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On Generalized Electromagnetism and Dirac Algebra*

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ABSTRACT

Using a framework of Dirac algebra, the Clifford algebra appropriate for Minkowski space-time, the formulation of classical electromagnetism including both electric and magnetic charge is explored. Employing the two-potential approach of Cabibbo and Ferrari, a Lagrangian is obtained that is dyality invariant and from which it is possible to derive by Hamilton's principle both the symmetrized Maxwell's equations and the equations of motion for both electrically and magnetically charged particles. This latter result is achieved by defining the variation of the action associated with the cross terms of the interaction Lagrangian in terms of a surface integral. The surface integral has an equivalent path integral form, showing that the contribution of the cross terms is local in nature. The form of these cross terms derives in a natural way from a Dirac algebraic formulation, and, in fact, the use of the geometric product of Dirac algebra is an essential aspect of this derivation. No kinematic restrictions are associated with the derivation, and no relationship between magnetic and electric charge evolves from the (classical) formulation. However, it is indicated that in bound states quantum mechanical considerations will lead to a version of Dirac's quantization condition. A discussion of parity violation of the generalized electromagnetic theory is given, and a new approach to the incorporation of this violation into the formalism is suggested. Possibilities for extensions are mentioned.

1. Introduction

Dirac[1] was the first to suggest the possibility of a particle that carries magnetic charge. In developing a theory of electrodynamics that included magnetic monopoles, Dirac introduced the notion of a singular vector potential. As a consequence, he found that magnetic monopoles would have (singularity) strings attached to them. In addition, his analysis required that electrically charged particles could not contact these strings; this constraint is known as the "Dirac veto." While considerable effort has been expended in trying to solve these difficulties and considerable progress has been made, no fully satisfactory solution has yet been found.#1

In a paper that employs the use of two vector potentials, Cabibbo and Ferrari[3] eliminate the need for strings but were unable to establish a Lagrangian formulation. It has been shown[4] that, given certain assumptions, rather severe restrictions apply to a Lagrangian formulation of electromagnetism when both electric and magnetic charges are present. In a different approach, with a technique using ideas borrowed from the mathematics of fiber bundles, Wu and Yang[5] have defined a Lagrangian using the singular vector potential of Dirac that circumvents the problem of the Dirac veto. Nevertheless, the use of the singular vector potential remains questionable.#2 Also, this result loses some generality because the action must be defined modulo eg. While for the classical theory the significance of this restriction is not clear, in quantum theory it leads to Dirac's quantization condition[1] $\frac{eg}{\hbar c} = \frac{n}{2}$, where n is an integer.

Recently, an analysis using Clifford algebras as a framework in which to incorporate magnetic monopoles into a generalized electromagnetic theory has been

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#1 Extensive reviews of these efforts and their various difficulties have been published.[2]

#2 Even setting aside an intuitive distrust of singular functions in physics, there still appear to be problems. For example, it has been pointed out[6] that even though one can move the (singular) strings about by suitable gauge transformations, in many body situations (a charge-monopole plasma, say) the strings may become tangled, leading to a confusing and obscure topological situation for which the standard dynamical equations might no longer apply.

published.[7] This analysis#3 follows that of Cabibbo and Ferrari, using two vector potentials. A new idea in Ref. [7], which is facilitated by the use of Dirac algebra—the Clifford algebra of Minkowski space-time[8]—is that the interaction term of the Lagrangian should be written as the product of a generalized electromagnetic current times a generalized vector potential. As a result, one obtains not only the "standard" interactions of the electric and magnetic current densities j^μ and k^μ with their respective vector potentials A^μ and M^μ (in tensor language, the $j_\mu A^\mu$ and $k_\mu M^\mu$ terms) but also the "cross" interactions $j_\mu M^\mu$ and $k_\mu A^\mu$. However, the Lagrangian in Ref. [7] specifically includes (Dirac) pseudoscalar pieces, which are unsuitable for a derivation of the Lorentz equations of motion for electrically and magnetically charged particles; the corresponding free particle contribution to the Lagrangian does not have any appropriate pseudoscalar pieces. Furthermore, the Lagrangian of Ref. [7] is not dyality invariant.#4

The purpose of this paper is to explore more fully the applicability of Dirac algebra to the formulation of a generalized electromagnetism in an effort to overcome some of the difficulties mentioned above. Some well-known results in the tensor formulation are included for the sake of completeness and to set the analysis using Dirac algebra in context. While the analysis here is classical, some possible implications for theories of elementary particles are given.

#3 Unfortunately, Ref. [7] is rather difficult to follow because it contains some typographical errors and the authors use conventional notation and analytical techniques in novel ways without the benefit of any clarifying definitions or discussion. Hence, the correctness of their analysis is not obvious.

#4 This is not uncommon. For example, the Lagrangian used in a rather general mathematically oriented study of electromagnetism including electric and magnetic charge using the calculus of exterior forms is not dyality invariant.[9]

2. Tensor Formulation

a. Electric Charges

To proceed, it is useful first to write down the Lagrangian in the more familiar covariant tensor form. The metric is $g_{\mu\nu} =$ diag $[1, -1, -1, -1]$. The summation rule for repeated indices is used; Greek indices range from 0 to 3.

The Lagrangian density (in Gaussian units) from which one can derive the standard set of Maxwell's equations is given by[10]

$$\mathcal{L}_e = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} j_\alpha A^\alpha, \tag{1}$$

where $F_{\alpha\beta}$ is the usual electromagnetic field tensor, $A^\alpha = (\phi, \boldsymbol{A})$ is the usual electromagnetic vector potential, and the electric current density $j^\mu = (\rho c, \boldsymbol{j})$. Boldface letters are used here to denote vectors in three-dimensional Euclidean-space.

The subscript e on $\mathcal{L}_e$ signifies "electromagnetic," which denotes the standard electricity and magnetism associated with electric sources j^μ. In this paper a distinction is made between the standard electromagnetism and "magnetoelectricity," which denotes the magnetism and electricity associated with the magnetic sources k^μ. This notation furnishes a framework in which a physical (as well as mathematical) distinction can be made between electrically and magnetically generated quantities. As long as the source currents are taken together with their associated fields, there is no ambiguity in partitioning the fields into their electromagnetic and magnetoelectric parts, even in source-free regions. Free fields, not associated with a source, cannot be treated rigorously[#5] in this framework in its present state of development. Hence, their discussion will be deferred to the final section.

#5 In a classical electromagnetic theory that includes magnetic monopoles, the partitioning of the free field into electromagnetic and magnetoelectric quantities is ambiguous. This problem is related to the two-photon question in the quantum theory of free electromagnetic fields using an analysis having two potentials.

The action associated with $\mathcal{L}_e$ is given by

$$S_E + S_I = \frac{1}{c}\int \mathcal{L}_e d^4x, \tag{2}$$

where the contribution of each term in the Lagrangian is explicitly denoted: S_E associated with the electromagnetic field, and S_I associated with the interaction between field and source. The $\frac{1}{c}$ factor is included in Eq. (2) to render the action in units of energy $\times$ time.

The field tensor and vector potential are related by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3}$$

which is the tensor equivalent of the vector#6 equations

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{c\partial t}, \text{ and } \boldsymbol{H} = \nabla \times \boldsymbol{A}. \tag{4}$$

Equations (3) and (4) dictate that $F_{oi} = E_i$ and $F_{ij} = -H_k$, where $i\ j\ k$ are cyclic and range from 1 to 3.

Substituting Eq. (3) into (1) and using Hamilton's principle to obtain the Euler-Lagrange equations associated with the variations δA^α yields the inhomogeneous Maxwell's equations[10]

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu. \tag{5}$$

Since $F^{\mu\nu}$ is antisymmetric, it follows that current is conserved; *i.e.*, $\partial_\nu j^\nu = 0$. Defining the dual of the field tensor

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{6}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, one can use Eq. (3) to derive

#6 $\boldsymbol{E}$ and $\boldsymbol{H}$ are used here as the fundamental (microscopic) fields (*e.g.*, see Katz[11]); the quantities $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$ are redundant, for in Gaussian units $\epsilon = \mu = 1$.

the homogeneous equation

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{7}$$

where choosing $\epsilon_{0123} = 1 = -\epsilon^{0123}$ dictates that $\tilde{F}^{oi} = H_i$ and $\tilde{F}^{ij} = -E_k$, $i\ j\ k$ cyclic.

In order to obtain the equations of motion for electrically charged particles, one adds S_P, the contribution of a free particle,#7 to the action and rewrites the current#8 in terms of the coordinates $x^\mu = x^\mu(\tau)$ that specify the particle's equilibrium path, yielding[13]

$$S_P + S_I = \int_a^b (-mc\ ds - \frac{e}{c} A_\mu dx^\mu), \tag{8}$$

where τ is the proper time and ds is the increment of invariant distance along the particle path. The integral, then, is taken along the particle's equilibrium path from point a to point b; the $F_{\alpha\beta}F^{\alpha\beta}$ term has been omitted here since it is not functionally dependent upon the particle trajectory.

The principle of least action states[14] that

$$\delta_x S = \delta_x \int_a^b \big(-mc\ ds - \frac{e}{c} A_\mu dx^\mu\big) = \int_a^b \Big[- mc\ \delta_x ds - \frac{e}{c} \delta_x \big(A_\mu dx^\mu\big)\Big] = 0, \tag{9}$$

where δ_x means that the path of integration from a to b is displaced from equilibrium by the arbitrary (infinitesimal) function δx^μ; $\delta x^\mu = 0$ at a and at b. The geometry associated with these action integrals is depicted in Fig. 1.

#7 When one adds the S_P term to the action, one can assume that it can take into account the divergent self-energy terms in Lagrangian. This is not done explicitly here, but a technique for doing so has been demonstrated by Rohrlich.[12]

#8 The current term in Eq. (8) can also be written[12] as $-\frac{e}{c}\int\int A_\mu(x)\delta^4(x - z(\lambda))\frac{dz^\mu}{d\lambda} d\lambda d^4x$, where $z^\mu = z^\mu(\lambda)$ is the particle trajectory as a function of the parameter λ. This form is useful because the Lagrangian density is explicitly manifest. (The expressions transcribed here are adjusted to be consistent with the notation and metric of this paper.)

$\delta_x S_P$, after integration by parts, leads[15] to the term $\int_b^a m\, dv_\mu \delta x^\mu$, where v^μ is the four-velocity of the (electrically charged) particle. $\delta_x S_I$, after integration by parts, leads to the expression[13]

$$\delta_x S_I = \frac{-e}{c} \int_a^b \left(\partial_\mu A_\nu dx^\nu \delta x^\mu - \partial_\nu A_\mu \delta x^\mu dx^\nu\right) = \frac{-e}{c} \int_a^b F_{\mu\nu} dx^\nu \delta x^\mu. \tag{10}$$

Combining the contributions from S_P and S_I and converting both to an integral over proper time yields

$$\int_a^b (m\frac{dv_\mu}{d\tau} - \frac{e}{c} F_{\mu\nu} v^\nu) \delta x^\mu d\tau = 0. \tag{11}$$

Since the δx^μ is arbitrary, it follows that

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c} F^{\mu\nu} v_\nu. \tag{12}$$

Equation (12) is the covariant form of the Lorentz force law

$$\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{H}), \tag{13}$$

where $\boldsymbol{p}$ is the particle momentum. When the source is described by the current density j_ν, then one can define a Lorentz force density[16]

$$f^\mu \equiv \frac{1}{c} F^{\mu\nu} j_\nu; \tag{14}$$

the relationship of Eq. (14) to Eq. (12) is obvious.

Now, $\delta_x S_I$ can, by the relativistic generalization of Stokes' theorem, be developed as an integral over a surface spanning the closed loop formed by the displaced path and the equilibrium path (See Fig. 1.), rather than as a path integral from a to b [Eq. (10)]. Intermediate steps in the derivation of Eq. (10) were included above to facilitate the comparison of $\delta_x S_I$ written as a path integral with $\delta_x S_I$ written as a surface integral. In Sec. 3d, we shall see that the use of the surface integral form for the contribution of the cross terms to $\delta_x S_I$ enables the formulation of the requisite (Dirac) scalar action integral from which the equations of motion of electrically and magnetically charged particles can be derived.

To develop $\delta_x S_I$ as a surface integral, we first observe that

$$\delta_x S_I = (S + \delta_x S)_I - S_I, \tag{15}$$

where $(S + \delta_x S)_I$ is the integral from a to b along a path that is displaced from the equilibrium path by δx^μ:

$$(S + \delta_x S)_I = -\frac{e}{c} \int_a^b A_\nu(x + \delta x) d(x + \delta x)^\nu. \tag{16}$$

Since in Eq. (15) the quantity S_I, which is associated with the line integral from a to b along the equilibrium path, enters with a minus sign, it can be considered as the integral from b to a and entered with a plus sign. Thus,

$$\delta_x S_I = \frac{-e}{c} \oint A_\nu dx^\nu, \tag{17}$$

where the integration path is shown in Fig. 1.

By the relativistic generalization of Stokes' theorem,[17] Eq. (17) becomes

$$\delta_x S_I = \frac{-e}{c} \int d\sigma^{\mu\nu} \partial_\mu A_\nu = \frac{-e}{2c} \int d\sigma^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{-e}{2c} \int d\sigma^{\mu\nu} F_{\mu\nu}, \tag{18}$$

where $d\sigma^{\mu\nu}$ is the incremental area of a surface spanning in Minkowski space-time the closed path of Eq. (17). Comparison of Eq. (18) with Eq. (10) indicates that one can make the identification:

$$\int_{\text{surface}} d\sigma^{\mu\nu} \Longleftrightarrow 2 \int_{\text{path}} dx^{\nu} \delta x^{\mu}. \tag{19}$$

Equation (19) mathematically describes the geometry of the situation; the surface spanning the loop formed by the displaced path and the equilibrium path is "linear" in the function δx^{μ} and hence is of infinitesimal width. (The length is macroscopic, going from a to b.)

b. Magnetic Charges

In analogy to the above analysis, one can for a "magnetic world" employ a second (or "magnetoelectric") vector#9 potential $M^{\mu} \equiv (\psi, \boldsymbol{M})$ and an associated field tensor $G_{\mu\nu}$ such that

$$G_{\mu\nu} = \partial_{\mu} M_{\nu} - \partial_{\nu} M_{\mu}. \tag{20}$$

If we assert that to satisfy dyality (or duality)#10 exchange, $\boldsymbol{E} \to \boldsymbol{H}'$, $\boldsymbol{H} \to -\boldsymbol{E}', \boldsymbol{A} \to \boldsymbol{M}, etc.$ (see Sec. 4b), then from Eq. (4) we obtain

$$\boldsymbol{H}' = -\nabla\psi - \frac{\partial \boldsymbol{M}}{c\partial t} \text{ and } \boldsymbol{E}' = -\nabla \times \boldsymbol{M}, \tag{21}$$

where $\mathbf{H}'$ and $\boldsymbol{E}'$ are the magnetic and electric fields associated with the magnetic current density $k^{\mu} = (\sigma c, \boldsymbol{k})$ as a source. The primes on the electric and magnetic field vectors associated with k^{μ} are consistent with the partitioning of the generalized electromagnetic fields into electrically and magnetically generated quantities. The tensor $G_{\mu\nu}$, then, is strictly magnetoelectric, with $G_{oi} = H'_k$ and $G_{ij} = E'_k, i\ j\ k$ cyclic.

#9 The vector versus pseudovector nature of M^{μ} and other magnetic quantities will be discussed in Section 4c.

#10 The word "duality" derives from the fact that when one takes the dual $\tilde{F}^{\mu\nu}$ of the tensor $F^{\mu\nu}$, the role of $\boldsymbol{E}$ and $\boldsymbol{H}$ are interchanged. Han and Biedenharn[3] introduced the term "dyality" to avoid confusion with hadronic duality. We continue to use it here for that reason and because in our formulation, as is seen in Sec. 4b, the dyality exchange relations, though related to, are not identical to the use of the tensor dual.

Of course, one can form the (magnetoelectric) Lagrangian density

$$\mathcal{L}_m = -\frac{1}{16\pi} G_{\alpha\beta} G^{\alpha\beta} - \frac{1}{c} k_\alpha M^\alpha \tag{22}$$

and use the Euler-Lagrange equations associated with the variations δM^α to obtain

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c} k^\nu, \tag{23}$$

and

$$\partial_\mu \tilde{G}^{\mu\nu} = 0. \tag{24}$$

As with $F^{\mu\nu}$, the antisymmetry of $G^{\mu\nu}$ leads to (magnetic) current conservation; *i.e.*, $\partial_\nu k^\nu = 0$. Equations (22), (23), and (24) can also be obtained directly from Eqs. (1), (5), and (7), respectively, by the dyality exchange relations.

In analogy to Eq. (9), one can, for the magnetic world, write down a least action principle for magnetic charges,

$$\delta_x S = \delta_x \int_a^b (-mc\ ds - \frac{g}{c} M_\mu dx^\mu) = 0, \tag{25}$$

where g is the monopole charge, and obtain

$$m\frac{dv^\mu}{d\tau} = \frac{g}{c} G^{\mu\nu} v_\nu, \tag{26}$$

which, as expected, is equivalent to the Lorentz force law

$$\frac{d\boldsymbol{p}}{dt} = g(\boldsymbol{H}' - \frac{\boldsymbol{v}}{c} \times \boldsymbol{E}'). \tag{27}$$

Here again, we see that Eq. (27) can also be obtained directly from Eq. (13) by using dyality exchange.

c. The Symmetrized Maxwell's Equations

Using the formalism presented above, electromagnetism and magnetoelectricity can be adjoined by adding Eq. (24) to Eq. (5) and subtracting Eq. (23) from Eq. (7). This step yields the symmetrized set of Maxwell's equations:#11

$$\partial_\mu \mathcal{F}^{\mu\nu} = \frac{4\pi}{c} j^\nu, \tag{28}$$

$$\partial_\mu \tilde{\mathcal{F}}^{\mu\nu} = \frac{-4\pi}{c} k^\nu, \tag{29}$$

where the total electromagnetic tensor

$$\mathcal{F}^{\mu\nu} \equiv F^{\mu\nu} + \tilde{G}^{\mu\nu} \tag{30}$$

and its dual

$$\tilde{\mathcal{F}}^{\mu\nu} = \tilde{F}^{\mu\nu} - G^{\mu\nu}. \tag{31}$$

(Recall that $\approx\!\!G^{\mu\nu} = -G^{\mu\nu}$.) Equation (30) is equivalent to

$$\mathcal{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + \epsilon^{\mu\nu\rho\sigma}\partial_\rho M_\sigma, \tag{32}$$

the Cabibbo-Ferrari-Shanmugadhasan#12 relation. In terms of the electric and magnetic field vectors, we see that

$$\mathcal{F}^{oi} = -(\boldsymbol{E} + \boldsymbol{E}')_i \text{ and } \mathcal{F}^{ij} = -(\boldsymbol{H} + \boldsymbol{H}')_k, \tag{33}$$

$i\ j\ k$ cyclic.#13

#11 Symmetrized Maxwell's equations were published[18] as long ago as 1893. The magnetic source terms included by Heaviside were not included for the purpose of describing magnetic charge, however, but rather as a convenient way to describe the "magnetification" of matter.

#12 Shanmugadhasan,[19] who introduced a second vector potential in a study of the dynamics of electric and magnetic charges, also obtained Eq. (32).

#13 In the approach taken in this paper, it is an arbitrary matter of definition whether one adds Eqs. (24) and (5) and subtracts Eqs. (23) and (7) or vice versa. The option that yields Eq. (33) was chosen; that is, the fields due to electric sources and those due to magnetic sources add.

The above analysis indicates that by using Hamilton's principle one can derive the symmetrized set of Maxwell's equations from the Lagrangian density $\mathcal{L}_e + \mathcal{L}_m$.#14 It is easily verified that $\mathcal{L}_e + \mathcal{L}_m$ is invariant under dyality exchange. However, it is obvious that electromagnetism and magnetoelectricity have not yet been unified;[20] the Lagrangian interaction terms $j_\alpha A^\alpha$ and $k_\alpha M^\alpha$ cannot lead to the "correct" #15 equations of motion in which the fields associated with electrically charged particles will exert forces on magnetically charged particles, and vice versa; terms describing the cross interactions are required. These cross interaction terms appear in a natural way when generalized electromagnetism is formulated using Dirac algebra.

3. Formulation using Dirac Algebra

a. Preliminaries

Dirac algebra is regarded[8] as the natural algebra to use for the description of events in four-dimensional Minkowski space-time. Only elements of Dirac algebra essential to this paper will be introduced here; readers interested in a more comprehensive discourse should consult the literature, for example, Ref. [8] or [21].

Four linearly independent vectors γ_μ are used as a basis set for Dirac algebra. These vectors satisfy the same multiplication rules as the familiar Dirac matrices[22] :

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}. \tag{34}$$

A Clifford or geometric product of two vectors has a symmetric part (or dot

#14 That one can write down a Lagrangian density from which one can derive the symmetrized set of Maxwell's equations was pointed out by Rohrlich.[4] In fact, $\mathcal{L}_e + \mathcal{L}_m$ is in essence equivalent to the example cited by Rohrlich.

#15 "Correct" means having Lorentz forces of the form $e[(\boldsymbol{E}+\boldsymbol{E}') + \frac{\boldsymbol{v}}{c} \times (\boldsymbol{H}+\boldsymbol{H}')]$ and $g[(\boldsymbol{H}+\boldsymbol{H}') - \frac{\boldsymbol{v}}{c} \times (\boldsymbol{E}+\boldsymbol{E}')]$.

product) and an antisymmetric part (or wedge product). That is,

$$\gamma_\mu \gamma_\nu = \gamma_\mu \cdot \gamma_\nu + \gamma_\mu \wedge \gamma_\nu, \tag{35}$$

where

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu}, \tag{36}$$

and

$$\gamma_\mu \wedge \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \gamma_{\mu\nu}. \tag{37}$$

$\gamma_{\mu\nu}$ is called a bivector. Similarly a trivector

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \equiv \gamma_{\mu\nu\rho} \tag{38}$$

and a quadrivector

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma \equiv \gamma_{\mu\nu\rho\sigma} \tag{39}$$

can be defined. (Clifford multiplication is associative.) These products yield 16 linearly independent quantities, which exhaust the possibilities.[23]

Any number D in Dirac algebra can be expanded as a sum of homogeneous multivectors D_r as follows:[24]

$$D = \sum_{r=0}^{4} D_r = d + d^\mu \gamma_\mu + \frac{d^{\mu\nu}\gamma_{\mu\nu}}{2!} + \frac{d^{\mu\nu\rho}\gamma_{\mu\nu\rho}}{3!} + \frac{d^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}}{4!}, \tag{40}$$

where the $d, d^\mu, d^{\mu\nu}$, etc. are the (Dirac) scalar coefficients of their associated homogeneous multivector. The number of indices r indicates the grade of the (homogeneous) multivector. The metric $g_{\mu\nu}$ can be used in the usual way to raise and lower indices in these expansions.[25] Using the relationships[8] $\gamma_{\mu\nu\rho\sigma} = \gamma_5 \epsilon_{\mu\nu\rho\sigma}, \gamma_{\mu\nu\rho} = \gamma_{\mu\nu\rho\sigma}\gamma^{\sigma}$, and

$$\gamma_5 \equiv \gamma_{0123} = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_3, \tag{41}$$

where γ_5 is the unit pseudoscalar#16 for the Dirac algebra, Eq. (40) can be put into the form

$$D = S + V^{\mu}\gamma_{\mu} + \frac{1}{2}T^{\mu\nu}\gamma_{\mu\nu} + C_{\sigma}\gamma_5\gamma^{\sigma} + P\gamma_5, \tag{42}$$

where S, V^{μ}, and $T^{\mu\nu}$ have been written for d, d^{μ}, and $d^{\mu\nu}$, respectively, and $C_{\sigma} \equiv \frac{d^{\mu\nu\rho}\epsilon_{\mu\nu\rho\sigma}}{3!}$, and $P \equiv \frac{d^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}}{4!}$.

The coefficients $S, V^{\mu}, T^{\mu\nu}$, etc. can be viewed as the components of the associated tensor description of D. Thus, Eq. (42) shows that the 16 linearly independent product forms in Dirac algebra partition into scalar, vector, tensor, axial vector (or pseudovector), and pseudoscalar objects in complete analogy to the bilinear forms constructed using solutions to the Dirac equation.[26] We note here that the words "scalar," "vector," etc. are used in their (Dirac) algebraic or geometric sense, without reference to any electromagnetic properties. Consideration of the parity of electromagnetic charges will by deferred to Sec. 4c.

The dual $\tilde{D}$ of D can be defined by

$$\tilde{D} \equiv \gamma_5 D. \tag{43}$$

Since $(\gamma_5)^2 = -1$, one has at once that

$$\overset{\approx}{D} = \gamma_5(\tilde{D}) = \gamma_5(\gamma_5 D) = (\gamma_5)^2 D = -D. \tag{44}$$

Of course, Eqs. (43) and (44) hold for the homogeneous multivector D_r and are fully consistent with the usual tensor definitions of duals. For example,

#16 Note that to be consistent with the geometric purpose of employing the γ_{μ}, we use Hestenes' definition[8] of γ_5 (which differs from that of Bjorken and Drell[22] by a factor $i = \sqrt{-1}$). Thus, $\gamma_5\gamma^5 = 1$ and $(\gamma^5)^2 = (\gamma_5)^2 = -1$.

the identity#17 $\gamma_5\gamma^{\mu\nu} = \frac{1}{2}\epsilon^{\rho\sigma\mu\nu}\gamma_{\rho\sigma}$ enables one to show for the expansion $\tilde{T} = \frac{1}{2}\tilde{T}^{\mu\nu}\gamma_{\mu\nu}$ that $\tilde{T}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}T_{\rho\sigma}$, in accordance with Eq. (6). Similarly, the identities $\gamma_5\gamma^{\mu} = -\frac{1}{6}\epsilon^{\alpha\beta\delta\mu}\gamma_{\alpha\beta\delta}$ and $\gamma_5\gamma^{\alpha\beta\delta} = \epsilon^{\mu\alpha\beta\delta}\gamma_{\mu}$ yield for the expansions $\tilde{V} = \frac{1}{6}\tilde{V}^{\alpha\beta\delta}\gamma_{\alpha\beta\delta}$ and $\tilde{K} = \tilde{K}^{\mu}\gamma_{\mu}$ the familiar tensor relationships[27] $\tilde{V}^{\alpha\beta\delta} = -\epsilon^{\alpha\beta\delta\mu}V_{\mu}$ and $\tilde{K}^{\mu} = \frac{1}{6}\epsilon^{\mu\alpha\beta\delta}K_{\alpha\beta\delta}$ for the duals of vectors and trivectors, respectively. Thus, we see that the dual of a vector is a trivector and vice versa.

b. Maxwell's Equations#18

We are now in a position to recapitulate, using Dirac algebra, the results of Sec. 2. The standard set of Maxwell's equations are written

$$\partial F = \frac{4\pi}{c} j, \tag{45}$$

where $\partial \equiv \gamma^{\mu}\partial_{\mu}, F = \frac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}$, and $j = j^{\mu}\gamma_{\mu}$ are the expansions of ∂, F, and j in the Dirac basis. Equation (45) partitions into a vector part,

$$\partial \cdot F = \frac{4\pi}{c} j, \tag{46}$$

and a trivector or pseudovector part,

$$\partial \wedge F = 0, \tag{47}$$

which are recognized as re-statements of Eqs. (5) and (7), respectively. Equation (45) demonstrates the economy of expression afforded by Dirac algebra—even better than differential geometry, which takes two equations [*i.e.*, the equivalent of Eqs. (46) and (47)] to say the same thing.[28]

#17 These identities are easy to derive from the formulae given in Appendix A of Ref. [8]. Some caution is advised, however, since some minus signs are missing.

#18 For the standard (electromagnetic) Maxwell's equations, this section follows Hestenes[8]; our treatment of the magnetoelectric Maxwell's equations, which was not covered by Hestenes, differs in some important respects from that of Ref. [7].

The relationship between the field tensor and vector potential are written as

$$F = \partial A, \tag{48}$$

where $A \equiv A^{\mu}\gamma_{\mu}$. Equation (48) partitions into a bivector part,

$$F = \partial \wedge A, \tag{49}$$

and a scalar part,

$$0 = \partial \cdot A. \tag{50}$$

Equation (49) is a re-statement of Eq. (3), and Eq. (50) is recognized as the Lorentz condition. It is also evident that

$$\partial^2 A = \frac{4\pi}{c}\, j, \tag{51}$$

where $\partial^2 \equiv \partial \cdot \partial$ is a scalar operator.

Similarly, for the magnetoelectric Maxwell's equations one writes

$$\partial G = \frac{4\pi}{c} k, \tag{52}$$

$$G = \partial M, \tag{53}$$

and

$$\partial^2 M = \frac{4\pi}{c} k. \tag{54}$$

By using γ_5 as a bookkeeping device, Eqs. (45) and (52) can be combined into a single equation. First, multiply Eq. (52) on the left by γ_5 to obtain

$$-\partial \gamma_5 G = \frac{4\pi}{c} \gamma_5 k. \tag{55}$$

(Note that $\gamma_5 \partial = -\partial \gamma_5$.) Setting $\tilde{G} = \gamma_5 G$ and $\tilde{k} = \gamma_5 k$, Eq. (55) becomes

$$-\partial \tilde{G} = \frac{4\pi}{c} \tilde{k}. \tag{56}$$

Then, subtracting Eq. (56) from Eq. (45) yields

$$\partial \mathcal{F} = \frac{4\pi}{c} \mathcal{J}, \tag{57}$$

where

$$\mathcal{F} \equiv F + \gamma_5 G = F + \tilde{G}, \tag{58}$$

and

$$\mathcal{J} \equiv j - \gamma_5 k = j - \tilde{k}. \tag{59}$$

Equation (57) includes both Eq. (28) as the vector part and Eq. (29) as the trivector part. The homogeneous equations, Eqs. (7) and (24), are also implied. Equation (57), representing the symmetrized set of Maxwell's equations, is the epitome of economy afforded by Dirac algebra.

The potential

$$\mathcal{A} \equiv A - \gamma_5 M = A - \tilde{M}, \tag{60}$$

which satisfies

$$\mathcal{F} = \partial \mathcal{A}, \tag{61}$$

may be formed. The relationship

$$\partial^2 \mathcal{A} = \frac{4\pi}{c} \mathcal{J} \tag{62}$$

also obtains.

c. Equations of Motion

It is easy to see that using Dirac algebra, Eqs. (12) and (26) are written as

$$ma = \frac{e}{c} F \cdot v, \tag{63}$$

and

$$ma = \frac{g}{c} G \cdot v, \tag{64}$$

respectively, where $a \equiv \frac{dv^\mu}{d\tau} \gamma_\mu$.

The desired generalizations of Eqs. (63) and (64) that include the cross interaction terms are

$$ma = \frac{e}{c}(F + \tilde{G}) \cdot v = \frac{e}{c}\mathcal{F} \cdot v \tag{65}$$

and

$$ma = \frac{g}{c}(G - \tilde{F}) \cdot v = -\frac{g}{c}\tilde{\mathcal{F}} \cdot v; \tag{66}$$

charged particles feel a force proportional to the total electromagnetic field as defined by Eq. (58) or, equivalently, by Eqs. (30) and (31). Noting that [using $(\gamma_5)^2 = -1$] Eq. (66) can also be written as

$$ma = \frac{-g}{c}(F + \tilde{G}) \cdot \tilde{v} = -\frac{g}{c}\mathcal{F} \cdot \tilde{v}, \tag{67}$$

we see that Eq. (14) may be generalized to

$$f = \frac{1}{c}\mathcal{F} \cdot J. \tag{68}$$

d. Lagrangian Formulation

In order to construct a Lagrangian formulation for generalized electromagnetism, we first look for a Lagrangian density from which Eq. (46) may be derived. It is immediately evident that the Lagrangian given in Eq. (1) is equivalent to

$$< \frac{1}{8\pi}FF - \frac{1}{c}jA >_S \tag{69}$$

in Dirac algebra,[#19] where the symbol $< >_S$ means take the scalar part. Taking the scalar part eliminates the bivector and pseudoscalar parts of the geometric products. (There are no vector or pseudovector parts to eliminate.) This is essential for Lorentz invariance. (Actually, the pseudoscalar part is also Lorentz invariant, except under inversions.)

#19 Note the factor -2 when one compares the (quadratic bivector) FF term of Eq. (69) with that of Eq. (1). In contrast, the product of two Dirac vectors has the factor $+1$ relative to the analogous tensor product.

Analogous to Eq. (2), the action is formed by integrating the Lagrangian density over all space-time. In Dirac algebra the differential volume of space-time can be written as the (Dirac) pseudoscalar

$$d\Omega = c\ dt \wedge dx \wedge dy \wedge dz, \tag{70}$$

where dt, dx, etc. are vectors. Properly combining the scalar Lagrangian density with the pseudoscalar $d\Omega$, the electromagnetic action is

$$S_e = \frac{1}{c} \int < (\frac{1}{8\pi} FF - \frac{1}{c} jA) d\Omega >_P, \tag{71}$$

where $<>_P$ means keep the pseudoscalar piece. Equivalently, we can use a scalar Lagrangian density by adopting a hybrid notational form employing the tensor (scalar) d^4x for the differential 4-volume and write the action as

$$S_e = \frac{1}{c} \int \mathcal{L}_e d^4x, \tag{72}$$

where

$$\mathcal{L}_e = \frac{1}{8\pi} < FF >_S - \frac{1}{c} < jA >_S . \tag{73}$$

It is straightforward to demonstrate that the condition that $\delta_A S_e$ be stationary, independent of δA, yields Eq. (46). Analogously,

$$S_m = \frac{1}{c} \int \mathcal{L}_m d^4x, \tag{74}$$

where

$$\mathcal{L}_m = \frac{1}{8\pi} < GG >_S - \frac{1}{c} < kM >_S, \tag{75}$$

will, through the (arbitrary) variations δM, yield the vector part of Eq. (52). As in the tensor formalism, the homogeneous Maxwell's equations [Eq. (47) and the trivector part of Eq. (52)] are not derived from an action principle but rather are viewed as deriving from the definitions of F and G in terms of A and M.[29] Thus, as before, we have a Lagrangian from which we can obtain by a least action principle the symmetrized Maxwell's equations.

The question now is how to extend this result and find a Lagrangian from which one can also derive the correct equations of motion for electrically and magnetically charged particles. Since we have already seen that $\mathcal{L}_e + \mathcal{L}_m$ will yield the symmetrized Maxwell's equations, it is reasonable to look for a Lagrangian that includes all the terms of $\mathcal{L}_e + \mathcal{L}_m$. Such a Lagrangian is

$$\mathcal{L} = \frac{1}{8\pi} < \mathcal{F}^{\star}\mathcal{F} >_S - \frac{1}{c} < \mathcal{J}\mathcal{A} >_S, \tag{76}$$

where $\mathcal{F}^{\star} = F - \gamma_5 G$; that is, the $\star$ symbol (in analogy to complex conjugation) changes the sign of the γ_5 (which should be explicitly written out[#20]).

When we multiply out the interaction term $-\mathcal{J}\mathcal{A}$, we obtain

$$-\mathcal{J}\mathcal{A} = -(j - \tilde{k})(A - \tilde{M}) = -(\mathcal{J}\mathcal{A} + \tilde{k}\tilde{M} - j\tilde{M} - \tilde{k}A). \tag{77}$$

The scalar part of $-\mathcal{J}\mathcal{A}$ is $-(j \cdot A + \tilde{k} \cdot \tilde{M}) = -j \cdot A - k \cdot M$, the same interaction terms as in $\mathcal{L}_e + \mathcal{L}_m$. This scalar part, then, comprises what we call the standard interaction terms. The cross terms $(j\tilde{M} + \tilde{k}A) = (j\tilde{M} - k\tilde{A})$ comprise a sum of bivectors and pseudoscalars, and hence do not survive the $< >_S$ operation. Thus, we see that, in fact, $\mathcal{L} = \mathcal{L}_e + \mathcal{L}_m$.

In order to utilize these cross terms, something new is necessary. We observe that the pseudoscalar piece of $\mathcal{J}\mathcal{A}$ cannot be used to obtain suitable equations of motion because S_P has no corresponding pseudoscalar piece.[#21] Thus, we have to look for a way to obtain scalars from the cross terms. As indicated in Sec. 2a, this is possible by using a surface integral formulation[#22] in Dirac algebra.

#20 Actually, the $\star$ symbol as it is employed here should be viewed as a notational convenience rather than as a rigorous operator of Dirac algebra.

#21 Since the Lagrangian of Ref. [7] specifically includes pseudoscalar pieces, it is not appropriate to use for the derivation of the equations of motion for electrically and magnetically charged particles. Even dropping the pseudoscalar pieces will not yield a suitable Lagrangian because of the requirement of dyality invariance. See Footnote #25.

#22 This option was precluded in Ref. [4] by maintaining the definition of the action in terms of a path integral.

To see how this might be done, it is instructive to write a sequence of equivalent expressions in Dirac algebra[#23] for the variation in the action associated with the standard interaction terms:

$$
\begin{aligned}
\delta_x S_{I_S} &= -\frac{1}{c}\oint dx \cdot (eA + gM) = \frac{1}{c}\int (d\sigma \cdot \partial) \cdot (eA + gM) \\
&= \frac{1}{c}\int d\sigma \cdot \Big[\partial \wedge (eA + gM)\Big] = \frac{1}{c}\int < d\sigma\partial(eA + gM) >_S .
\end{aligned} \tag{78}
$$

We can see that the penultimate expression in Eq. (78) is equal to

$$
\frac{1}{c}\int d\sigma \cdot (eF + gG) = -\frac{1}{2c}\int d\sigma^{\mu\nu}(eF_{\mu\nu} + gG_{\mu\nu}), \tag{79}
$$

which, using Eq. (19), can be converted to path integral form:

$$
\Longrightarrow -\frac{1}{c}\int_a^b (eF_{\mu\nu} + gG_{\mu\nu})dx^\nu \delta x^\mu, \tag{80}
$$

the appropriate term to combine with $\delta_x S_P$ to obtain the Euler-Lagrange equations of motion for electrically and magnetically charged particles—but without the cross terms.

Using Eq. (77), the cross terms (in their appropriate form) can be consistently entered with the standard terms into the final expression in Eq. (78):

$$
\delta_x S_I = \delta_x(S_{I_S} + S_{I_G}) = \frac{1}{c}\int < d\sigma\partial(eA + gM - e\tilde{M} + g\tilde{A}) >_S . \tag{81}
$$

The cross term contribution to the variation in the action is

---

#23 To be consistent with our tensor expansions, a minus sign is included in the boundary theorem[8] in going in Dirac algebra from the path integral (a Dirac vector product) to the surface integral (a Dirac bivector product). That this is appropriate is verified by the fact that Eq. (80) yields the correct result for the standard interaction terms.

$$\delta_x S_{I_G} = \frac{-1}{c}\int < d\sigma(e\partial\tilde{M} - g\partial\tilde{A}) >_S . \tag{82}$$

Taking the scalar part, Eq. (82) becomes

$$\delta_x S_{I_G} = \frac{-1}{c}\int d\sigma\cdot(e\partial\cdot\tilde{M} - g\partial\cdot\tilde{A}) = \frac{1}{c}\int d\sigma\cdot(e\tilde{G} - g\tilde{F}). \tag{83}$$

In tensor notation the final expression in Eq. (83) is equal to

$$\frac{-1}{2c}\int(e\tilde{G}_{\mu\nu} - g\tilde{F}_{\mu\nu})d\sigma^{\mu\nu} \Longrightarrow \frac{-1}{c}\int_a^b(e\tilde{G}_{\mu\nu} - g\tilde{F}_{\mu\nu})dx^\nu\delta x^\mu, \tag{84}$$

where we have again used Eq. (19). The $\delta_x S_{I_G}$ given in path integral form by Eq. (84) combines with the $\delta_x S_{I_S}$ of Eq. (80) and $\delta_x S_P$ to yield the correct equations of motion, Eqs. (65) and (66).

We observe that (in the absence of magnetic charges) the usual least-action formulation for $\delta_x S_I \sim \delta_x e\int_a^b A_\mu dx^\mu$ or $\sim e\int F_{\mu\nu}d\sigma^{\mu\nu}$ places a condition upon the (derivatives of the) vector potential A_μ along the path of the electrically charged particle. This condition is such that $\boldsymbol{E}$ along the path accelerates an electric charge and $\boldsymbol{H}$ across the path deflects it, in accordance with Eq. (63). The cross term $\sim e\int\tilde{G}_{\mu\nu}d\sigma^{\mu\nu}$ consistently combines the magnetoelectric field into this relationship with $\boldsymbol{E}'$ adding to $\boldsymbol{E}$ and $\boldsymbol{H}'$ adding to $\boldsymbol{H}$, in accordance with Eq. (65). However, in the case of $e\int\tilde{G}_{\mu\nu}d\sigma^{\mu\nu}$, the condition upon the (derivatives of the) vector potential M_μ differs from that placed upon A_μ; using the surface integral formulation, the relationship of M_μ to path geometry is "inverted." The γ_5 effects this inversion (in Dirac algebra) by performing an exchange (analogous to dyality exchange) of the roles of the dx^ν and the δx^μ in the (equivalent) path integral form for the cross interaction.

By the same token, for the cross term $\sim g\int\tilde{F}_{\mu\nu}d\sigma^{\mu\nu}$, the relationship of the electromagnetic potential A_μ to path geometry is inverted. As a consequence, as is required for an electromagnetic field acting on a magnetic charge, $\boldsymbol{H}$ along the path accelerates the monopole and $\boldsymbol{E}$ across the path deflects it. And $\tilde{F}$ combines with G in the appropriate way, as given by Eq. (66).

It is now necessary to go back and see if there is a contribution of the cross terms in the action associated with variations in δA and δM; these variations were used to obtain Maxwell's equations (from the standard interaction terms). Whereas the standard interaction terms make a nonzero contribution to the action prior to the variational procedure, the cross terms do not; this is true either when one takes the pseudoscalar piece of the entire action integral, as in Eq. (71), or when one takes the scalar part of the Lagrangian density only, as in Eq. (72). (Recall that $< j\tilde{M} + \tilde{k}A >_S = 0$.) And since in the surface integral formulation, prior to the variational procedure, we have a null surface area ($\delta x^\mu = 0$), we conclude that in any case

$$S_{I_G} = 0. \tag{85}$$

It also follows from these considerations that whatever form we choose for the cross terms, these cross terms have null coefficients in the relationships ensuing from the variations of δA and δM. That is, while $\delta_x S_{I_G} \neq 0$, as given by Eq. (84), we have at the same time

$$\delta_A S_{I_G} = \delta_M S_{I_G} = 0, \tag{86}$$

and the inclusion of the cross terms in the action integral does not impair the original derivation of the symmetrized Maxwell's equations. Thus, by using Dirac algebra and the surface integral form for S_{I_G}, we have found a Lagrangian formulation that unifies electromagnetism and magnetoelectricity.

e. Symmetric Stress Tensor

In source-free regions the source terms in Maxwell's equation are null, and one may write for such regions

$$\partial F = 0, \tag{87}$$

which is equivalent to

$$\overleftarrow{F}\,\overleftarrow{\partial} = 0, \tag{88}$$

where the over-arrow denotes the reversion operator, reversing the order of the γ vectors. Thus, $\overleftarrow{F} = -F$ and $\overleftarrow{\partial} = \partial$; in the latter case, the arrow also dictates differentiating to the left rather than to the right.

Using Eqs. (87) and (88), one can construct (still for a source-free region) the quadratic form

$$\overleftarrow{F}\,\partial F + \overleftarrow{F}\,\overleftarrow{\partial}\,F = \partial_\mu(\overleftarrow{F}\,\gamma^\mu F) = 0. \tag{89}$$

The Dirac vector (actually, four Dirac vectors labeled by μ),

$$S^\mu \equiv \frac{1}{8\pi}\overleftarrow{F}\,\gamma^\mu F, \tag{90}$$

has been called the stress-energy vector[8] and can be used to define an energy-momentum tensor,[30]

$$S^{\mu\nu} \equiv S^\mu \cdot \gamma^\nu = \gamma^\nu \cdot S^\mu = S^{\nu\mu}, \tag{91}$$

a symmetric 4-by-4 matrix of Dirac scalers. Using the rules of Dirac algebra, it is straightforward to demonstrate that

$$S^{\mu\nu} = \frac{1}{4\pi}(g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) = \Theta^{\mu\nu}, \tag{92}$$

where $\Theta^{\mu\nu}$ is the usual symmetric stress tensor[31] of electromagnetism. In a source-free region, $S^{\mu\nu}$ is conserved.[8] That is,

$$\partial_\nu S^{\nu\mu} = \partial_\nu(\gamma^\nu \cdot S^\mu) = \partial \cdot S^\mu = 0, \tag{93}$$

or, equivalently,

$$\partial_\mu S^\mu = 0. \tag{94}$$

When sources are present, Eq. (45) replaces Eq. (87) in the construction of the quadratic form of Eq. (89), and Eq. (94) becomes

$$\partial_\mu S^\mu = \frac{1}{c} j \cdot F = -\frac{1}{c} F \cdot j. \tag{95}$$

When one incorporates monopoles into this analysis, Eq. (57) replaces Eq. (45), in which case Eqs. (90) and (95) generalize in a straightforward way to

$$S^\mu \equiv \frac{1}{8\pi} \overleftarrow{\mathcal{F}} \gamma^\mu \mathcal{F} \tag{96}$$

and

$$\partial_\mu S^\mu = \frac{1}{c} \mathcal{J}^\star \cdot \mathcal{F} = -\frac{1}{c} \mathcal{F} \cdot \mathcal{J}, \tag{97}$$

respectively. Equation (97) justifies the generalized Lorentz force vector density given in Eq. (68).

4. Invariance Relationships and All That

a. Gauge Invariance

It is well known that the field tensor F is invariant under the gauge transformation

$$A \rightarrow A' = A + \partial\chi, \tag{98}$$

where χ is a scalar. This is easily verified using Dirac algebra:

$$F \rightarrow F' = \partial \wedge A' = \partial \wedge (A + \partial\chi) = F + \partial \wedge \partial\chi = F; \tag{99}$$

the (antisymmetric) wedge product of a vector with itself is zero.[8] To maintain the Lorentz condition for A', χ must satisfy $\partial^2\chi = 0$. Analogous results hold for G and M, under the gauge transformation

$$M \rightarrow M' = M + \partial\lambda, \tag{100}$$

where λ is a scalar obeying $\partial^2\lambda = 0$.

Now the term $j \cdot A$ in the Lagrangian is not gauge invariant as it stands, but the contribution of $\partial\chi$ to the variation in action $\delta_x S_{I_S}$ is null. This is easily seen when one goes back to Eq. (18) and writes for $-e \oint dx \cdot A'$ the equivalent expression from Eq. (78):

$$e \int \; < d\sigma \partial A' >_S = e \int d\sigma \cdot \Big[\partial \wedge (A + \partial\chi) \Big] = e \int d\sigma \cdot (\partial \wedge A + \partial \wedge \partial\chi). \tag{101}$$

Again, the $\partial \wedge \partial\chi = 0$ drops out, leaving the action integral gauge invariant. This result is routine for classical electrodynamics[32] and, of course, holds in similar fashion for the standard magnetoelectric term because $\partial \wedge \partial\lambda = 0$.

In analogous fashion, one can also verify that $\delta_x S_{I_G}$ is gauge invariant. We write for the cross term involving g and $\tilde{A}'$,

$$g\int < d\sigma\partial\tilde{A}' >_S = g\int < d\sigma\partial(\tilde{A}+\gamma_5\partial\chi) >_S = g\int d\sigma\cdot\Big[\partial\cdot(\tilde{A}+\gamma_5\partial\chi)\Big]. \quad (102)$$

The extra term here,

$$g\int d\sigma\cdot\Big[\partial\cdot(\gamma_5\partial\chi)\Big] = -g\int d\sigma\cdot\Big[\partial\cdot(\partial\cdot\gamma_5\chi)\Big] = -g\int d\sigma\cdot\Big[(\partial\wedge\partial)\cdot\gamma_5\chi\Big] = 0, \quad (103)$$

again because $\partial\wedge\partial = 0$. Similarly, the cross term involving e and $\tilde{M}'$ will be gauge invariant under the transformation given by Eq. (100). Thus, we see that the unified electromagnetism developed in this paper satisfies gauge invariance under the transformations defined by Eqs. (98) and (100).

b. Dyality Exchange Relations and Dyality Rotations

It is well known that the symmetrized set of Maxwell's equations (in vector form) are invariant under the dyality exchange relations:[33]

$$\boldsymbol{E}\rightarrow\pm\boldsymbol{H},\ \boldsymbol{H}\rightarrow\mp\boldsymbol{E}, \quad (104)$$

and

$$\begin{aligned}\rho\rightarrow\pm\sigma,\ \sigma\rightarrow\mp\rho,\\ \boldsymbol{j}\rightarrow\pm\boldsymbol{k},\ \boldsymbol{k}\rightarrow\mp\boldsymbol{j}.\end{aligned} \quad (105)$$

Partitioning the electric and magnetic fields into those associated with $(\rho c,\boldsymbol{j})$ and those associated with $(\sigma c,\boldsymbol{k})$, as is done in Sec. 2, leads to no difficulty with the concept of dyality exchange relations. For example, Eq. (104) becomes

$$(\boldsymbol{E}+\boldsymbol{E}')\rightarrow\pm(\boldsymbol{H}+\boldsymbol{H}'),\ (\boldsymbol{H}+\boldsymbol{H}')\rightarrow\mp(\boldsymbol{E}+\boldsymbol{E}'). \quad (106)$$

More specifically, (the electric) Coulomb field interchanges with (the magnetic) Coulomb field, and (the electrically generated) solenoidal field interchanges with (the magnetically generated) solenoidal field. That is,

$$\boldsymbol{E} \rightarrow \pm \boldsymbol{H}', \ \boldsymbol{H}' \rightarrow \mp \boldsymbol{E}, \ \text{and} \ \boldsymbol{H} \rightarrow \mp \boldsymbol{E}', \boldsymbol{E}' \rightarrow \pm \boldsymbol{H}. \tag{107}$$

Similarly, by extension, one obtains

$$\boldsymbol{A} \rightarrow \pm \boldsymbol{M}, \ \boldsymbol{M} \rightarrow \mp \boldsymbol{A}, \ \text{and} \ \phi \rightarrow \pm \ \psi, \ \psi \rightarrow \mp \ \phi. \tag{108}$$

The partitioning given in Eqs. (107) and (108) logically accompanies the source exchange relations of Eq. (105).

Specifying the upper signs as the (arbitrary) convention for this paper leads to Eq. (21). This specification leads for the field tensors to the dyality exchange relation: $F \rightarrow G, G \rightarrow -F$. Thus, we see that when separate field tensors are used for electromagnetism and magnetoelectricity, a distinction needs to be made between the mathematical definition of the tensor dual, *e.g.*, Eq. (6), and the dyality exchange relations for Maxwell's equations; though closely related, these two concepts are not identical. [Also, see Eqs. (114) and (115).]

Rainich [34] in 1925 pointed out that one can describe by an arbitrary angle a continuous symmetry between electric and magnetic quantities. This symmetry generalizes the concept of dyality exchange relations. Following Rainich, we can define (for the partitioned fields) "rotated" electric and magnetic quantities:

$$\begin{aligned} \boldsymbol{E}^R &= \boldsymbol{E} \cos\theta + \boldsymbol{H}' \sin\theta, \\ \boldsymbol{H}'^R &= -\boldsymbol{E} \sin\theta + \boldsymbol{H}' \cos\theta \end{aligned} \tag{109}$$

for the Coulomb fields;

$$\begin{aligned} \boldsymbol{E}'^R &= \boldsymbol{E}' \cos\theta + \boldsymbol{H} \sin\theta, \\ \boldsymbol{H}^R &= -\boldsymbol{E}' \sin\theta + \boldsymbol{H} \cos\theta \end{aligned} \tag{110}$$

for the solenoidal fields; and

$$\begin{aligned} \rho^R &= \rho \cos\theta + \sigma \sin\theta, \\ \sigma^R &= -\rho \sin\theta + \sigma \cos\theta \end{aligned} \tag{111}$$

for the sources. When one collects all the rotated terms (which relate to a newly chosen reference direction in the electromagnetic charge plane), one again obtains a set of symmetrized Maxwell's equations, but this time in terms of the rotated quantities. A dyality rotation of $\theta = \mp\pi/2$, then, generates the dyality exchange relations, Eqs. (107) and (105). Partitioning the fields into electrically generated quantities and magnetically generated quantities entails no difficulties.#24 One just rotates the Coulomb and solenoidal fields separately as a generalization of Eq. (107).

It is straightforward to incorporate the concept of dyality rotations into the Dirac algebraic Maxwell's equations. These rotations, induced[7] by $e^{\gamma_5\theta}$, are

$$j \to j^R = e^{\gamma_5\theta} j \tag{112}$$

for vectors (and trivectors) and

$$F \to F^R = e^{-\gamma_5\theta} F \tag{113}$$

for bivectors. The origin of the minus sign in Eq. (113) stems from the fact that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$, which can be seen to operate, for example, in Eq. (55). In general, then, one writes

$$J \to J^R = e^{\gamma_5\theta} J \tag{114}$$

and

$$\mathcal{F} \to \mathcal{F}^R = e^{-\gamma_5\theta} \mathcal{F}. \tag{115}$$

It is obvious that the generalized Eqs. (57), (61), (62), and (68) are invariant under an arbitrary dyality rotation as defined by Eqs. (114) and (115). Similarly,

---

#24 On this point one also notes that Lorentz transformations do not mix electromagnetic and magnetoelectric quantities.

it is straightforward to demonstrate that $\mathcal{L}$ of Eq. (76) is dyality invariant:

$$\mathcal{L} \Longrightarrow \mathcal{L}^R = \frac{1}{8\pi} < (e^{-\gamma_5\theta}\mathcal{F})^\star(e^{-\gamma_5\theta}\mathcal{F}) >_S - \frac{1}{c} < (e^{\gamma_5\theta}\mathcal{J})(e^{\gamma_5\theta}\mathcal{A}) >_S = \mathcal{L}. \quad (116)$$

(The bivector and pseudoscalar parts of $\mathcal{F}^\star\mathcal{F}$ and $\mathcal{J}\mathcal{A}$ are dyality invariant as well as the scalar part.)[#25] Likewise, Eqs. (96) and (97) are dyality invariant. In short, all of the unified electromagnetic theory developed in this paper is dyality invariant under arbitrary dyality rotations.

c. Parity Violation

It was observed long ago that when one incorporates magnetic monopoles into electromagnetism, there is difficulty with parity conservation.[35] The usual approach to the resolution of this difficulty is to assert that magnetic charge is pseudoscalar and electric charge is scalar[35–36] (or vice versa). This approach finds its basis in an assumption about the generalized electromagnetic field. It is assumed, for example, that the external magnetic dipole field made by a north pole and a south pole at some (small) separation should be indistinguishable from a dipole due to the circulation of electric charge.[#26] By this assumption, while the components of the generalized electromagnetic field tensor will behave in the same way under the parity reflection operator as do those of the "electric" electromagnetic field, for consistency electric and magnetic charge must have opposite parity. But, following the ideas behind the the analysis in this paper, if the generalized electromagnetic field partitions into electromagnetic and magnetoelectric parts physically as well as mathematically, then the basis for this

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#25 We note here that neither of the two terms in the Lagrangian of Ref. [7] (which in our notation would be written as $< \mathcal{F}\mathcal{F} >_{S,P}$ and $< \overleftarrow{\mathcal{J}}\mathcal{A} >_{S,P}$, where $<>_{S,P}$ means keep both the scalar and pseudoscalar parts of the geometric products) is invariant under dyality exchange. For example, $\overleftarrow{\mathcal{J}}\mathcal{A} = \mathcal{J}^\star\mathcal{A} \Rightarrow (e^{\gamma_5\theta}\mathcal{J})^\star(e^{\gamma_5\theta}\mathcal{A}) = e^{-2\gamma_5\theta}\mathcal{J}^\star\mathcal{A} \neq \overleftarrow{\mathcal{J}}\mathcal{A}$. Hence, that Lagrangian cannot be used to derive consistent equations of motion for electrically and magnetically charged particles.

#26 It is recognized, of course, that on the scale of the configuration of the sources of the dipole fields, there are structural differences in the magnetic dipole due to magnetic charges and the one due to electric currents.[37]

assumption does not obtain, and one can take a different approach. That is, instead of assuming that the place to connect electromagnetism to magnetoelectricity is the field tensor (identical properties of F and $\tilde{G}$ under parity reflection), one can assume that the junction of electricity and magnetism should be at the source#27 (identical properties of j and k under parity). We pursue the latter view here.

The dyality invariance of the unified electromagnetic theory developed above offers support for this view. If any given charge (where, for the sake of argument, we presume that both types exist) can be viewed as electric or magnetic or, in fact, a mixture of both, depending upon one's choice of the dyality angle, is it not reasonable to suppose that electric and magnetic charge are really different manifestations of the same essence?#28 If this be the case, then it follows that electric and magnetic charge would both be scalar (or both pseudoscalar).

Joining electricity to magnetism at the source leads to the following characterization with respect to parity reflection:

$$\begin{aligned} \text{scalars} &: e,\ g,\ \phi,\ \psi \\ \text{polar vectors} &: \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{E}, \boldsymbol{H}', \boldsymbol{A}, \boldsymbol{M} \\ \text{axial vectors} &: \boldsymbol{H}, \boldsymbol{E}'. \end{aligned} \tag{117}$$

With this construction, in the presence of both electric and magnetic charge, the (unified) electromagnetic field $\mathcal{F}$ and energy-momentum tensor $\mathcal{S}^{\mu\nu}$ would be objects of mixed parity, inconsistent with the usual assumption. Of course, when

#27 The question of which entity is primary, source or field, was raised by Misner, Thorne, and Wheeler,[38] who left the issue unresolved.

#28 The mathematical situation is paradoxical even if physically, there would be only one kind of charge (electric, say). One can, by a dyality rotation of $\theta = \pm\pi/2$ go from a description of electromagnetism in an electric world to one of magnetoelectricity in a magnetic world. But since the underlying physical situation is the same in both cases, why should there be a shift in the parity? The problem is that there is no natural way to relate the discrete operation, parity reflection, to the continuous variable, dyality angle. This difficulty is avoidable by defining the parity of electric and magnetic charge to be the same in the first place. A similar model has been proposed by Tevikyan.[39]

there are no magnetic charges, $G = 0$, and $\mathcal{F} = F$ and $\mathcal{S}^{\mu\nu} = \Theta^{\mu\nu}$ will have the usual parity assignments, and there will be no parity violation.

d. Stationary Action Integral

The symmetrized Maxwell's equations, as well as the equations of motion of electrically and magnetically charged particles, are derived above by the least action principle in a formalism of unified electromagnetism. In the past, it is this latter derivation that has been problematical. In our derivation, many of the difficulties are resolved, once we recognize that δx^μ is an infinitesimal function.

For the mathematical procedure to be valid, δx^μ must be (small enough) such that only the first term in the Taylor's expansion of the potentials (not counting the self-potentials) along the path from a to b is significant. This first term, of course, is the field tensor, which, as a consequence of the Euler-Lagrange equations, enters as a factor in the Lorentz force. For the standard interaction, for the Lorentz force on an electric charge the relevant term is the $\frac{-e}{c}\int_a^b F_{\mu\nu}dx^\nu\delta x^\mu$ of Eq. (10); for the cross term it is the $\frac{-e}{c}\int_a^b \tilde{G}_{\mu\nu}dx^\nu\delta x^\mu$ of Eq. (84). Both are linear in δx^μ.

If one is considering point particles, by inspection it is clear that δx^μ can be kept sufficiently small (The displaced path must be much closer to the equilibrium path than is any [other] charge.), and the derivation goes through, provided that two particles never occupy the same point in space-time. Since the practical possibility of point particles colliding is infinitesimal, we argue (as does Rohrlich[4]) that this should not be viewed as a serious difficulty. If one is still concerned about exact, head-on collisions, then the resolution must be found by going to such a small scale that the particles are no longer points but (smoothly) distributed in space.

If currents can be viewed as smoothly distributed, the derivation still goes through (but not by inspection). The argument is best made using the surface integral form for the relevant terms in the action integral and following the path of a point particle#29 under the influence of fields from a smoothly distributed source. In keeping with the spirit of the infinitesimal δx^{μ}, one must still limit the excursion of the (arbitrary) surfaces of the surface integral such that only the first term of the Taylor's expansion of the potential is relevant. But with an external current density in the locality of the equilibrium path, the equivalence of alternative choices for a surface becomes questionable. Whereas with the fields from point particles we could declare as illegitimate any displaced path or spanning surface that came too close to a passing charge (δx^{μ} could still be arbitrary, but small), it is conceivable that in the case of distributed currents, the equilibrium path of the charge we are following could pass right through another current distribution. This situation represents the head-on collision referred to above. In this case, the volume enclosed by two arbitrary surfaces spanning the same loop defined by the equilibrium path and a given δx^{μ} cannot be rendered devoid of charge, no matter how small (a nonzero) δx^{μ}.

For the standard interaction terms, the arbitrariness of the surface in the presence of a charge distribution near the equilibrium path is not a problem. To see this, we first convert the path integral to the surface integral $\frac{-e}{2c}\int d\sigma^{\mu\nu}F_{\mu\nu}$ as indicated by Eq. (18). Since we wish to know if this integral is a function of the placement of the arbitrary (infinitesimal) surface, we take the difference of two such surface integrals. But this is just the integral over the total surface σ of the enclosed 3-volume ω. Now this (total) surface integral is equal to the volume integral of a divergence:[40]

$$\frac{-e}{2c}\int d\sigma^{\mu\nu}F_{\mu\nu}=\frac{e}{2c}\int d\tilde{\sigma}^{\mu\nu}\tilde{F}_{\mu\nu}=\frac{e}{c}\int d\omega_{\mu}\partial_{\nu}\tilde{F}^{\mu\nu}, \tag{118}$$

where $d\omega_{\mu}$ is the increment of the μ^{th} 3-volume in Minkowski space-time. But by Eq. (7), $\partial_{\nu}\tilde{F}^{\mu\nu}=0$, and the derivation is independent of the placement of

#29 One could be more general and consider the world line of an increment of charge, but this extra complication is not necessary at this juncture.

the surface chosen to span the loop. Any charge distribution that may be in the vicinity of the equilibrium path makes no (direct) contribution to the Euler-Lagrange equations of motion (via the standard interaction terms).

However, following a cross term through this same sequence leads to

$$\frac{-e}{2c}\int d\tilde{\sigma}^{\mu\nu}G_{\mu\nu} = -\frac{e}{c}\int d\omega_\mu \partial_\nu G^{\mu\nu} = \frac{4\pi e}{c^2}\int d\omega_\mu k^\mu. \tag{119}$$

While there is now a possible difficulty because $k^\mu \neq 0$, we must remember that (by assumption) k^μ is finite and well behaved. This fact is crucial because, owing to the sliver-like nature of the volumes enclosed by two arbitrary surfaces spanning the closed loop, $d\omega_\mu$ is second order in δx^μ; in analogy to $d\sigma^{\mu\nu} \sim dx^\nu \delta x^\mu$, we have $d\omega_\mu \sim \epsilon_{\mu\nu\rho\sigma} dx^\nu \delta x^\rho \delta x^\sigma$. This means that the term given by Eq. (119), proportional to k^μ, is second order in the infinitesimal displacements δx^μ and can be neglected relative to the $\tilde{G}_{\mu\nu}$ term given by Eq. (84), which is linear in δx^μ. Again, any charge distribution that may be in the vicinity of the equilibrium path makes no (direct, first order) contribution to the Euler-Lagrange equations of motion. We have made no additional restrictions on δx^μ; it is still arbitrary, but suitably small.

This result shows that the derivation of the equations of motion for the electric and magnetic charges is not only unrestricted by any kinematic conditions, but that, in fact, the derivation of these equations remains valid even when two particles collide head-on, provided that at some small scale the particles can be viewed as having smoothly distributed non-singular charge distributions.[#30] In addition, because in this derivation the (direct) cross terms proportional to current density can be neglected, no relationship or restrictions between electric and magnetic charge evolves from the derivation. (It is indicated below, however, that for bound states involving both electric and magnetic charge, quantum mechanical considerations will lead to a version of the Dirac quantization condition.)

#30 Viewing particles on a scale at which they are no longer points also resolves the problem that the angular momentum proportional to eg will flip sign in a head-on charge-monopole collision. See R. A. Brandt and J. R. Primack, Ref. [5].

5. Summary and Discussion

The formulation of a generalized electromagnetism that includes both electric and magnetic charge is explored in the framework of Dirac algebra. Initially, using the two-potential approach of Cabibbo and Ferrari, the formulations of electromagnetism, associated with electric sources j^μ, and magnetoelectricity, associated with magnetic sources k^μ, are developed separately. The use of two potentials, of course, obviates the need for singular strings. These two formulations are then adjoined to yield the usual symmetrized set of Maxwell's equations, which in Dirac algebra is but one compact equation. In order to set this exposition in context, a detailed comparison is made with the corresponding tensor formulation. While certain aspects of this part of the analysis are facilitated by the use of Dirac algebra, in comparing the Dirac and tensor formulations, it can be seen that the use of Dirac algebra is thus far not essential.

It is then shown that it is possible to write an action integral from which one can derive by Hamilton's principle not only the symmetrized set of Maxwell's equations but also the equations of motion for both electrically and magnetically charged particles. Obtaining from an action principle proper equations of motion for both electric and magnetic particles unifies the electromagnetic and magnetoelectric formalisms. It is seen that there are no kinematic restrictions on the validity of the derivation of the equations of motion of the charged particles or current distributions, and no relationship between electric and magnetic charge obtains.

This derivation is made possible by employing a surface integral form for the contribution of the cross interaction terms to the action. Since the surface integral has an equivalent path integral form, the contribution of the potential in both the standard interaction terms and the cross interaction terms is local in nature. The proofs that a satisfactory Lagrangian does not exist[4] do not apply because a surface integral is not of the usual form assumed for the cross terms.

It is pointed out that a satisfactory (variation of the) action integral must be scalar in nature. The motivation to use a surface integral, then, derives from the need to accommodate the presence of γ_5, the pseudoscalar of the Dirac algebra, which appears in a natural way in the right places (and with the correct sign) in the cross terms of the generalized interaction Lagrangian. It is in this unification that the use of Dirac algebra is essential; it is the more general geometric product of Clifford algebra that enables a uniform treatment of the standard and the cross interaction terms (but only in the context of the surface integral). The surface integral form enables one to obtain Dirac scalars from the cross terms because it has two additional vectors entering into the geometric products; one vector is the ∂, the other comes from the incremental (bivector) surface element. The consistent unification of electromagnetism and magnetoelectricity is achieved only by forming inside of the $<>_S$ operator the geometric products of the vectors of the surface integral times the generalized electromagnetic quantities; this fact implies an intrinsic relationship between electromagnetism and the properties of space-time.#31

Although the set of symmetrized Maxwell's equations and the equations of motion for electric and magnetic charges obtained above are those that have been used since the turn of the century, a different perspective of the unification is offered. This perspective leads to the suggestion that electric and magnetic charge are different manifestations of the same essence, and therefore would have the same behavior under the parity operation. This has the consequence that the generalized field quantities such as $\mathcal{F}$ and $\mathcal{S}^{\mu\nu}$ would be objects of mixed parity. That a generalized electromagnetic theory should lead to objects that are not eigenstates of parity is actually not such a radical notion. First, Eq. (117) is just a different manifestation of the long-recognized fact that the coexistence (which cannot be eliminated by a dyality rotation) of electric and magnetic charge must involve some form of parity violation. And second, Nature is known to commonly

#31 Also using Dirac algebra, but based on somewhat different reasoning, Salingaros[41] likewise supports the view that there must be an intrinsic relationship between electromagnetism and space-time.

provide objects that are not eigenstates of parity, *e.g.*, the left-handed neutrino. On this latter point, if electromagnetism is indeed the fundamental interaction of physics,[42] then the mutual interaction of magnetic and electric charge in the dynamical construction of the elementary particles could lead in a natural way to the parity violation observed in weak interactions.#32 Neutrinos would not be defined *ab initio* as two-component objects,#33 but the experimental fact that they are always observed as two-component objects would instead derive from the underlying physics.

The notion that the field bivector $\mathcal{F}$ can be partitioned into electromagnetic and magnetoelectric parts implies that these parts are physically as well as mathematically distinguishable, *i.e.*, F and G are distinguishable. This is not a problem as long as there is a source current J to associate (directly) with the field $\mathcal{F}$. However, there is a problem in the case of free fields. Once radiated, the free field has no source to indicate the appropriate partitioning of $\mathcal{F}$, which step is an integral part of this analysis. Of course, we could assume that the free F and G fields are intrinsically distinguishable and that each represents a legitimate free field solution of the electromagnetic and magnetoelectric Maxwell's equations, respectively. We thus arrive at the classical analogue of the two-photon question of quantum electrodynamics (when one employs two potentials) and the reason that free fields were not covered in this analysis.

Rather than addressing this problem by imposing the "zero field conditions,"[3] which do not appear to be intrinsic to the formalism, we leave it as an open question for further study. In this context, the reasoning that led to the notion of a

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#32 In this context, one might even go so far as to argue that parity violation is evidence for magnetic charge.

#33 Expositions of the Standard Model (*e.g.* Ref. [43], which provides earlier references) generally (and reasonably) justify the assumption that neutrinos are two-component objects by noting that experimentally this is how they always are observed. But in this regard, the Standard Model must be characterized as descriptive rather than as explanatory. Of course, since the descriptive success of the Standard Model is so good, any theory that purports to explain this physics on some other basis must be shown to reduce to the Standard Model in some approximation.

distinguishable F and G has its genesis in the continuous dyality symmetry characterizing the classical equations of unified electrodynamics; nevertheless, from experiment one knows that dyality symmetry is broken: electrons are observed, but not "magnetons"; one photon is observed, not two. That Nature should be characterized by a spontaneously broken dyality symmetry, as seems quite probable, implies that quantum mechanics, which can better describe electrons, photons, and their interactions, will be an essential ingredient in the further study; the path integral formalism of Feynman[44] seems particularly well suited to this purpose.

One final comment. While the (classical) analysis in this paper does not reveal any general connection between electric and magnetic charge, one should not conclude#34 that we have lost Dirac's incisive insight: the existence of magnetic charge could account for the quantization of electric charge.[1] In this connection a distinction should be made between scattering or non-stationary states and bound or stationary states. It is evident that using the gauge invariant action from this paper in a quantum mechanical phase factor $e^{iS_I/\hbar}$ for a particle wavefunction will lead to the appropriate Aharonov-Bohm interference effects[46] for non-stationary or scattering states, including magnetic charge. In similar fashion, if one views a bound state wavefunction to be a stationary Aharonov-Bohm interference pattern resulting from a suitable summation of all possible orbits (or Feynman paths), then one can obtain an integral (approximation) of the form of Eq. (84) in which the equivalent of δx^μ will have spatial components comparable to the size of the bound state. In this situation, the 3-volume ω_0 of the (ensuing) equivalent of Eq. (119) comprises the volume of the bound state (orbits). Generalizing to a bound state of two particles of electric and magnetic charges e_1, g_1 and e_2, g_2, respectively, the appropriate self-consistency condition

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#34 It has been asserted that the two-potential approach entails the loss of Dirac's quantization that we have lost Dirac's incisive insight: condition.[20,45]

for the phase of particle 1 is $\Delta\bar{S}^{(1)}_{I_G} = 2\pi n\hbar$, where#35

$$\Delta\bar{S}^{(1)}_{I_G} = \frac{4\pi}{c^2}\int d\omega_0(e_1 k^0_2 - g_1 j^0_2). \tag{120}$$

For point particles,#36 the result is

$$e_1 g_2 - e_2 g_1 = n\hbar c/2, \tag{121}$$

the appropriate generalization of the Dirac quantization condition.#37 The same considerations for $\Delta\bar{S}^{(2)}_{I_G}$ lead to the same equation, Eq. (121), for the relationship between the charges of the two particles.

#35 Though the underlying physics arguments differ from those of Cabibbo and Ferrari,[3] Eqs. (120) and (121) are the same as theirs, but with the generalization to particles of dual charge.

#36 To the extent that the individual particle sizes are significant relative to the bound state size, it appears that a correction factor will be necessary in Eq. (121) to account for finite-sized (and possibly overlapping) distributions. An estimate for this correction can be made by using the semiclassical angular momentum argument of Saha[47] and Wilson;[48] Eq. (121) would read $<\cos\theta>(e_1 g_2 - e_2 g_1) = n\hbar c/2$, where θ is the angle between a unit vector along a line joining incrementals of charge and a unit vector along the axis of (cylindrical) symmetry, and $<\cos\theta>$ is the average value of $\cos\theta$ as determined by the charge density distributions as weighting factors. In principle, if the physics were known, a proper calculation could be made using the Feynman path integral formalism.

#37 One recalls that Schwinger[49] obtained the quantization condition $e_1 g_2 - e_2 g_1 = n\hbar c$ for objects of dual charge; Zwanziger[50] obtained that same result using group theoretical considerations.

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FIGURE CAPTION

1. Depiction of the geometry of the action integrals associated with the interaction terms of the Lagrangian. Only the three spatial dimensions are indicated; the time variation along the path is given by $x^\mu = x^\mu(\tau)$. A typical point $x^\mu(\tau)$ on the equilibrium path (the heavier line) is shown. The path integral of Eq. (8) goes from point a to point b along the equilibrium path (in the direction opposite to the arrows). The integral around the closed loop, Eq. (17), associated with the variation of the action (of the interaction terms) goes in the direction indicated by the arrows along the displaced path (the lighter line) and back along the equilibrium path. As indicated, the displaced path is an arbitrary, infinitesimal distance δx^μ from the equilibrium path. The surface integral of Eq. (18) is over a surface that spans the closed loop, *i.e.* the integration path, of Eq. (17).

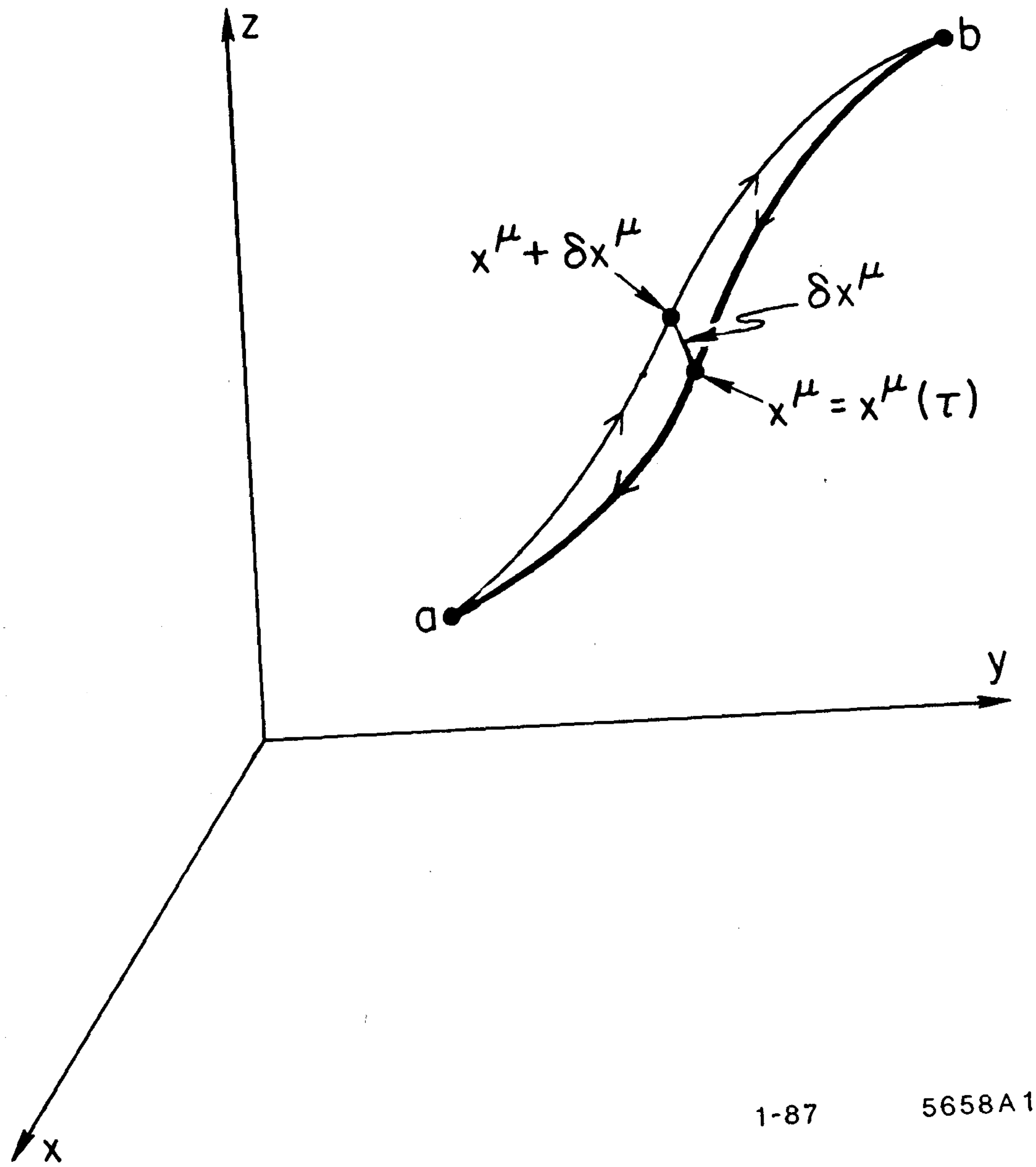

Fig. 1