$ and $F(f^1)=g^0=\bigcup_{i<\kappa}g^1_i$ and by (iv) these are not E_*-equivalent, a contradiction.

For $i=0$, we let $\alpha_i=\beta_i=0$ (and $f^0_i=f^1_i=g^0_i=g^1_i=\emptyset$) and at limits we take unions. Clearly these are as required.

So suppose that we have constructed these for $j\leq i$ and we construct them for $i+1$. First we want to find $h^0,h^1\colon\kappa\times\kappa\to 2$ such that $f^0_i\subseteq h^0$, $f^1_i\subseteq h^1$, $h^0,h^1\in U$ and for all $(\delta,\delta')\in(\kappa\times\kappa)-(\alpha_i\times\alpha_i)$, $h^0(\delta,\delta')=h^1(\delta,\delta')$ if and only if $\delta<\alpha_i$. For this we construct increasing sequences $(h^0_j)_{j<\kappa}$ and $(h^1_j)_{j<\kappa}$ of functions $h^0_j,h^1_j\colon\gamma_j\times\gamma_j\to 2$ as follows:

For $j=0$, we let $\gamma_j=\alpha_i$, $h^0_j=f^0_i$ and $h^1_j=f^1_i$ and at limits we take unions. For $j=2k+1$ choose the h^0_j, h^1_j as follows: We let $\gamma_j>\gamma_{2k}$ and $h^0_j\colon\gamma_j\times\gamma_j\to 2$ be such that $h^0_{2k}\subseteq h^0_j$ and $N_{h^0_j}\subseteq U_k$ and we let $h^1_j\colon\gamma_j\times\gamma_j\to 2$ be such that $h^1_{2k}\subseteq h^1_j$ and for all $(\delta,\delta')\in(\gamma_j\times\gamma_j)-(\gamma_{2k}\times\gamma_{2k})$ $h^1_j(\delta,\delta')=h^0_j(\delta,\delta')$ if and only if $\delta<\alpha_i$. For $j=2k+2$ we do the reverse i.e. we let $\gamma_j>\gamma_{2k+1}$ and $h^1_j\colon\gamma_j\times\gamma_j\to 2$ be such that $h^1_{2k+1}\subseteq h^1_j$ and $N_{h^1_j}\in U_k$ and we let $h^0_j\colon\gamma_j\times\gamma_j\to 2$ be such that $h^0_{2k+1}\subseteq h^1_j$ and for all $(\delta,\delta')\in(\gamma_j\times\gamma_j)-(\gamma_{2k+1}\times\gamma_{2k+1})$ $h^0_j(\delta,\delta')=h^1_j(\delta,\delta')$ if and only if $\delta<\alpha_i$. Then $h^0=\bigcup_{j<\kappa}h^0_j$ and $h^1=\bigcup_{j<\kappa}h^1_j$ are as wanted.

Since h^0 and h^1 are not E^*-equivalent and $h^0,h^1\in U$, $F(h^0)\supseteq g^0_i$ and $F(h^1)\supseteq g^1_i$ are not E_*-equivalent. So by choosing $\beta_{i+1}>\beta_i$ large enough, $g^0_{i+1}=F(h^0)\restriction\beta_{i+1}$ and $g^1_{i+1}=F(h^1)\restriction\beta_{i+1}$ satisfy (iv) and the relevant parts of (i). Since F is continuous on U, by choosing $\alpha_{i+1}>\alpha_i$ large enough the rest of the requirements can be satisfied. $\square$

We finish this paper with some open questions. But before this, we make some definitions and observations.

Let id be the set of pairs $(f,g) \in (2^\kappa)^2$ such that $f = g$ and E_0 be the set of pairs $(f,g) \in (2^\kappa)^2$ such that for some $\alpha < \kappa$, $f(\gamma) = g(\gamma)$ for all $\gamma > \alpha$. Then these are Borel equivalence relations and clearly $id \leq_B E_0$ and similarly $id \leq_B E_0'$ (Definition 3 (I)). As pointed out in [FHK], $E_0 \not\leq_B id$ since if F is a reduction of E_0 to id and continuous on a co-meager set U, one can find $\alpha < \kappa$ and $\eta, \xi\colon \alpha \to 2$ and $\alpha' < \kappa$ and $\eta', \xi'\colon \alpha' \to 2$ so that $\eta' \neq \xi'$ and $F(N_\eta \cap U) \subseteq N_{\eta'}$ and $F(N_\xi \cap U) \subseteq N_{\xi'}$. But this is impossible because there are $f \in N_\eta \cap U$ and $g \in N_\xi \cap U$ which are E_0-equivalent. Similarly $E_0' \not\leq_B id$ and by repeating this argument ω times, one can see the following lemma:

6 Lemma $E_0 \not\leq E_0'$**.**

Proof. (Sketch) For a contradiction, suppose $F\colon 2^\kappa \to 2^\kappa$ is a reduction, which is continuous on a co-meager set U. As in the proof of $E_0 \not\leq id$, one can find increasing sequences $(\alpha_i)_{i<\omega}$ and $(\gamma_i)_{i<\omega}$ of ordinals and increasing sequences of functions $\eta_i, \xi_i\colon \alpha_i \to 2$ and $\eta_i', \xi_i'\colon \gamma_i \to 2$ such that

(i) $F(N_{\eta_i} \cap U) \subseteq N_{\eta_i'}$ and $F(N_{\xi_i} \cap U) \subseteq N_{\xi_i'}$,

(ii) for all $i < \omega$ there are $\gamma_i \leq \beta < \beta' < \gamma_{i+1}$ such that $\eta_{i+1}'(\beta) = \xi_{i+1}'(\beta)$ and $\eta_{i+1}'(\beta') \neq \xi_{i+1}'(\beta')$.

But now we have a contradiction since there are $f \in U \cap \bigcap_{i<\omega} N_{\eta_i}$ and $g \in U \cap \bigcap_{i<\omega} N_{\xi_i}$ which are E_0-equivalent. ▫

7 Open question. *Is E_0' Borel reducible to E_0?*

8 Open question. *In the case $\kappa = \omega$, by the Glimm-Effros dichotomy, see e.g. [BK], for all Borel equivalence relations E above id, either $E \leq_B id$ or $E_0 \leq_B E$. By what is above, E_0' witnesses that this is not true for uncountable κ. However, notice that for $\kappa = \omega$, $E_0 = E_0'$ and one can ask, is Glimm-Effros true with E_0' in place of E_0 (for $\kappa > \omega$)?*

9 Open question. *Let us look at the structure $(BE, \leq_B)$ where BE is the set of all Borel equivalence relations on 2^κ. By what is said above, $(BE, \leq_B)$ contains antichains of length at least 2 and above id, chains of length at least 4 ($id <_B E_0' <_B E^* <_B$ "$E^* \times E_*$", essentially as in the proof of Lemma 5 one can show that E^* is not Borel reducible to E_0'). Can one find longer chains and antichains?*

In Open question 9 we mean equivalence relations that can be defined for all $\kappa = \kappa^{<\kappa} > \omega$. For large κ, the following gives a long chain: For all γ such that $\aleph_\gamma < \kappa$, let E_0^γ be the set of pairs $(f,g) \in (2^\kappa)^2$ such that there is an increasing and continuous sequence $(\alpha_i)_{i\leq\beta}$, $\beta < \aleph_{\gamma+1}$, such that $\alpha_0 = 0$, for all $\delta \geq \alpha_\beta$, $f(\delta) = g(\delta)$ and for all $i < \beta$, either for all $\alpha_i \leq \delta < \alpha_{i+1}$, $f(\delta) = g(\delta)$ or for all $\alpha_i \leq \delta < \alpha_{i+1}$, $f(\delta) \neq g(\delta)$. Then for $\alpha > 0$, $\aleph_\alpha < \kappa$, we define $E_0^{<\alpha}$ to be the set of all pairs $(f,g) \in (2^{\alpha\times\kappa})^2$ such that for all $\gamma < \alpha$, $f_\gamma E_0^\gamma g_\gamma$.

It is easy to see that these are Borel equivalence relations, for all $\gamma < \beta$ and $0 < \alpha < \beta$, $E_0^\gamma, E_0^{<\alpha} \leq_B E_0^{<\beta}$ and as in the proof of Lemma 6, one can see that $E_0^\alpha \not\leq_B E_0^{<\alpha}$ and thus $E_0^{<\beta} \not\leq_B E_0^{<\alpha}$.

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