

Modal Logics of Reactive Frames

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paper: DovPapers/306-TheoryGraphs/306logicsofreactiveframes.tex

Partial Draft
October 1, 2009

Abstract

A reactive graph generalizes the concept of a graph by making it dynamic, in the sense that the arrows coming out from a point depend on how we got there.

This idea was first applied to Kripke semantics of modal logic in [2]. In this paper we strengthen that unimodal language by adding a second operator. One operator corresponds to the dynamics relation and the other one relates paths with the same endpoint. We explore the expressivity of this interpretation by axiomatizing some natural subclasses of reactive frames.

The main objective of this paper is to present a methodology to study reactive logics using the existent classic techniques.

1 Introduction

Reactiveness. The reactive idea is very simple. Given a system with states and the possibility of transitions moving between states, we naturally can imagine a path beginning at an initial state and moving along the path following allowed transitions. If our starting point is s_0 , and the path is $s_0 \dots s_n$, then the system is an ordinary non-reactive system if the options available at s_n (i.e. which states t we can go to from s_n) do not depend on the path $s_0 \dots s_n$ (i.e. do not depend on how we got to s_n). Otherwise if there is such dependence then the system is *reactive*.

A simple example would be to consider as worlds the configurations on a chess board and the transitions the allowed moves. It is clear that this system is reactive in

*This work was partially supported by FCT and EU FEDER, namely via the KLog project PTDC/MAT/68723/2006 of SQIG-IT. Sérgio Marcelino also supported by FCT PhD fellowship SFRH/BD/27938/2006.

the above sense. To be able to castle one must not have moved either the the king or the rook, it is not enough that they are in their original positions. Moving the king or the rook corresponds to a higher order state transition, changing its nature.

It seems that the simple idea of taking existing systems and making them reactive in certain ways has many new applications. Indeed, there are applications of the reactive ideas in such diverse areas as modal logic, preferential non-monotonic logic, inheritance systems, context free grammars, automata theory, deontic logic and contrary to duty and argumentation and other networks, see papers [1, 2, 3, 4, 6, 7, 8, 9, 10, 11]. The list is increasing, see [13].

One can take a reactive system and turn it into an ordinary system by taking the new states as the paths. This is true but from the point of view of applications there is serious loss of information, as the applicability of the reactive system may come from the manner in which the change occurs along the path. In any specific application, the states have meaning, the transitions have meaning and the paths have meaning. Therefore the changes in the system as we go along a path can have very important meaning in the context, which enhances the usability of the model.

Reactive-switch graphs. The ideas and concepts involved are best explained through examples.

Figure 1 presents a simple transition system.

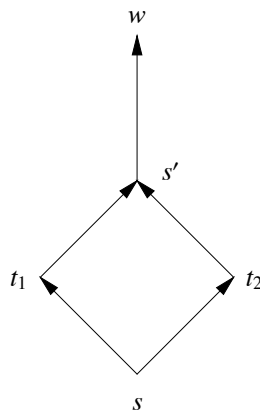


Figure 1:

We show only the transition options, without details of what causes the transitions. We have the transitions:

$$\begin{aligned}
 s &\rightarrow t_1 \\
 s &\rightarrow t_2 \\
 t_1 &\rightarrow s' \\
 t_2 &\rightarrow s' \\
 s' &\rightarrow w.
 \end{aligned}$$

The system is non-deterministic. Such systems are widely applicable in many areas and their exact meaning depends on the application area in which they are used.

It could be part of an automaton table (where the input letter causing the transition is not shown) or a set of context-free rules, or even part of a Kripke model where $x \rightarrow y$ means y is accessible from x .

A **reactive graph** (or reactive transition system) is a set of sequences of elements of a set, the **paths**, that contain all the sequences of length one and is closed under prefixes. It is clear it generalizes the static notion of a graph in which every possible extension of a path depends only on its endpoint (see proposition 3.12).

A system becomes reactive when the transition table changes as we move along the graph. We can make the above system reactive by, for example, saying that if we start in s , when we reach s' , it sees t' if and only if we pass through t_1 . A natural way of depicting this situation can be found in figure 2.

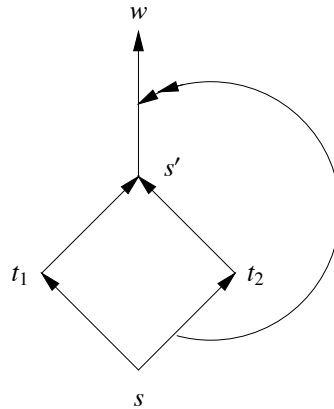


Figure 2:

The double arrow from the edge $s \rightarrow t_2$ to the edge $s' \rightarrow w$ indicates that if we go from s to t_2 then the connection $s' \rightarrow w$ is no longer available. The double arrow expresses the dependence on the path by codifying the transitions effects on the system configuration. We can make double arrows to act over other double arrows, getting higher and higher levels of dependence.

A **switch graph** (introduced in [2]) is such a graph enriched with higher order arrows, called the switches. This **local** way of presenting (generating) the relational dynamics, as it is proved in a paper in preparation, is enough to generate all reactive graphs. There are many possible variations on how to define it. In the mentioned result a switch is an arrow from an edge to another edge or a switch, and it may be a connecting or a disconnecting switch.

Reactive modal logic. As we observed above, a reactive system can be seen in a static point of view by considering their paths as the actual states.

Figure 3 shows the conversion of our example:

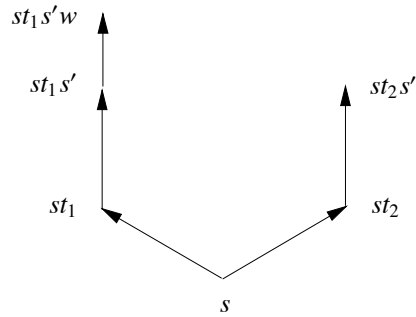


Figure 3:

The points in Figure 3 are the paths. So from the point st_1s' , there is a transition to $st_1s'w$ and from the point st_2s' there is no transition to $st_2s'w$. That the accessible points vary when in move is already a property of a static graph, what reactivity adds is the possibility of the accessible points from a point to be different depending on how we got there. In this sense the figure 3 is not a proper representation of Figure 2. The two paths st_1s' and st_2s' share an endpoint (i.e. same state s') and this can be important. So we need to indicate that by an equivalence relation Figure 4 does that, through the circle around the equivalent points, this is the unfolded version of Figure 2.

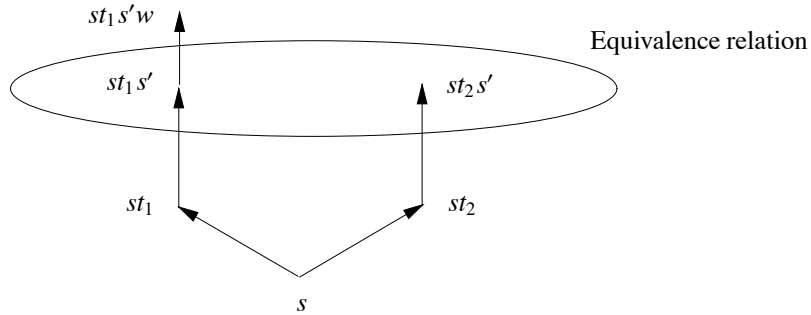


Figure 4:

We will introduce an interpretation of modal logic over reactive graphs (frames), in a way that generalizes the identification between Kripke frames and static graphs, taking in consideration these last remarks.

We were naturally led to the choice of a bimodal language, where \diamond_R deals, as usual, with the dynamics and \diamond_P with the relation that identifies all the paths with the same endpoint. As we stressed before, reactivity is about the change on the set of accessible points of a given point. \diamond_P ranges through the various relational states of each point. As we shall see, many reactive properties can be axiomatised by the interaction between \diamond_R and \diamond_P .

Like the accessibility relation, we let the value of (part of) the variables evolve

while we go through the graph. The subset X of variables which we consider fixed, possibly all or none of them, will be a parameter of our logics. So the value of the variables in X depends only on the position in the graph, i.e. the path endpoint. The inclusion of this feature in the models gives the results more generality, more possibility of application and allows us to understand better the influence of each component of the generalization in the results.

We can now ask the obvious questions

- what axioms on \Box_P, \Box_R correspond to this semantics?
- if we add the reflexivity or transitivity or other obvious axioms on \Box_R , what are the corresponding conditions on the models?
- which other properties can we express with this language?

As an answer we present a procedure to prove that, given a logic and some reactive properties, the first axiomatises the second. While soundness is proved directly, to prove completeness we will use the static view on reactivity by unfolding the models. This allows us to use the canonical model theorem and other classic techniques. When we unfold the notion of a reactive model we obtain a classic bimodal Kripke model, (W, R, P, V) (let R be the dynamics and P the equivalence relation relating the paths with the same end point), satisfying three additional properties:

- there is a family $I \subseteq W$ picking one element on each P -equivalence class that R -generates all graph;
- $xRy \ \& \ xRz \ \& \ yPz$ imply that $y \neq z$;
- the worlds related by P satisfy the same variables in X .

A **shattered** frame is a frame satisfying the first property and if it satisfies also the second we say it is a **coherently shattered (cs)** frame. The completeness proofs are done in two steps:

- we use the canonical model theorem to obtain completeness to a certain subclass of shattered frames corresponding to the reactive one (usually the first-order correspondent of the added axiom)
- we find a truth preserving model transformation that given shattered frame gives a *cs*-frame with that property.

The procedure proves successful in many cases (see in the end of the paper table 1) but we present also its limitations in the form of an open problem. We hope the techniques prove itself useful to characterize logics, possibly in extended languages, expressing reactivity properties coming from research areas where reactivity is being applied to, or to properties suggested by the switch graph research.

2 Reactive models

Definition 2.1 The modal language \mathcal{L}_r is defined by

$$\varphi = p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \diamond_R\varphi \mid \diamond_P\varphi.$$

where $p \in \Pi$ is a propositional variable. The other connectives: \top , \perp , \vee , \rightarrow , \leftrightarrow , \square_R and \square_P are introduced by the usual abbreviations.

Definition 2.2 A **path** over a set W is any finite sequence of points from W . A **prefix** of a path $w_0w_1 \dots w_n$ is any path of the form $w_0w_1 \dots w_k$ for $k \leq n$. Given a path $\lambda = w_0w_1 \dots w_n$, we let $t(\lambda) = w_n$ and n its length.

A **reactive frame** is a pair (W, Δ) , where W is a non-empty set and Δ is a set of paths over W that

- contains all one-element paths, i.e., W ,
- is closed under taking prefixes.

Given $X \subseteq \Pi$, a **X -reactive model** is a triple $\mathcal{M} = (W, \Delta, \nu)$, where (W, Δ) is a reactive frame and ν is a function $\nu : \Pi \rightarrow 2^\Delta$ such that for $p \in X$ and $\lambda w, \lambda' w \in \Delta$ we have $\lambda w \in \nu(p)$ iff $\lambda' w \in \nu(p)$. X corresponds to the subset of variables that we are fixed while we move, i.e., which value is determined by the current world.

Given a X -reactive model \mathcal{M} , for every $\lambda \in \Delta$ and every \mathcal{L}_r -formula φ , we define the notion ' **φ is true at λ in \mathcal{M}** ($\mathcal{M}, \lambda \models_X \varphi$)' inductively as follows:

- $\mathcal{M}, \lambda \models_X p$ iff $\lambda \in \nu(p)$ for variables p ,
- $\mathcal{M}, \lambda \models_X \neg\varphi$ iff $\mathcal{M}, \lambda \not\models_X \varphi$,
- $\mathcal{M}, \lambda \models_X \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}, \lambda \models_X \varphi_1$ and $\mathcal{M}, \lambda \models_X \varphi_2$,
- $\mathcal{M}, \lambda \models_X \diamond_R\varphi$ iff there is $w \in W$ such that $\lambda w \in \Delta$ and $\mathcal{M}, \lambda w \models_X \varphi$,
- $\mathcal{M}, \lambda \models_X \diamond_P\varphi$ iff there is $\gamma \in \Delta$ such that $t(\gamma) = t(\lambda)$ and $\mathcal{M}, \gamma \models_X \varphi$.

Notice that if $X = \Pi$, then ν can be seen as a function from the Π to 2^W and the first line of the definition of \models_X can be equivalently replaced by $\mathcal{M}, \lambda \models_\Pi p$ iff $t(\lambda) \in \nu(p)$.

We say that **φ is true in \mathcal{M}** iff $\mathcal{M}, \lambda \models_X \varphi$ for every $\lambda \in \Delta$. We say that **φ is X -valid in a reactive frame** if it is true in every X -reactive model over it.

When $X = \Pi$ we may omit X .

Definition 2.3 Given a reactive frame $F = (W, \Delta)$ we define $\sim_F \subseteq \Delta^2$ as:

$$\lambda \sim_F \gamma \text{ iff } t(\lambda) = t(\gamma) \ \& \ \forall \beta \in W^* \lambda\beta \in \Delta \leftrightarrow \gamma\beta \in \Delta.$$

We may omit F when it is clear from the context. We write $[\lambda]$ to refer to the equivalence class of $\lambda \in \Delta$.

Remark 2.4 We will be only interested in studying properties of X -reactive frames, that is, logics valid in the whole class of X -reactive models over reactive frames. Otherwise the notion of \sim could be adapted to the context of X -reactive models and many of the following results would be valid in its model version.

Proposition 2.5 Let (W, Δ) be a reactive frame.

1. If $\lambda \sim \gamma$ and $(W, \Delta, \nu), \lambda \models_{\Pi} \varphi$ then $(W, \Delta, \nu), \gamma \models_{\Pi} \varphi$.
2. If $\lambda, \gamma \in \Delta$ and $\lambda \not\sim \gamma$ then there exists φ and ν s.t. $(W, \Delta, \nu), \lambda \models_{\Pi} \varphi$ and $(W, \Delta, \nu), \gamma \models_{\Pi} \neg\varphi$.

Proof.

1. Let us prove it by induction on the structure of φ :
 - if φ is a variable this is trivial since $t(\lambda) = t(\gamma)$.
 - if $\varphi = \neg\psi$ then $\mathcal{M}, \lambda \not\models_{\Pi} \psi$ and so by IH and symmetry of \sim , $\mathcal{M}, \gamma \not\models_{\Pi} \psi$ thus $\mathcal{M}, \gamma \models_{\Pi} \neg\psi = \varphi$. The $\varphi = \varphi_1 \wedge \varphi_2$ case is trivial.
 - if $\varphi = \diamond_R \psi$ then it exists $w \in W$ s.t. $\lambda w \in \Delta$ and $\mathcal{M}, \lambda w \models_{\Pi} \psi$. It is clear that we also have $\lambda w \sim \gamma w \in \Delta$ and so by I.H. $\mathcal{M}, \gamma w \models_{\Pi} \psi$ hence $\mathcal{M}, \lambda \models_{\Pi} \diamond_R \psi = \varphi$.
 - the case $\varphi = \diamond_P \psi$ also comes from $t(\lambda) = t(\gamma)$.
2. If $t(\lambda) = t(\gamma)$ and exists $\beta = w_1 \dots w_n$ s.t. we do not have $\lambda\beta \in \Delta$ iff $\gamma\beta \in \Delta$. Without loss of generality let us assume $\lambda\beta \in \Delta$ and $\gamma\beta \notin \Delta$ and pick a valuation ν that distinguishes all w_i , i.e., let $p_i \in \Pi$ with $1 \leq i \leq n$ and $\alpha \in \nu(p_i)$ iff $t(\alpha) = w_i$. Let $\varphi = \diamond_R \varphi_\beta$ where φ_β is defined recursively by:

$$\begin{aligned}\varphi_{w_n} &= p_n \\ \varphi_{w_i \lambda} &= p_i \wedge \diamond_R \varphi_\lambda.\end{aligned}$$

It is clear that $(W, \Delta, \nu), \lambda \models_{\Pi} \varphi$ but $(W, \Delta, \nu), \gamma \not\models_{\Pi} \varphi$.
If $t(\lambda) \neq t(\gamma)$ we pick a valuation that distinguishes them. ■

Definition 2.6 $L_{r,X}$ is the logic of all reactive frames:

$$L_{r,X} = \{\varphi : \varphi \text{ is } X\text{-valid in every reactive frame}\}.$$

2.1 Reactiveness unfolded

Definition 2.7 1. A **shattered frame** is a bimodal frame (W, R, P) such that P is an equivalence relation over W . Given a shattered frame, we say $I \subseteq W$ is an **initial family** if it picks one element from each P -class that R -generates the whole frame.

2. A **cs-frame** (coherently shattered) is a shattered frame (W, R, P) that admits an initial family and that is **coherent**, i.e., such that for all $w, w', w'' \in W$, if wRw' , wRw'' and $w'Pw''$ then $w' = w''$.
3. A **X-shattered model** is a Kripke model $\mathcal{M} = (W, R, P, V)$ over a shattered frame (W, R, P) and V is **X-admissible**, i.e., for all $w, w' \in W$, if wPw' then

$$w \in V(p) \text{ iff } w' \in V(p) \text{ for all } p \in X.$$

We say φ is **X-valid** in a shattered frame if it is true in every X-shattered model over it.

4. An **X-cs-model** is a X-shattered model over a cs-frame.

Notice that in order for the restrictions on the valuations in a Π -shattered model (W, R, P, V) to correspond to the restrictions on a general frame (W, R, P, A) , where A is the boolean algebra generated by the equivalence classes of P , A has to be closed under m_R (and m_P). As we shall see in proposition 3.12 the *cs*-frames that satisfy this requirement are the ones coming from unfolding static reactive frames. But in this case there is no restriction to deal with!

Definition 2.8 $L_{cs,X}$ is the logic of *cs*-frames:

$$L_{cs,X} = \{\varphi : \varphi \text{ is } X\text{-valid in every cs-frame}\}.$$

Remark 2.9 It is straightforward to see that every reactive frame $F = (W, \Delta)$ can be regarded as *cs*-frame $F_{cs} = (\Delta, R^\Delta, P^\Delta)$ where W is an initial family and

- $\lambda R^\Delta \gamma$ iff there is some $w \in W$ such that $\gamma = \lambda w$,
- $\lambda P^\Delta \gamma$ iff $t(\lambda) = t(\gamma)$.

We call it the **unfolding** of F . Furthermore it is easy to see that there is a bijective correspondence between X -reactive models $\mathcal{M} = (W, \Delta, \nu)$ and X -*cs*-models $\mathcal{M}_{cs} = (\Delta, R^\Delta, P^\Delta, \nu)$ that preserves truthness, i.e., for every $\lambda \in \Delta$ and every \mathcal{L}_r -formula φ ,

$$\mathcal{M}, \lambda \models_X \varphi \quad \text{iff} \quad \mathcal{M}_{cs}, \lambda \models_X \varphi,$$

and so we have $L_{cs,X} \subseteq L_{r,X}$.

The converse is also true.

Proposition 2.10 *Let (W, R, P) *cs*-frame with I as initial family. There is a reactive frame s.t. there is bijective correspondence between X -*cs*-model over the first and X -reactive models over the second that preserves truthness.*

Proof. We call an R -path $\lambda = w_0 \dots w_n$ an **I -initial path** if $w_0 \in I$. For every such I -initial path λ , let

$$l_I(\lambda) = i_0 \dots i_n$$

for the unique sequence of $i_j \in I$ with $w_j P i_j$ for $1 \leq j \leq n$ (see Def. 2.7).

Moreover l_I is an injective function from the set of all \mathcal{M} -initial paths into paths over I , that is, if $l_I(\lambda) = l_I(\gamma)$ then $\lambda = \gamma$. Indeed, as l clearly preserves the length of a path, we can do induction on the length n of $\lambda = w_0 \dots w_n$, $\gamma = v_0 \dots v_n$. If $n = 0$ then $w_0 = l_I(w_0) = l_I(v_0) = v_0$ follows. Now suppose that $l_I(w_0 \dots w_n w_{n+1}) = l_I(v_0 \dots v_n v_{n+1}) = i_1 \dots i_n i_{n+1}$ for some $i_j \in I$, $1 \leq j \leq n+1$. Then $l_I(w_0 \dots w_n) = l_I(v_0 \dots v_n) = i_1 \dots i_n$ and $w_{n+1} P i_{n+1} P v_{n+1}$. By the IH, we have $w_j = v_j$ for $1 \leq j \leq n$. Therefore not only $w_n R w_{n+1}$, but also $w_n R v_{n+1}$. So $w_{n+1} = v_{n+1}$ follows by coherence.

Let $\Delta^I = \{l_I(\lambda) : \lambda \text{ is an } I\text{-initial path}\}$, (I, Δ^I) is clearly a reactive frame, we call it a **folding** of (W, R, P) .

Given a X -cs-model $\mathcal{M} = (W, R, P, V)$ (with I as an initial family) let $\mathcal{M}' = (I, \Delta^I, \nu)$ with

$$\nu(p) = \{l_I(\lambda) \in \Delta^I : t(\lambda) \in V(p)\}, \text{ for each variable } p.$$

It is straightforward to see that \mathcal{M}' is a X -reactive model. We will show that $\mathcal{M} \models_X \varphi$ iff $\mathcal{M}' \models_X \varphi$.

According to Remark 2.9 we can regard \mathcal{M}' as a X -cs-model \mathcal{M}'_{cs} preserving truthness. Hence, to conclude, it is enough to prove that \mathcal{M} is a bounded morphic image of \mathcal{M}'_{cs} :

We define a function $f : \Delta^I \rightarrow W$ by taking

$$f(l_I(\lambda)) = t(\lambda).$$

which is well defined since l_I is injective. We claim that f is a surjective bounded morphism from \mathcal{M}'_{cs} onto \mathcal{M} :

- f is surjective since by Def. 2.7(1) we have that I R -generates all the frame
- p -morphism in R . First, if $l_I(\lambda), l_I(\lambda)i \in \Delta^I$ then there is some $w \in W$ such that $t(\lambda)Rw$ and $l_I(\lambda)i = l_I(\lambda w)$. So $t(\lambda)Rw = f(l_I(\lambda)i)$. Second, if $f(l_I(\lambda)) = t(\lambda)Rw$ then $l_I(\lambda w) \in \Delta^I$ and $f(l_I(\lambda w)) = w$.
- p -morphism in P . First, if $t(l_I(\lambda)) = t(l_I(\gamma))$ then $t(\lambda)Pt(\gamma)$. Second, if $f(l_I(\lambda)) = t(\lambda)Pw$ then, by Def. 2.7(1), there is some \mathcal{M} -initial path γ such that $t(\gamma) = w$. Then $l_I(\gamma) \in \Delta^I$ and $t(l_I(\lambda)) = t(l_I(\gamma))$ follows.
- p -morphism in V : $l_I(\lambda) \in \nu$ iff $t(\lambda) = f(l_I(\lambda)) \in V(p)$.

■

Corollary 2.11 $L_{cs,X} = L_{r,X}$.

Proof. From remark 2.9 we get one direction. For the other, suppose that $\varphi \notin L_{cs,X}$, that is, there is some X -cs-model $\mathcal{M} = (W, R, P, V)$ having I as initial family and such that $\mathcal{M} \not\models_X \varphi$. \mathcal{M}' is a X -reactive model and $\mathcal{M}' \not\models \varphi$ follows from the previous proposition.

■

Given a reactive frame we can obtain a cs-frame by unfolding (and vice-versa by folding) where each X -model over the first corresponds to a X -model over the second and modal satisfaction is preserved.

With shattered frames we are back in the classic setting even if, in the case of $X \neq \emptyset$, we are not at the frame level. There are restrictions on the models over the correspondent Kripke frames, the situation is closer to the case of general frames.

Folding and unfolding will be the bridge from reactivity to this classical setting and will allow us to use some known techniques in the study the axiomatisation of logics over reactive frames. The most immediate consequence of this connection is that the logics formed by formulas valid over reactive frames and cs-frames coincide.

3 Axiomatizations

In this section we will deal with axiomatizing logics over some classes of reactive frames.

Definition 3.1 Let $L \cup L' \cup \{\varphi\}$ be a set of formulas.

- $L \oplus L'$ is the closure by the rules of modus ponens and necessity of the set

$$\{\psi : \psi \text{ is a propositional instance of } \varphi \in L \cup L'\}.$$

If L, L' have only one element we may omit the curly brackets.

- $L +_X \varphi$ is the closure by the rules of modus ponens and necessity of the set

$$L \cup \{\psi : \psi \text{ is the result of substituting the variables on } \varphi \text{ by variables in } X\}.$$

Let

$$L_X = K_R \oplus S5_P +_X p \leftrightarrow \Box_P p,$$

meaning that L_X is the closure by the rules of modus ponens and necessity (for \Box_R and \Box_P) of the set containing

$$p \leftrightarrow \Box_P p$$

for every propositional variable $p \in X$, the substitution instances of all propositional tautologies and of the following axioms:

1. $\Box_R(p \rightarrow q) \rightarrow (\Box_R p \rightarrow \Box_R q)$
2. $\Box_P(p \rightarrow q) \rightarrow (\Box_P p \rightarrow \Box_P q)$
3. $\Box_P p \rightarrow p$
4. $\Box_P p \rightarrow \Box_P \Box_P p$
5. $p \rightarrow \Box_P \Diamond_P p$

It is clear that $L_X \subseteq L_{cs,X} = L_{r,X}$ since all the L_X axioms are sound in respect to shattered frames. Notice that the \diamond_R fragment of L_X is just K_R , i.e. K . It is obvious that $K \subseteq L_X$ and from the fact that every Kripke model is a trivial X -reactive model we get the other inclusion.

We want to show that L_X is complete with respect to all reactive frames. For that we will prove that L_X 's canonical model is a X -shattered model and then prove that any such model is the bounded image of a generated subframe of a X -cs-frame. Hence concluding that $L_X = L_{cs,X} = L_{r,X}$.

If $X \neq \emptyset$ then L_X is not closed under structural substitution. One can define the canonical model of a normal logic L the usual way, $\mathcal{M}_L = (W^L, R^L, P^L, V^L)$ where:

$$\begin{aligned} W^L &= \{s : s \text{ is } L\text{-MCS}\} \\ sR^L t &\text{ iff } \{\varphi : \Box_R \varphi \in s\} \subseteq t \text{ iff } \{\Diamond_R \varphi : \varphi \in r\} \subseteq s \\ sP^L t &\text{ iff } \{\varphi : \Box_P \varphi \in s\} \subseteq t \text{ iff } \{\Diamond_P \varphi : \varphi \in r\} \subseteq s \\ V^L(p) &= \{s \in W^L : p \in s\}. \end{aligned}$$

and prove the well-known truth lemma about it:

$$\{\varphi : \mathcal{M}_{L_X} \models \varphi\} = L_X$$

even if the logic in question is not closed under the rule of substitution, see e.g. [14].

Proposition 3.2 *If a logic L contains $p \leftrightarrow \Box_P p$ and for $p \in X \subseteq \Pi$ then given $vP^{M_L} w$ we have that:*

$$v \in V^{M_L}(p) \Rightarrow w \in V^{M_L}(p) \text{ for all } p \in X.$$

Proof. If $sP^{M_L} t$ and $p \in s$ then $\Box_P p \in s$ and so $p \in t$. ■

Clearly, if P^{M_L} is a equivalence class then all the worlds related by P^{M_L} satisfy the same variables in X .

Corollary 3.3 \mathcal{M}_{L_X} is X -shattered model and so

$$\{\varphi : \varphi \text{ is } X\text{-valid in every shattered frame}\} = L_X.$$

In other words we have that L_X is sound and complete with respect to the class of X -models over shattered frames. Next, by showing that is also complete with respect to its subclass of cs-frames we conclude the axioms generating L_X axiomatise the minimal logic over reactive frames.

Theorem 3.4 $L_X = L_{r,X}$

Proof. Since $L_{r,X} = L_{cs,X}$ it is equivalent to prove that $L_X = L_{cs,X}$.

Every cs-frame is also a shattered frame, hence we have that $L_{cs,X} \subseteq L_X$.

Let $\varphi \notin L_X$ then exists some X -shattered model $\mathcal{M} = (W, R, P, V)$ that does not satisfy φ .

We define $\mathcal{N} = (W', R', P', V')$, where:

$$\begin{aligned} W' &= \{(z, y) : yPz\} \cup W \times \{*\} \\ (x, i)R'(y, j) &\text{ iff } i = * \ \& \ x = y \text{ or } i \neq * \ \& \ xRy \ \& \ y = j \\ (x, i)P'(y, j) &\text{ iff } i = j \ \& \ (i = * \rightarrow x = y) \\ (x, i) \in V'(p) &\text{ iff } x \in V(p). \end{aligned}$$

\mathcal{N} is a X -cs-model and \mathcal{M} is a bounded morphic image of the submodel generated by $W'' = W' - W \times \{*\}$ (and therefore $\varphi \notin L_{cs,X}$):

- let I be a any family of elements of W' picking one element from each P -class. It is clear that $W \times \{*\} \subseteq I$ and so I generates the \mathcal{N} . V' is an X -admissible valuation: $(x, i)P'(y, j)$ imply $i = j$ hence xPy and therefore if $p \in X$,

$$(x, i) \in V'(p) \text{ iff } z \in V(p) \text{ iff } y \in V(p) \text{ iff } (y, j) \in V'(p).$$

Let us check that it is coherent. Given $(x, i), (y, j), (z, k) \in W'$. If $(y, j)P'(z, k)$ then $j = k$. Let us assume also $(x, i)R'(y, j)$ and $(x, i)R'(z, k)$. If $i = *$ then $x = y = z$ and $i \neq *$ then $y = j = k = z$. So $(y, j) = (z, k)$.

- let $f : W'' \rightarrow W$ be defined by: $f((x, i)) = x$.

It is straightforward to see that f is surjective, let us see that it is actually a p-morphism:

- f is a p-morphism in R
 - * $(x, i)R'(y, j)$ then $f((x, i)) = xRy = f((y, j))$.
 - * $f((x, i))Ry$ then $(x, i)R'(y, y)$.
- f is a p-morphism in P
 - * $(x, i)P'(y, j)$ then $i = j$ and so, by construction, we have $f((x, i)) = xPy = f((y, j))$.
 - * $f((x, i))Py$ then $(x, i)P'(y, i)$
- f is a p-morphism in V
 - $(x, y) \in V'(p)$ iff $f((x, i)) = x \in V(p)$.

To this way of generating a X -cs-model from a given X -shattered model we will call the *blow up trick* and it is inspired by the standard extensions used in [15]. ■

If we consider the basic properties in graphs like reflexivity, symmetry or transitivity, we see that there are many ways of generalising it to the reactive graph level. This properties refer to the accessibility of points without referring to the changes in their relational state since it is always the same. In the reactive case this is not true and

they may mean different things. Reflexivity can mean for example that we can always access loop without any change on the set of accessible points or we may require anything else. All the mentioned properties have similar variants, they are the *generalized notions of the originals*.

Just as the (static) graphs properties are studied in classic modal logic, we dedicate the rest of the paper to study the logics of the subclasses of reactive frames satisfying some of these properties.

Our strategy will be the same as we used for the general case. Given a class of reactive frames and set of candidate axioms, we first check if they correspond to the reactive frame property in question (in some cases we only have soundness). Then we prove completeness in steps:

- we establish that the logic they originate is complete with respect to class of shattered frames with a certain property by analysing its canonical model;
- we check that its subclass of cs-frames validates exactly the same formulas by showing the existence of a transformation from shattered frame to cs-frames (using the blow up trick) that preserves the class we are considering and showing the first one is a bounded morphic image of a generated submodel of the second;
- we show that folding these cs-frames gives origin to reactive frames with the required property, thus obtaining the result by applying of proposition 2.10 like in corollary 2.11.

3.1 Reflexivity and transitivity

Now we will study some subclasses of reactive frames obtained by imposing properties that generalize the notions of reflexivity and transitivity in the static case and axiomatise them.

Let us introduce some variations of the blow up trick that will work for all the next cases.

Proposition 3.5 Given φ and some X -shattered model $\mathcal{M} = (W, R, P, V)$ we define $t_i(\mathcal{M}) = (W_i, R_i, P_i, V_i)$ for $i = 1, 2, 3, 4$:

1.
 - $W_1 = \{(x, (y, 0)), (x, (y, 1)) : xPy\} \cup W \times \{*\}$
 - $(x, i)R_1(y, j)$ iff $i = * \ \& \ x = y$ or
 $(i = (x', a) \ \& \ j = (y', b) \ \& \ (b = |a - 1| \ \& \ y = t'(x, y') \ \text{or} \ i = j \ \& \ y = t'(x, x)))$
 - $(x, i)P_1(y, j)$ iff $i = j \ \& \ (i = 1 \rightarrow x = y)$
 - $(z, i) \in V_1(p)$ iff $z \in V(p)$.
2.
 - $W_2 = \{(t^n(c), n, c, d, i) : cPd; i = 0, 1, *, n < \omega\}$
 - $(a, b, c, d, i)R_2(a', b', c', d', i')$ iff $(c, d) = (c', d') \ \& \ (i = * \ \& \ (a, b) = (a', b') \ \text{or} \ i \neq * \ \& \ aRa' \ \& \ (a = t(a') \ \& \ b = b' + 1 \ \& \ i = i' \ \text{or} \ a' = c' = d' \ \& \ b' = 0 \ \& \ i = |i' - 1|))$

- $w = (a, b, c, d, i)P_2(a', b', c', d', i') = w'$ iff $(d, i) = (d', i')$ & $(i = * \rightarrow w = w')$
 - $(a, b, c, d, i) \in V_2(p)$ iff $a \in V(p)$
- $(t^0(w) = w)$
- 3.
- $W_3 = \{(x, y, n), (x, n) : xPy, n < \omega\}$
 - $(x, i)R_3(y, j)$ iff $y = (y_1, y_2)$ & $(x \in W \ \& \ (x = y_1 \ \& \ i = j \ \text{or} \ j > i \ \& \ y_1 = y_2) \ \text{or} \ x = (x_1, x_2) \ \& \ j > i \ \& \ x_1Ry_1 = y_2)$
 - $(x, i)P_3(y, j)$ iff $i = j \ \& \ (x \in W \ \& \ x = y \ \text{or} \ x = (x_1, w) \ \& \ y = (y_1, w))$
 - $(x, i) \in V_3(p)$ iff $x \in W \ \& \ x \in V(P) \ \text{or} \ x = (x_1, x_2) \ \& \ x_1 \in V(p)$.
- 4.
- $W_4 = W_3, P_4 = P_3 \ \& \ V_4 = V_3$
 - $(x, i)R_4(y, j)$ iff $y = (y_1, y_2)$ & $(x \in W \ \& \ (x = y_1 \ \& \ i = j \ \text{or} \ j > i \ \& \ y_1 = y_2) \ \text{or} \ x = (x_1, x_2) \ \& \ j > i \ \& \ t'(x_1, y_2) = y_1)$

where t, t' are defined by

- $t(w) = v$ be s.t. $vRwPv$ if there is such v , otherwise it is undefined,
- $t'(w, v) = v'$, s.t. $wRv'Pv$ and $wRv \rightarrow v' = v$ if there is such a v' , and is undefined otherwise.

$t_i(\mathcal{M})$ is a X -cs-model and \mathcal{M} is a bounded morphic image of a generated submodel of $t_i(\mathcal{M})$ for $i = 1, 2, 3, 4$.

Remark 3.6 Any of these constructions could have been used in theorem 3.4 instead of \mathcal{N} .

Proof.

1. $t_1(\mathcal{M})$ is a X -cs-model and \mathcal{M} is a bounded morphic image of the submodel of $t_1(\mathcal{M})$ generated by $W'_1 = \{(x, (y, 0)), (x, (y, 1)) : xPy\}$:

- the existence of an initial family and that V_1 is an X -admissible valuation is dealt just like in prop. 3.4.
Let us check that it is coherent. Given $(x, i), (y, j), (z, k) \in W_1$. If $(y, j)P_1(z, k)$ then $j = k$. Let us assume that we also have $(x, i)R_1(y, j)$ and $(x, i)R_1(z, k)$. If $i = *$ we get $x = y = z$. If $i = (z, a)$ then $j = (w, b) = k$. If $a = b$ then $y = t(x, x) = z$ and if $a \neq b$ then $y = t(x, w) = z$. In any case $(y, j) = (z, k)$.
- let $f : W'_1 \rightarrow W$ be defined by: $f((x, i)) = x$. It is straightforward to see that it is a surjective function, let us see that it is actually a p -morphism:
 - f is a p -morphism in R
 - * $(x, (x', i))R_1(y, (y', j))$ then or $y = t'(x, y')$ or $y = t'(x, x)$. In both cases $f((x, i)) = xRy = f((y, j))$,

- * $f((x, (t, a)))Ry$ then we have that $t'(x, y) = y$,
hence $(x, (t, a))R_1(y, (y, |a - 1|))$;
 - f is a p -morphism in P
 - * $(x, i)P_1(y, j)$ then $i = j$ and so $f((x, i)) = xPy = f((y, j))$,
 - * $f((x, i))Py$ then $(x, i)P_1(y, i)$;
 - f is a p -morphism in V , is trivial since by definition $(x, i) \in V_1(p)$ iff $f((x, i)) = x \in V$.
2. $t_2(\mathcal{M})$ is a X - cs -model and \mathcal{M} is a bounded morphic image of the submodel of $t_2(\mathcal{M})$ generated by $W'_2 = \{(t^n(c), n, c, d, i) : cPd; i = 0, 1; n < \omega\}$:
- it is easy to verify that V_2 is admissible and that any choice of I , picking one element from each P -class and containing $\{(t^n(c), n, c, d, *) : cPd; n < \omega\}$, works as initial family.
Let us check that it is coherent. Given $w = (a, b, c, d, i), w' = (a', b', c', d', i')$ and $w'' = (a'', b'', c'', d'', i'')$ in W_2 . If $w'Pw''$ then $(d', i') = (d'', i'')$. Let us assume that we also have wR_2w' and wR_2w'' so $(c', d') = (c, d) = (c'', d'')$. If $i = *$ we get $(a, b) = (a', b') = (a'', b'')$. Otherwise, if either $i = i'$ or $i = i''$ then $i = i' = i''$, so $b' = b - 1 = b''$ and $a = t^{b'}(c) = t^{b''}(c) = a''$. If $i \neq i'$ then $b = b' = 0, a' = c' = d' = d'' = c'' = a''$. Hence, in any case, $w' = w''$.
 - let $f : W'_2 \rightarrow W$ be defined by $f((a, b, c, d, i)) = a$.
It is straightforward to see that it is a surjective function, let us see that it is actually a p -morphism:
The condition in the valuation is trivial as before;
 - f is a p -morphism in R
 - * $w = (a, b, c, d, i)R_2(a', b', c', d', i') = w'$ then $f((x, i)) = aRb = f((y, j))$,
 - * $f((a, b, c, d, i))Ry$ then or $(a, b, c, d, i)R_2(y, 0, y, y, |i - 1|)$;
 - f is a p -morphism in P
 - * $w = (a, b, c, d, i)P_2(a', b', c', d', i') = w'$ then $aPcPd = d'Pc'Pa'$.
Thus, by transitivity of P , we obtain $f(w) = aPa' = f(w')$.
 - * $f((a, b, c, d, i))Py$ then $(a, b, c, d, i)P_2(y, 0, y, d, i)$.
3. $t_3(\mathcal{M})$ is a X - cs -model and \mathcal{M} is a bounded morphic image of the submodel of $t_3(\mathcal{M})$ generated by $W'_3 = \{(x, y), n) : xPy, n < \omega\}$:
- the existence of an initial family and the admissibility of V_3 are dealt as before (in this case the generator is $\{(x, n) : n < \omega\}$).
Let us check that it satisfies coherence. Given $(x, i), (y, j), (z, k) \in W_3$. Let us assume that we have $(x, i)R_3(y, j), (x, i)R_3(z, k)$ and $(y, j)P_3(z, k)$ then $y = (y_1, w), z = (z_1, w)$ and $j = k$. So, or $i = j = k$, in which case, $x \in W$ and $y_1 = x = z_1$; or $i < j = k$ and $y_1 = w = z_1$.

- let $f : W'_3 \rightarrow W$ be defined by: $f((x, y), i) = x$.
It is straightforward to see that it is a function let us see that it is also a p -morphism (the condition in V is dealt as before):
 - f is a p -morphism in R
 - * $((x_1, x_2), i)R_3((y_1, y_2), j)$ then

$$f(((x_1, x_2), i)) = x_1Ry_1 = f(((y_1, y_2), j)),$$
 - * $f(((x_1, x_2), i))Ry$ then $((x_1, x_2), i)R_3((y, y), j)$ for any $j > i$;
 - f is a p -morphism in P
 - * $((x_1, x_2), i)P_3((y_1, y_2), j)$ then

$$f(((x_1, x_2), i)) = x_1Px_2 = y_2Py_1 = f(((y_1, y_2), j)),$$
 - * $f((x_1, x_2), i)Py$ then $((x_1, x_2), i)P_3((y, x_2), i)$.

4. Let us consider the morphism as in 3., we just need to check the conditions involving R_4 :

- *initiality is dealt as in 3.*
Coherence: Let us assume that we have $(x, i)R_4(y, j)$, $(x, i)R_4(z, k)$ and $(y, j)P_4(z, k)$. Then $y = (y_1, w)$, $z = (z_1, w)$ and $j = k$. If $x \in W$ then, if $i = j = k$ then $x = y_1 = z_1$ and if $i < j = k$ then $y_1 = w = z_1$. Otherwise $x = (x_1, x_2)$ and $y_1 = t(x_1, w) = z_1$.
- $((x_1, x_2), i)R_4((y_1, y_2), j)$ then $y_1 = t'(x_1, y_2)$ hence

$$f(((x_1, x_2), i)) = x_1Ry_1 = f(((y_1, y_2), j)),$$
- $f(((x_1, x_2), i))Ry$ then $t'(x_1, y) = y$ thus $((x_1, x_2), i)R_4((y, y), j)$ for any $j > i$. ■

We start by analysing the subclasses of reactive frames (or models) that correspond to the usual axioms for reflexivity and transitivity (with one operator).

Let us prove a lemma that will be useful throughout the paper:

Lemma 3.7 Given a cs -frame (W, R, P) admitting as initial family I . Let γs and $\gamma' s$ be two I -initial paths,

$$l_I(\gamma s) \sim_{(I, \Delta_I)} l_I(\gamma' s).$$

Proof. $l_I(\gamma s)\alpha \in \Delta_I$ iff there is a β s.t. $\gamma s\beta$ is an I -initial path (iff $\gamma' s\beta$ is an I -initial path) and $l_I(\beta) = \alpha$ iff $l_I(\gamma' s)\alpha \in \Delta_I$. Since clearly $t(l_I(\gamma s)) = l_I(s) = t(l_I(\gamma' s))$ we get $l_I(\gamma s) \sim l_I(\gamma' s)$. ■

Proposition 3.8 1. A reactive frame (W, Δ) (II)-validates $p \rightarrow \diamond_{RP}$ iff it is **strongly reflexive**, i.e. satisfies:

$$\lambda w \in \Delta \rightarrow \lambda w \sim \lambda ww \in \Delta.$$

Let $L_X^T = L_X \oplus p \rightarrow \diamond_{RP}$.

2. L_{Π}^T is sound and complete with respect to the class of (all Π -reactive models over) strongly reflexive frames.
3. If $X \neq \Pi$, L_X^T is not sound and complete with respect to (all X -models over) any class of reactive frames.

Proof.

1.

- Given a strongly reflexive Π -reactive model $\mathcal{M} = (W, \Delta, \nu)$ and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models_{\Pi} \varphi$. Thus $\lambda w \sim \lambda w w$ and, by proposition 2.5, $\mathcal{M}, \lambda w w \models_{\Pi} \varphi$ and so $\mathcal{M}, \lambda w \models_{\Pi} \diamond_R \varphi$.
- Given a non strongly reactive frame (W, Δ) . There exists some $\lambda w \in \Delta$ s.t. $\lambda w \not\sim \lambda w w$, so we have two cases.
Or $\lambda w w \notin \Delta$ so if we take $p \in \Pi$ and pick ν s.t. $\lambda w \in \nu(p)$ and $\lambda w w' \notin \nu(p)$ for all $\lambda w w' \in \Delta$ in which case

$$(W, \Delta, \nu), \lambda w \models p \wedge \neg \diamond_R p.$$

Or else, exists $\beta = w_1 \dots w_n$ s.t. we don't have $\lambda w \beta \in \Delta \leftrightarrow \lambda w w \beta \in \Delta$. We consider $w_0 = t(\lambda) = t(\gamma), p_0, \dots, p_n$ and pick ν s.t. $\alpha \in \nu(p_i)$ iff $t(\alpha) = w_i$. Let $\varphi = \diamond_R \varphi_{\beta}$ be as defined in proposition (2.5.2). If $\lambda w \beta \in \Delta$ but $\lambda w w \beta \notin \Delta$ then λw satisfies $\psi = (p_0 \wedge \varphi) \vee \neg p_0$ and $\lambda w w$ does not. If $\lambda w \beta \notin \Delta$ but $\lambda w w \beta \in \Delta$ then λw satisfies $\neg \psi = \neg(p_0 \wedge \varphi) \wedge p_0$ and $\lambda w w$ does not. Since no $\lambda w w' \in \Delta$ with $w' \neq w$ satisfies p_0 and both ψ and $\neg \psi$ imply p_0 (which is only satisfied at paths with w as end point), we know that either $(W, \Delta, \nu), \lambda w \not\models \psi \rightarrow \diamond_R \psi$ or $(W, \Delta, \nu), \lambda w \not\models \neg \psi \rightarrow \diamond_R \neg \psi$

Imposing the usual axiom for reflexivity forces a very strong notion of reflexivity in reactive frames. Strong reflexivity imposes that, no matter which path we have covered, we can always loop without any change to the accessible worlds.

2. Soundness of L_{Π}^T comes easily from the soundness of the new axiom which has just been established.

Let us prove that $L_{\Pi}^T = \{\varphi : \varphi \text{ is valid in every strongly reflexive reactive frame}\}$:

- using lemma (3.2) it is easy to check that the canonical model for L_{Π}^T is a Π -shattered model (W, R, P, V) where R reflexive. Since every cs-frame is also a shattered frame we have that:

$$\begin{aligned} L_{\Pi}^T &= \{\varphi : \varphi \text{ is true in every } \Pi\text{-shattered model } (W, R, P, \nu) \text{ where } R \text{ reflexive}\} \\ &\subseteq \{\varphi : \varphi \text{ is true in every } \Pi\text{-cs model } (W, R, P, \nu) \text{ where } R \text{ reflexive}\}. \end{aligned}$$

- t_1 preserves R -reflexivity:

Given (x, i) the case $i = *$ is trivial and if $i = (x', a)$ we have

$$(x, (x', a))R_2(t'(x, x), (x', a)) = (x, (x', a))$$

since $t'(x, x) = x$. Hence

$$L_{\Pi}^T = \{\varphi : \varphi \text{ is true in every } \Pi\text{-cs model } (W, R, P, \nu) \text{ where } R \text{ reflexive}\}.$$

- given a R -reflexive cs -frame $F = (W, R, P)$ with initial family I , (I, Δ^I) is a strongly reflexive reactive frame:

Let $\lambda w \in \Delta^I$, so there is $\gamma \in W^*$ and $s \in W$ s.t. γs is an I -initial path and $l_I(\gamma s) = \lambda w$. Since F is R -reflexive we have that $\gamma s s$ is also an I -initial path and clearly $l_I(\gamma s s) = \lambda w w \in \Delta$. Applying lemma 3.7 we conclude that $\lambda w \sim \lambda w w$.

3. Let $X \subsetneq \Pi$ and a class of reactive frames \mathcal{F} .

If there is a reactive frame $F = (W, \Delta) \in \mathcal{F}$ then let $\lambda \in \Delta$ and $p \in \Pi - X$. We pick v s.t. $\lambda \in v(p)$ and $\lambda w \notin v(p)$ for all $\lambda w \in \Delta$. So

$$(W, \Delta, v), \lambda w \models p \wedge \neg \diamond_R p.$$

If \mathcal{F} is the empty class it validates \perp . Since there are Π -reactive models over strongly reflexive frames and $L_X^T \subseteq L_\Pi^T$ we conclude that

$$L_X^T \subseteq \{\varphi : \varphi \text{ is valid in all } F \in \mathcal{F}\} \ni \perp.$$

■

Notice that to have reactive frame completeness we have to impose that $X = \Pi$. This will also happen in the next case.

Proposition 3.9 1. A reactive frame (W, Δ) (Π -)validates $\diamond_R \diamond_R p \rightarrow \diamond_R p$ iff it is strongly transitive, i.e. satisfies:

$$\lambda w w' w'' \in \Delta \rightarrow \lambda w w' w'' \sim \lambda w w'' \in \Delta$$

Let $L_X^4 = L_X \oplus \diamond_R \diamond_R p \rightarrow \diamond_R p$.

2. L_Π^4 is sound and complete with respect to the class of Π -reactive models over strongly transitive frames.
3. If $X \neq \Pi$, L_X^4 is not sound and complete with respect to (all X -models over) any class of reactive frames.

Proof. 1.

- Given a strongly transitive reactive model $\mathcal{M} = (W, \Delta, v)$ and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \diamond_R \diamond_R \varphi$ then there are $w', w'' \in W$ s.t. $\lambda w w' w'' \in \Delta$ and $\mathcal{M}, \lambda w w' w'' \models \varphi$. Thus $\lambda w w' w'' \sim \lambda w w'' \in \Delta$ and, by proposition 2.5, $\mathcal{M}, \lambda w w'' \models \varphi$ hence $\mathcal{M}, \lambda w \models \diamond_R \varphi$.
- Given a non strongly transitive reactive frame (W, Δ) . There exists some $\lambda w w' w'' \in \Delta$ s.t. $\lambda w w' w'' \not\sim \lambda w w''$ then we have two cases:
Or $\lambda w w'' \notin \Delta$ which implies $w'' \neq w'$. Hence, if we take $p \in \Pi$ and pick v s.t. $\lambda w w' w'' \in v(p)$ and $\lambda w v \notin v(p)$ for all $\lambda w v \in \Delta$, in which case

$$\lambda w \models \diamond_R \diamond_R p \wedge \neg \diamond_R p.$$

Or exists β s.t. we don't have $\lambda w w' w'' \beta \in \Delta \leftrightarrow \lambda w w'' \beta \in \Delta$. In which case we pick $w_0 = w''$ and define ψ as in proposition (3.8.1). As before we conclude that either $(W, \Delta, v), \lambda w \not\models \diamond_R \diamond_R \psi \rightarrow \diamond_R \psi$ or $(W, \Delta, v), \lambda w \not\models \diamond_R \diamond_R \neg \psi \rightarrow \diamond_R \neg \psi$.

Similarly to the case of strong reflexivity, strong transitivity imposes that, regardless of the path we have covered, every world accessible in two steps is accessible in one and that the set of accessible worlds is the same in both cases.

2. Soundness of L_{Π}^4 comes easily from the soundness of the new axiom which has just been established.

Let us prove that $L_{\Pi}^4 = \{ \varphi : \varphi \text{ is valid in every strongly reflexive reactive frame} \}$:

- using lemma (3.2) it is easy to check that the canonical model for L_X^4 is a shattered model (W, R, P, V) where R transitive.
- t_3 preserves R -transitivity:
If $(x_1, i_1)R_3(x_2, i_2)R_3(x_3, i_3)$ then $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$, $i_1 \leq i_2 < i_3$ and $x_{3,1} = x_{3,2}$. If $x_1 \in W$ then immediately we conclude $(x_1, i_1)R_3(x_3, i_3)$. If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$. Hence $x_{1,1}Rx_{3,1} = x_{3,2}$ and $(x_1, i_1)R_3(x_3, i_3)$.
- given a R -transitive cs -frame with initial family I , (I, Δ^I) is a strongly transitive reactive frame:
Let $\lambda ww'w'' \in \Delta^I$, so there is $\gamma \in W^*$ and $s, s', s'' \in W$ s.t. $\gamma ss's''$ is an I -initial path and $l_I(\gamma ss's'') = \lambda ww'w''$. Since F is R -transitive we have that $\gamma ss''$ is also an I -initial path and clearly $l_I(\gamma ss'') = \lambda ww'' \in \Delta$. Applying lemma 3.7 we conclude that $\lambda ww'w'' \sim \lambda ww''$.

3. Let $X \subseteq \Pi$ and a class of reactive frames \mathcal{F} .

If there is a reactive frame $F = (W, \Delta) \in \mathcal{F}$ with a path of length three, $w_0w_1w_2$, let ν be s.t. $w_0w_1w_2 \in \nu(p)$ and $w_0\nu \notin \nu(p)$ for all $w_0\nu \in \Delta$ then $(W, \Delta, \nu), w_0 \models \diamond_R \diamond_{RP} \wedge \neg \diamond_{RP}$.

Given a reactive frame (W, Δ) s.t. there are no $w_0, w_1, w_2 \in W$ s.t. $w_0w_1w_2 \in \Delta$ then it validates $\diamond_R \square_{R\perp}$ but $\diamond_R \square_{R\perp} \notin L_X^T$ since

$$L_X^4 \subseteq L_{\Pi}^4 \subseteq \{ \varphi : (\{a\}, \{a\}^*, \nu) \models \varphi \ \& \ \nu(p) = \{a\}^* \text{ for all } p \in \Pi \} \not\subseteq \diamond_R \square_{R\perp}.$$

So if there is no frame with a path of length three in \mathcal{F} (in particular if \mathcal{F} is empty) then

$$L_X^4 \subseteq \{ \varphi : \varphi \text{ is valid in all } F \in \mathcal{F} \}.$$

■

Let us consider some variants of these axioms and see that they axiomatise other generalized notions of reflexivity and transitivity.

Proposition 3.10 1. A reactive frame (W, Δ) X -validates $p \rightarrow \diamond_R \diamond_{PP}$ iff it is **outwardly reflexive**, i.e. satisfies:

$$\lambda w \in \Delta \rightarrow \lambda ww \in \Delta.$$

$L_X^{T_o} = L_X \oplus p \rightarrow \diamond_R \diamond_{PP}$ is sound and complete with respect to the class of X -reactive models over outwardly reflexive frames.

2. Let $L_X^{T_i} = L_X \oplus p \rightarrow \diamond_P \diamond_R p$. If $X = \Pi$, $L_\Pi^{T_i}$ is sound and complete with respect to the class of Π -reactive models over **inwardly reflexive** frames, i.e. satisfying:

$$\lambda w \in \Delta \rightarrow \exists \lambda' \lambda w \sim \lambda' w w \in \Delta.$$

If $X \subsetneq \Pi$, $L_X^{T_i}$ is not sound with respect to (all X -models over) any class of reactive frames.

3. A reactive frame (W, Δ) X -validates $\Box_P p \rightarrow \diamond_P \diamond_R p$ iff it is **lightly reflexive**, i.e. satisfies:

$$w \in W \rightarrow \exists \lambda \lambda w w$$

$L_X^{T_i} = L_X \oplus \Box_P p \rightarrow \diamond_P \diamond_R p$ is sound and complete with respect to the class of X -reactive models over inwardly reflexive frames.

It is clear that strong reflexivity implies inward and outward reflexivity and both imply light reflexivity.

Proof.

1. Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a outwardly reflexive reactive frame and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \varphi$ then since $\lambda w w \in \Delta$ and $t(\lambda w) = t(\lambda w w)$ we have $\mathcal{M}, \lambda w w \models \diamond_P \varphi$ and so $\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi$.

If a reactive frame (W, Δ) is not outwardly reflexive then there exists some $\lambda w \in \Delta$ s.t. $\lambda w w \notin \Delta$. Picking a $p \in \Pi$ and choosing a ν s.t. $\lambda' \in \nu(p)$ iff $t(\lambda') = w$. Now we have that $(W, \Delta, \nu), \lambda w \models p \wedge \neg \diamond_R \diamond_P p$.

Soundness of $L_X^{T_o}$ comes easily from the soundness of the new axiom which has just been established.

Using the same strategy as before we will obtain the equality:

$$L_X^{T_o} = \{\varphi : \varphi \text{ is } X\text{-valid in every outwardly reflexive reactive frame}\}.$$

- $q \rightarrow \diamond_R \diamond_P q$ is a Sahlqvist formula and so the canonical frame (W, R, P) for $L_X^{T_o}$ satisfies:

$$\forall w \exists w' w R w' P w \tag{a}$$

since it is the first-order correspondent to the new axiom.

- t_1 preserves property (a):
Given (x, i) , if $i = *$ then $(x, i) R_1(x, i) P_1(x, i)$ and if $i = (x', a)$ then by property (a) there exists y s.t. $x R y P x$ thus $t'(x, x)$ is defined and $(x, i) R_1(t'(x, x), i) P_1(x, i)$.
- let (W, R, P) be a cs-frame satisfying property (a) and I an initial family. (I, Δ^I) is a outward reflexive reactive frame:
Let $\lambda w \in \Delta^I$ then there are $\gamma \in W^*$ and $s \in W$ s.t. γs is an I -initial path and $l_I(\gamma s) = \lambda w$. By property (a) there exists $t \in W$ s.t. $s R t P s$ (which implies $t P w$) thus $\gamma s t$ is also an I -initial path and $l_I(\gamma s t) = \lambda w w \in \Delta^I$.

2. To establish soundness we have just to check if the added axiom is sound: given a Π -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a inwardly reflexive reactive frame and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \varphi$ then since exists λ' s.t. $\lambda' w w \in \Delta$ and $\lambda w \sim \lambda' w w$. Thus by proposition (2.5) we have $\mathcal{M}, \lambda' w w \models \varphi$ and since $t(\lambda w) = t(\lambda' w)$ we have that $\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi$.

For completeness we proceed as before:

- $p \rightarrow \diamond_P \diamond_R q$ is a Sahlqvist formula and so the canonical frame $F = (W, R, P)$ for $L_{\Pi}^{T_i}$ satisfies:

$$\forall w \exists w' w P w' R w \quad (b)$$

since it is its first-order correspondent.

- t_2 preserves property (b): given $w = (a, b, c, d, i) \in W_2$, if $i = *$ then $w P_2 w R_2 w$ and if $i \neq *$ then $(a, b, c, d, i) P_2 (t(a), b + 1, c, d, i) R_2 (a, b, c, d, i)$.
- let (W, R, P) be a cs-frame satisfying property (b) and I an initial family. (I, Δ^I) is a inwardly reflexive reactive frame:
Let $\lambda w \in \Delta^I$, so there is $\gamma \in W^*$ and $s \in W$ s.t. γs is an I -initial path and $I_1(\gamma s) = \lambda w$. By property (b) there exists $t \in W$ s.t. $t R s P t$. From the initiality of I we know that there is some γ' s.t. $\gamma' t$ is an I -initial path, thus $\gamma' t s$ is also an I -initial and $I_1(\gamma' t s) = I_1(\gamma) w w = \lambda' w w \in \Delta^I$. Applying lemma 3.7 we conclude that $\lambda w \sim \lambda' w w$.

So

$$L_{\Pi}^{T_i} = \{\varphi : \varphi \text{ is valid in every inwardly reflexive reactive frame}\}.$$

Given $X \subseteq \Pi$ and a reactive frame (W, Δ) there is a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$, $\lambda \in \Delta$ and φ s.t. $\mathcal{M}, \lambda \not\models \varphi \rightarrow \diamond_R \varphi$: given $\lambda w \in \Delta$ and some $p \in \Pi - X$, let ν s.t. $\lambda w \in \nu(p)$ and $\lambda' w w \notin \nu(p)$ for all $\lambda' w w \in \Delta$. So

$$(W, \Delta, \nu), \lambda w \models p \wedge \neg \diamond_R \diamond p.$$

3. Let $\mathcal{M} = (W, \Delta, \nu)$ be a X -reactive model over a lightly reflexive reactive frame and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \Box_P \varphi$. Since there exists λ' s.t. $\lambda' w w \in \Delta$ and $t(\lambda w) = t(\lambda' w w)$ we have $\mathcal{M}, \lambda' w w \models \varphi$ and so $\mathcal{M}, \lambda' w \models \diamond_R \varphi$ hence $\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi$.

If a reactive frame (W, Δ) is not lightly reflexive then there exists some $w \in W$ for which there is no $\lambda \in \Delta$ s.t. $\lambda w w \in \Delta$. So we pick a $p \in \Pi$ and choose a ν s.t. $\lambda' \in \nu(p)$ iff $t(\lambda') = w$. Now we have that $(W, \Delta, \nu), w \models \Box_P p \wedge \neg \diamond_P \diamond_R p$.

Soundness of $L_X^{T_i}$ comes easily from the soundness of the new axiom which has just been established.

For completeness, as before:

- $\Box p \rightarrow \diamond_P \diamond_R q$ is a Sahlqvist formula and (just as before) the canonical model (W, R, P) for $L_X^{T_i}$ satisfies:

$$\forall w \exists w' w'' w P w' R w'' P w \quad (c)$$

since it is its first-order correspondent.

- t_1 preserves property (c): given $(x, i) \in W_1$, if $i = 1$ then

$$(x, i)P_1(x, i)R_1(x, i)P_1(x, i)$$

and if $i = (x', a)$ then by property (c) there exist y, z s.t. $xPyRzPx$ thus $t'(y, y)$ is defined and

$$(x, i)P_1(y, i)R_1(t(y, y), i)P_1(x, i).$$

- let (W, R, P) be a cs-frame satisfying property (c) and I an initial family. (I, Δ^I) is a lightly reflexive reactive frame:
Let $\lambda w \in \Delta^I$ then there are s, t s.t. $I_I(s) = w = I_I(t)$ and sRt . By initiality of I there is an I -initial path ending in s , γs , making γst also an I -initial path, hence $I_I(\gamma)ww = \lambda'ww \in \Delta$.

So

$$L_X^{T_1} = \{\varphi : \varphi \text{ is } X\text{-valid in every lightly reflexive reactive frame}\}.$$

■

Proposition 3.11 1. A reactive frame (W, Δ) X -validates $\diamond_R \diamond_{RP} \rightarrow \diamond_R \diamond_{PP}$ iff it is **left transitive**, i.e. satisfies:

$$\lambda ww'w'' \in \Delta \rightarrow \lambda ww'' \in \Delta$$

$L_X^{4_l} = L_X \oplus \diamond_R \diamond_{RP} \rightarrow \diamond_R \diamond_{PP}$ is sound and complete with respect to the class of X -reactive models over left transitive reactive frames.

2. A reactive frame (W, Δ) X -validates $\diamond_R \diamond_{RP} \rightarrow \diamond_P \diamond_{RP} \rightarrow \diamond_P \diamond_{PP}$ iff it is **middle transitive**, i.e. satisfies:

$$\lambda ww'w'' \in \Delta \rightarrow \exists \lambda' \lambda'ww'' \in \Delta$$

$L_X^{4_m} = L_X \oplus \diamond_R \diamond_{RP} \rightarrow \diamond_P \diamond_{RP} \rightarrow \diamond_P \diamond_{PP}$ is sound and complete with respect to the class of X -reactive models over middle transitive reactive frames.

3. Let $L_X^{4_r} = L_X \oplus \diamond_R \diamond_{RP} \rightarrow \diamond_P \diamond_{RP}$. $L_\Pi^{4_r}$ is sound and complete with respect to the class of Π -reactive models over **right transitive** reactive frames, i.e. satisfying:

$$\lambda ww'w'' \in \Delta \rightarrow \exists \lambda' \lambda ww'w'' \sim \lambda'ww'' \in \Delta$$

If $X \subseteq \Pi$, $L_X^{4_r}$ is not sound and complete with respect to (all X -models over) any class of reactive frames.

4. A reactive frame (W, Δ) X -validates $\diamond_R \diamond_P \diamond_{RP} \rightarrow \diamond_R \diamond_{PP}$ iff it is **globally left transitive**, i.e. satisfies:

$$\lambda ww', \lambda'w'w'' \in \Delta \rightarrow \lambda ww'' \in \Delta$$

$L_X^{4_{gl}} = L_X \oplus \diamond_R \diamond_P \diamond_{RP} \rightarrow \diamond_R \diamond_{PP}$ is sound and complete with respect to the class of X -reactive models over globally left transitive reactive frames.

5. A reactive frame (W, Δ) X -validates $\diamond_R \diamond_P \diamond_{RP} \rightarrow \diamond_P \diamond_R \diamond_{PP}$ iff it is **globally middle transitive**, i.e. satisfies:

$$\lambda w w', \lambda' w' w'' \in \Delta \rightarrow \exists \lambda'' \lambda'' w w'' \in \Delta.$$

$L_X^{4gm} = L_X \oplus \diamond_R \diamond_P \diamond_{RP} \rightarrow \diamond_P \diamond_R \diamond_{PP}$ is sound and complete with respect to the class of X -reactive models over globally transitive reactive frames.

6. Let $L_X^{4gr} = L_X \oplus \diamond_R \diamond_P \diamond_{RP} \rightarrow \diamond_P \diamond_{RP}$. L_{Π}^{4gr} is sound and complete with respect to the class of Π -reactive models over **globally right transitive** reactive frames, i.e. satisfies:

$$\lambda w w', \lambda' w' w'' \in \Delta \rightarrow \exists \lambda'' \lambda' w' w'' \sim \lambda'' w w'' \in \Delta$$

If $X \subsetneq \Pi$, L_X^{4gr} is not sound and complete with respect to (all X -models over) any class of reactive frames.

It is clear that strong transitivity implies left, middle and right transitivity. Both left and right transitivity imply middle transitivity and all of them are implied by its global version.

Proof.

1. Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a left transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_{RP} \varphi,$$

i.e. there are w', w'' s.t. $\lambda w w' w'' \in \Delta$ and $\mathcal{M}, \lambda w w' w'' \models \varphi$. By left transitivity we get that $\lambda w w'' \in \Delta$ and since $t(\lambda w w' w'') = t(\lambda w w'')$ we have

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi.$$

If a reactive frame (W, Δ) is not left transitive then there exists $\lambda w w' w'' \in \Delta$ s.t. $\lambda w w'' \notin \Delta$. So we pick a $p \in \Pi$ and choose a ν s.t. $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Now we have that

$$(W, \Delta, \nu), \lambda w \models \diamond_R \diamond_{RP} \wedge \neg \diamond_R \diamond_P p.$$

Soundness of L_X^{4t} follows from the soundness of the new axiom which has just been established.

Let us prove that $L_X^{4t} = \{\varphi : \varphi \text{ is } X\text{-valid in every left transitive reactive frame}\}$:

- $\diamond_R \diamond_{RP} \rightarrow \diamond_R \diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for L_X^{4t} satisfies:

$$\forall t \nu w \exists w' t R \nu R w \rightarrow t R w' P w \quad (LT)$$

since it is its first-order correspondent.

- t_4 preserves property (LT):
Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3$ s.t. $(x_1, i_1) R_4 (x_2, i_2) R_4 (x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$. If $x_1 \in W$ we have that

$$(x_1, i_1) R_4 ((x_{3,2}, x_{3,2}), i_3) P_4 (x_3, i_3).$$

If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$ hence, by property (LT), there is x s.t. $x_{1,1}RxPx_{3,1}(Px_{3,2})$. Thus $t'(x_{1,1}, x_{3,2})$ is defined and

$$(x_1, i_1)R_4((t'(x_{1,1}, x_{3,2})), i_3)P_4(x_3, i_3).$$

- let (W, R, P) be a *cs*-frame satisfying property (LT) and I an initial family. (I, Δ^I) is a left transitive reactive frame:
Let $\lambda ww'w'' \in I$ then there are some $\gamma \in W^*$ and $s, s', s'' \in W$ s.t. $\gamma ss's''$ is an I -initial path and $l_I(\gamma ss's'') = \lambda ww'w''$. By (LT) there exists some $t \in W$ s.t. $sRtPs''Pw$ and so γst is an I -initial path. Hence $l_I(\gamma st) = \lambda ww'' \in \Delta^I$.

2. Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a middle transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_R \varphi,$$

i.e. there are w', w'' s.t. $\lambda ww'w'' \in \Delta$ and $\mathcal{M}, \lambda ww'w'' \models \varphi$. By middle transitivity, there is λ' s.t. $\lambda' ww'' \in \Delta$. Since $t(\lambda ww'w'') = t(\lambda' ww'')$ and $t(\lambda w) = t(\lambda' w)$ we have

$$\mathcal{M}, \lambda w \models \diamond_P \diamond_R \diamond_P \varphi.$$

If a reactive frame (W, Δ) is not middle transitive then there exists some $\lambda ww'w'' \in \Delta$ and no λ' s.t. $\lambda' ww'' \in \Delta$. So we pick a $p \in \Pi$ and choose a ν s.t. $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Obtaining

$$(W, \Delta, \nu), \lambda w \models \diamond_R \diamond_R p \wedge \neg \diamond_P \diamond_R \diamond_P p.$$

Soundness of $L_X^{A_1}$ follows from the soundness of the new axiom which has just been established.

We prove completeness as before:

- $\diamond_R \diamond_R p \rightarrow \diamond_P \diamond_R \diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{A_m}$ satisfies:

$$\forall t \nu w \exists t' w' tR\nu R w \rightarrow tPt'Rw'Pw \quad (\text{MT})$$

since it is its first-order correspondent.

- t_4 preserves property property (MT):
Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3$ s.t. $(x_1, i_1)R_4(x_2, i_2)R_4(x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$. If $x_1 \in W$ we have that

$$(x_1, i_1)P_4(x_1, i_1)R_4((x_{3,2}, x_{3,2}), i_3)P_4(x_3, i_3).$$

If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Rx_{3,1}$ hence, by property (MT), there are x, x' s.t. $x_{1,1}PxRx'Px_{3,1}(Px_{3,2})$ and so $t'(x, x_{3,2})$ is defined and

$$(x_1, i_1)P_4((x, x_{1,2}), i_1)R_4((t'(x, x_{3,2})), i_3)P_4(x_3, i_3).$$

- let (W, R, P) be a *cs*-frame satisfying property (MT) and I an initial family. (I, Δ^I) is a middle transitive reactive frame:
Let $\lambda ww'w'' \in I$ then there are some $\gamma \in W^*$ and $s, s', s'' \in W$ s.t. $\gamma ss's''$ is an initial path and $l_I(\gamma ss's'') = \lambda ww'w''$. From property (MT) follows that there are some $t, t' \in W$ s.t. $sPtRt'Ps''$. By initiality of I there is an I -initial path $\gamma't$, making $\gamma'tt'$ also an I -initial thus $l_I(\gamma'tt') = l_I(\gamma')ww'' = \lambda' ww'' \in \Delta^I$.

So $L_X^{4m} = \{\varphi : \varphi \text{ is } X\text{-valid in every middle transitive reactive frame}\}.$

3. To establish soundness of L_Π^{4r} it is enough to verify soundness of the added axiom: given a Π -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a right transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_R \varphi,$$

so there are w', w'' s.t. $\lambda w w' w'' \in \Delta$ and $\mathcal{M}, \lambda w w' w'' \models \varphi$. By right transitivity there is $\lambda' w w'' \in \Delta$ s.t. $\lambda w w' w'' \sim \lambda' w w''$ and so, by proposition 2.5, $\mathcal{M}, \lambda' w w'' \models \varphi$. Thus

$$\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi.$$

For completeness:

- $\diamond_R \diamond_R p \rightarrow \diamond_P \diamond_R p$ is a Sahlqvist formula and so the canonical model (W, R, P) for L_Π^{4r} satisfies:

$$\forall t v w \exists t' t R v R w \rightarrow t P t' R w \quad (\text{RT})$$

since it is its first-order correspondent.

- t_3 preserves property (RT):
Given $(x_k, i_k) \in W_3$ for $k = 1, 2, 3$ s.t. $(x_1, i_1) R_3 (x_2, i_2) R_3 (x_3, i_3)$. So $i_1 \leq i_2 < i_3$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3$ and $x_{3,1} = x_{3,2}$. If $x_1 \in W$ then immediately we conclude that $(x_1, i_1) P_3 (x_1, i_1) R_3 (x_3, i_3)$. If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1} R x_{2,1} R x_{3,1} = x_{3,2}$ so by property (RT) exists a x s.t. $x_{1,1} P x R x_{3,1}$ and so $(x_1, i_1) P_3 ((x, x_{1,2}), i_1) R_3 (x_3, i_3)$.
- let (W, R, P) be a cs-frame satisfying property (RT) and I an initial family. (I, Δ^I) is a right transitive reactive frame:
Let $\lambda w w' w'' \in \Delta^I$ then there is an I -initial path $\gamma s_1 s_2 s_3$ ($s_i \in W$) s.t. $l_I(\gamma s_1 s_2 s_3) = \lambda w w' w''$. By property (RT) exists $s \in W$ s.t. $s_1 P s R s_3$. From the initiality of I we know that there is some γ' s.t. $\gamma' s$ is an I -initial path and so $\gamma' s s_3$ is also an I -initial. Applying lemma 3.7 we conclude that $\Delta^I \ni l_I(\gamma' s t) = l_I(\gamma') w w'' = \lambda' w w'' \sim \lambda w w' w''$.

Hence,

$$L_\Pi^{4r} = \{\varphi : \varphi \text{ is } \Pi\text{-valid in every right transitive reactive frame}\}.$$

Given $X \subsetneq \Pi$ and a (non-empty) class of reactive frames \mathcal{F} :

- if there is a reactive frame $(W, \Delta) \in \mathcal{F}$ with a path $\lambda w_0 w_1 w_2$, where $w_0 \neq w_1$, we pick $p \in \Pi - X$ and ν s.t. $\gamma \in \nu(p)$ iff $\gamma = \lambda w_0 w_1 w_2$. Thus

$$(W, \Delta, \nu), \lambda w_0 \models \diamond_R \diamond_R p \wedge \neg \diamond_P \diamond_R p.$$

- if \mathcal{F} contains only reactive frames with paths of length bigger 2 of the form $\lambda w w w'$ then it validates $\diamond_R(\varphi \wedge \diamond_R \top) \rightarrow \diamond_P \varphi$. Consider that $L_X^{4r} \subseteq L_\Pi^{4r}$ and the right transitive Π -reactive model: $\mathcal{M} = (\{0, 1\}, \{0, 1, 01, 011\}, \nu)$ s.t. $\lambda \in \nu(p)$ iff $t(\lambda) = 1$. It is clear that $\mathcal{M}, 0 \models \diamond_R(p \wedge \diamond_R \top) \wedge \neg \diamond_P p$ and so:

$$L_X^{4r} \subsetneq \{\varphi : \varphi \text{ is valid in all } F \in \mathcal{F}\}.$$

- if \mathcal{F} contains only reactive frames with paths of length smaller than 3, following the same reasoning as in proposition (3.9.3), together with the fact that the strong transitive reactive frame used there - $(\{a\}, \{a\}^*)$ - is also a right transitive reactive frame, we conclude that L_X^{4r} is not complete with respect to \mathcal{F} .

4. Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a globally left transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_P \diamond_R \varphi,$$

i.e. there are $w', w'' \in W$ and λ' s.t. $\lambda w w', \lambda' w' w'' \in \Delta$ and $\mathcal{M}, \lambda' w' w'' \models \varphi$. By light left transitivity, there is $\lambda w w'' \in \Delta$ and since $t(\lambda w w'') = t(\lambda' w' w'')$ we have

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi.$$

If a reactive frame (W, Δ) is not globally left transitive then there exists some $\lambda w w', \lambda' w' w'' \in \Delta$ s.t. $\lambda w w'' \notin \Delta$. So we pick a $p \in \Pi$ and choose a ν s.t. $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Hence

$$(W, \Delta, \nu), \lambda w \models \diamond_R \diamond_P \diamond_R p \wedge \neg \diamond_R \diamond_P p.$$

Soundness of L_X^{4l} comes easily from the soundness of the new axiom which has just been established.

We establish completeness as before:

- $\diamond_R \diamond_P \diamond_R p \rightarrow \diamond_R \diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for L_X^{4sl} satisfies:

$$\forall t v v' w \exists w' t R v P v' R w \rightarrow t R w' P w \quad (LLT)$$

since it is its first-order correspondent.

- t_4 preserves property (LLT):
Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3, 4$ s.t. $(x_1, i_1) R_4 (x_2, i_2) P_4 (x_3, i_3) R_4 (x_4, i_4)$. So $i_1 \leq i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$. If $x_1 \in W$ we have that

$$(x_1, i_1) R_4 ((x_{4,2}, x_{4,2}), i_4) P_4 (x_4, i_4).$$

If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1} R x_{2,1} P x_{3,1} R x_{4,1}$ hence, by property (LLT), there is x s.t. $x_{1,1} R x P x_{4,1} (P x_{4,1})$ and so $t'(x_{1,1}, x_{4,2})$ is defined and

$$(x_1, i_1) R_4 ((t'(x_{1,1}, x_{4,2})), i_3) P_4 (x_4, i_4).$$

- given a cs -frame (W, R, P) satisfying property (LLT) with initial family $I, (I, \Delta^I)$ is a light left transitive reactive frame: Let $\lambda w w', \lambda' w' w'' \in \Delta^I$ then there are I -initial paths $\gamma s_1 s_2$ and $\gamma' s_3 s_4$ ($s_i \in W$) s.t. $l_I(\gamma s_1 s_2) = \lambda w w'$ and $l_I(\gamma' s_3 s_4) = \lambda' w' w''$ (so $s_1 R s_2 P s_3 R s_4$). By property (LLT) there exists $s \in W$ s.t. $s_1 R s P s_4$. So $\gamma s_1 s$ is also an I -initial path and $l_I(\gamma s_1 s) = \lambda w w'' \in \Delta^I$.

Hence

$$L_X^{4sl} = \{\varphi : \varphi \text{ is } X\text{-valid in every light left transitive reactive frame}\}.$$

5. Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a light transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_P \diamond_R \varphi,$$

i.e. there are $w', w'' \in W$ and λ' s.t. $\lambda w w', \lambda' w' w'' \in \Delta$ and $\mathcal{M}, \lambda' w' w'' \models \varphi$. By globally middle transitivity, there is λ'' s.t. $\lambda'' w w'' \in \Delta$, thus

$$\mathcal{M}, \lambda w \models \diamond_P \diamond_R \diamond_P \varphi.$$

If a reactive frame (W, Δ) is not globally middle transitive then there exists some $\lambda w w', \lambda' w' w'' \in \Delta$ s.t. for all λ'' we have $\lambda'' w w'' \notin \Delta$. So we pick $p \in \Pi$ and ν s.t. $\gamma \in \nu(p)$ iff $t(\gamma) = w''$. Hence

$$(W, \Delta, \nu), \lambda w \models \diamond_R \diamond_P \diamond_R p \wedge \neg \diamond_P \diamond_R \diamond_P p.$$

Soundness of L_X^{4l} comes easily from the soundness of the new axiom which has just been established.

Completeness:

- $\diamond_R \diamond_P \diamond_R p \rightarrow \diamond_P \diamond_R \diamond_P p$ is a Sahlqvist formula and so the canonical model (W, R, P) for L_X^{4l} satisfies:

$$\forall t, v, v', w \exists t', w' t R v P v' R w \rightarrow t P t' R w' P w \quad (\text{GM})$$

since it is the first-order correspondent to the new axiom.

- t_4 preserves property (GM):
Given $(x_k, i_k) \in W_4$ for $k = 1, 2, 3, 4$ s.t. $(x_1, i_1) R_4 (x_2, i_2) P_4 (x_3, i_3) R_4 (x_4, i_4)$. So $i_1 \leq i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$. If $x_1 \in W$ we have that

$$(x_1, i_1) P_4 (x_1, i_1) R_4 ((x_{4,2}, x_{4,2}), i_4) P_4 (x_4, i_4).$$

If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1} R x_{2,1} P x_{3,1} R x_{4,1}$ hence, by property (GM), there is x, x' s.t. $x_{1,1} P x R x' P x_{4,1}$ and so $t'(x, x_{4,2})$ is defined and

$$(x_1, i_1) P_4 ((x, x_{1,2}), i_1) R_4 ((t'(x, x_{4,2})), i_3) P_4 (x_4, i_4).$$

- given a cs -frame (W, R, P) satisfying property (GM) with initial family $I, (I, \Delta^I)$ is a globally middle transitive reactive frame:
Let $\lambda w w', \lambda' w' w'' \in \Delta^I$ then there are I -initial paths $\gamma s_1 s_2$ and $\gamma' s_3 s_4$ ($s_i \in W$) s.t. $l_I(\gamma s_1 s_2) = \lambda w w'$ and $l_I(\gamma' s_3 s_4) = \lambda' w' w''$ (in particular $s_1 R s_2 P s_3 R s_4$). By property (GM) there exists $t, t' \in W$ s.t. $s_1 P t R t' R s_4$. By initiality of I , there is some γ'' s.t. $\gamma'' t$ is an I -initial path and so $\gamma'' t'$ is also an I -initial path and $l_I(\gamma'' t') = \lambda'' w w'' \in \Delta^I$.

Hence $L_X^{4gm} = \{\varphi : \varphi \text{ is } X\text{-valid in every globally middle transitive reactive frame}\}$.

6. To establish soundness of L_Π^{4gr} it is enough to verify soundness of the added axiom: given a Π -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a globally right transitive reactive frame and $\lambda w \in \Delta$ s.t.

$$\mathcal{M}, \lambda w \models \diamond_R \diamond_P \diamond_R \varphi,$$

i.e. there are $w', w'' \in W$ and λ' s.t. $\lambda ww', \lambda'w'w'' \in \Delta$ and $\mathcal{M}, \lambda'w'w'' \models \varphi$. By globally right transitivity we get that there is λ'' s.t. $\lambda''ww'' \sim \lambda'w'w''$, so by proposition 2.5 we get $\mathcal{M}, \lambda''ww'' \models \varphi$ and so

$$\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi.$$

For completeness:

- $\diamond_R \diamond_P \diamond_R p \rightarrow \diamond_P \diamond_R p$ is a Sahlqvist formula and so the canonical model (W, R, P) for $L_X^{4_{gr}}$ satisfies:

$$\forall tvv'w \exists t' tRvPv'Rw \rightarrow tPt'Rw \quad (\text{GR})$$

since it is its first-order correspondent.

- t_3 preserves property (GR):
Given $(x_k, i_k) \in W_3$ for $k = 1, 2, 3, 4$ s.t. $(x_1, i_1)R_3(x_2, i_2)P_3(x_3, i_3)R_3(x_4, i_4)$. So $i_1 \leq i_2 = i_3 < i_4$, $x_k = (x_{k,1}, x_{k,2})$ for $k = 2, 3, 4$ and $x_{4,1} = x_{4,2}$. If $x_1 \in W$ then immediately we conclude $(x_1, i_1)P_3(x_1, i_1)R_3(x_4, i_4)$. If $x_1 = (x_{1,1}, x_{1,2})$ then $x_{1,1}Rx_{2,1}Px_{3,1}Rx_{4,1} = x_{4,2}$ so by property (GR) exists a x s.t. $x_{1,1}PxRx_{4,1}$ and so $(x_1, i_1)P_3((x, x_{1,2}), i_1)R_3(x_4, i_4)$.
- given a cs -frame (W, R, P) satisfying property (GR) with initial family I , (I, Δ^I) is a globally right transitive reactive frame:
Let $\lambda ww', \lambda'w'w'' \in \Delta^I$ then there are I -initial paths $\gamma_{s_1s_2}$ and $\gamma'_{s_3s_4}$ ($s_i \in W$) s.t. $l_I(\gamma_{s_1s_2}) = \lambda ww'$ and $l_I(\gamma'_{s_3s_4}) = \lambda'w'w''$ (so $s_1Rs_2Ps_3Rs_4$). By property (GR) there exists $s \in W$ s.t. s_1PsRs_4 . By initiality of I , there is some γ'' s.t. $\gamma''s$ is an I -initial path and so $\gamma''s s_4$, thus $l_I(\gamma''s s_4) = \lambda''ww'' \in \Delta$. Applying lemma 3.7 we conclude that $\lambda'w'w'' \sim \lambda''ww''$.

So

$$L_{\Pi}^{4_{gr}} = \{\varphi : \varphi \text{ is valid in every globally right transitive reactive frame}\}.$$

Given $X \subseteq \Pi$, $L_X^{4_{gr}}$ is not sound and complete for any class of reactive frames:

It follows from the proof in 3. for $L_X^{4_r}$. In the first case we can also conclude that the frame does not validate $\diamond_R \diamond_P \diamond_R p \rightarrow \diamond_P \diamond_R p$ thus $L_X^{4_{ri}} \subset L_X^{4_r}$ it is not sound in relation any class of frames containing a reactive frame with a path of the form $\lambda w_0 w_1 w_2$ with $w_0 \neq w_1$. Neither it is complete with respect to the other two cases considered since both reactive frames used in the proof are also globally right transitive.

■

3.2 Static and quasi-static

Proposition 3.12 We have that $(\diamond_P \diamond_R p \rightarrow \diamond_R \diamond_P p) \leftrightarrow (\diamond_R \square_P p \rightarrow \square_P \diamond_R p) \in L_X$ and $L_X^S = L_X \oplus \diamond_P \diamond_R p \rightarrow \diamond_R \diamond_P p = L_X \oplus \diamond_R \square_P p \rightarrow \square_P \diamond_R p$ is sound and complete with respect to the class of X -reactive models over **static** reactive frames, i.e. satisfying:

$$\lambda w, \lambda w' \in \Delta \rightarrow \lambda w \sim \lambda' w$$

Notice that if $X = \Pi$, the new axioms are equivalent to $\diamond_P p \rightarrow p$ and that if we impose this axiom instead we get a result limited to this case. This tells us that in the other cases where we have this kind of restriction there may be better axiomatisations valid for all X .

Proof. Let $com_{PR}^f = \diamond_P \diamond_R P \rightarrow \diamond_R \diamond_P P$ and $chr_{PR} = \diamond_R \square_P P \rightarrow \square_P \diamond_R P$.

- Using the equality $L_{r,X} = L_X$:

- chr_{PR} implies com_{PR}^f

$\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi$, since $\varphi \rightarrow \square_P \diamond_P \varphi$ of P we have $\mathcal{M}, \lambda w \models \diamond_P \diamond_R \square_P \diamond_P \varphi$. Applying chr_{PR} we get $\mathcal{M}, \lambda w \models \diamond_P \square_P \diamond_R \diamond_P \varphi$ and, again from $\psi \rightarrow \square_P \diamond_P \psi$ (equivalent to $\diamond_P \square_P \psi \rightarrow \psi$), we obtain $\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi$.

- com_{PR}^f implies chr_{PR}

$\mathcal{M}, \lambda w \models \diamond_P \square_P \varphi$, since $\varphi \rightarrow \square_P \diamond_P \varphi$, we have $\mathcal{M}, \lambda w \models \diamond_P \square_P \square_P \diamond_P \varphi$. Applying com_{PR}^f we get $\mathcal{M}, \lambda w \models \diamond_P \square_P \square_P \diamond_R \diamond_P \varphi$ and, again from $\diamond_P \square_P \psi \rightarrow \psi$, we obtain $\mathcal{M}, \lambda w \models \square_P \diamond_R \varphi$.

- Soundness of L_X^S comes easily from the soundness of the new axiom:

Given a X -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a static frame and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi$ then there exists $\lambda' w w' \in \Delta$ s.t. $\mathcal{M}, \lambda' w w' \models \varphi$ thus $\lambda w w' \in \Delta$ (since $\lambda w \sim \lambda' w$) and $\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi$ (since $t(\lambda w w') = t(\lambda' w w')$).

- For completeness we proceed as before:

- $\diamond_P \diamond_R P \rightarrow \diamond_R \diamond_P P$ is a Sahlqvist formula and so the canonical model $F = (W, R, P)$ for L_X^S satisfies PR -commutativity:

$$\forall x y z \exists y' x P y R z \rightarrow x R y' P z$$

since it is the first-order correspondent to the new axiom.

- PR -commutativity is preserved by t_1 :

Let $(x, i), (y, j), (z, k) \in W'$ s.t. $(x, i) P' (y, j) R' (z, k)$. If $i = *$ then $(x, i) = (y, j)$ and $(x, i) R' (z, k) P' (z, k)$. If $i \neq *$ then $x P y R z$, $i = j = (x', a)$ and $k = (z', b)$. So exists y' s.t. $x R y' P z$, and so $t'(x, v)$ is defined for any v s.t. $v P z$. Or $i = j = k$, so $t'(x, x)$ is defined and $(x, i) R' (t'(x, x), k) P' (z, k)$, or $i = j \neq k$, so $t'(x, z')$ is defined and $(x, i) R' (t'(x, z'), k) P' (z, k)$.

- Given a PR -commutative cs -frame with initial family I then (I, Δ^I) is a static reactive frame:

Let $\lambda w, \lambda' w \in \Delta^I$ so there is βb_1 and $\beta' b'_1$ I -initial paths s.t. $l_I(\beta b_1) = \lambda w$ and $l_I(\beta' b'_1) = \lambda' w$. Let us see that for every $\gamma = w_1 \dots w_n$ we have $\beta b_1 \gamma$ iff exist $\gamma' = v_1 \dots v_n$ s.t. $\beta' b'_1 \gamma'$ and $v_i P w_i$ for $i = 1, \dots, n$. By induction on n (we do it only in one direction, the other is the same):

- $n = 1$

$\beta b_1 w_1$ is an I -initial path so by PR -commutativity exists v_1 s.t. $b_2 R v_1 P w_1$ and so $\beta' b'_1 v_1$ is an I -initial path.

– $n + 1$

$\beta b_1 w_1 \dots w_n w_{n+1}$ is an I -initial path so by PR -commutativity exists v_{n+1} s.t. $v_n R v_{n+1} P w_{n+1}$ and so $\beta' b_2 v_1$ is an I -initial path.

Hence we conclude that $\lambda w \sim \lambda' w$.

■

A static frame is a reactive frame that does not react, that is, the accessible worlds depend only on the current world and not on how you get there. It is clear that from such a reactive frame (W, Δ) we can obtain a classic Kripke frame (W, R) where

$$R = \{(w, v) : wv \in \Delta\} = \{(w, v) : \lambda wv \in \Delta\}.$$

It is straightforward to see that in such a reactive frame all the variants of transitivity and reflexivity on reactive frames coincide with the usual notions on the correspondent Kripke frame.

It is easy to see that a Kripke model over general frame (W, R, P, A) , where P is a equivalence class, is a Π -shattered model iff $A \subseteq 2^W$ – the boolean algebra generated by the P equivalence classes – is closed for the operators: $m_R(X) = \{x \in W : \exists y \in W \text{ s.t. } xRy\}$ and $m_P(X) = \{x \in W : \exists y \in W \text{ s.t. } xPy\}$. Being P an equivalence relation, A is trivially closed under m_P . It is easy to see that A being closed under m_R , means that if a world is in $m_R(X)$ so it must be all its P -class, which is corresponds to PR -commutativity. From the point of view of reactivity, the use of (shattered) general frames to deal with the restrictions over the valuations (even if only in the case of $X = \Pi$) does not help, it demands a very strict interaction between R and P , it corresponds, in the reactive level to ask it not to react! See [16] for an equivalent presentation of Π -shattered models (with only \diamond_R) and its relation with general frames.

Proposition 3.13 Let $L_X^{qS} = L_X \oplus \diamond_R \diamond_P P \rightarrow \diamond_P \diamond_R P$.

L_Π^{qS} is sound and complete with respect to the class of (Π -shattered models over) quasi-static reactive frames, i.e. that satisfy:

$$\lambda w w', \lambda' w' \in \Delta \rightarrow \exists \lambda'' \lambda' w' \sim \lambda'' w w' \in \Delta$$

If $X \subsetneq \Pi$, L_X^{qS} is not sound and complete with respect to (all X -models over) any class of reactive frames.

Proof.

- Soundness:

Given a Π -reactive model $\mathcal{M} = (W, \Delta, \nu)$ over a quasi-static reactive frame and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \diamond_R \diamond_P \varphi$ then there exists $\lambda w w', \lambda' w' \in \Delta$ s.t. $\mathcal{M}, \lambda' w' \models \varphi$ thus $\lambda' w' \sim \lambda'' w w' \in \Delta$ and $\mathcal{M}, \lambda'' w w' \models \varphi$ (by lemma 2.5). Hence $\mathcal{M}, \lambda w \models \diamond_P \diamond_R \varphi$ (since $t(\lambda w w') = t(\lambda' w w')$).

- Completeness:

$\diamond_R \diamond_P P \rightarrow \diamond_P \diamond_R P$ is a Sahlqvist formula and so the canonical frame $F = (W, R, P)$ for L_X^{qS} satisfies RP -commutativity:

$$\forall xyz \exists y' xRyPz \rightarrow xPy'Rz$$

since it is the first-order correspondent to the new axiom.

Given $M = (W, R, P, V)$ a RP -commutative shattered frame, let $M' = (W', R', P', V')$ be defined by:

- $W' = (W^+)^3 \cup (W^+)^2 \times \{*\}$
- $(a, b, c)R'(a', b', c')$ iff $c = * \& a = a'$ or $c \neq * \neq c' \& o(a)Ro(a') \& (a \in W \& a' = b' = c' \text{ or } a \notin W \& (b' \neq a' \neq c' \& r(a) = a' \text{ or } r(a) = b' \& a' = c' \text{ or } r(a) = c' \& a' = b'))$
- $(a, b, c)P'(a', b', c')$ iff $(b, c) = (b', c') \& o(a)Po(a')$
- $(a, b, c) \in V'(p)$ iff $o(a) \in V(p)$,

where $o(w_1 \dots w_n) = w_1$, $t(w_1 \dots w_n) = w_n$ and $r(w_1 \dots w_n) = w_2 \dots w_n$.

We have that (W', R', P') is a RP -commutative cs-frame and the M' is a bounded morphic image of M :

- (W', R', P') is a RP -commutative Let $(a, b, c)R'(a', b', c')P'(a'', b'', c'')$. If $c = *$ then $a = a'$ and $o(a) = o(a')Po(a'')$ thus

$$(a, b, c)P'(a'', b, c)R'(a'', b'', c'').$$

If $c \neq * \neq c'$ then $o(a)Ro(a')Po(a'')$ so exists w s.t. $o(a)PwRo(a'')$. We have three possibilities:

or $b' = b'' \neq a'' \neq c'' = c'$ and so $(a, b, c)P'(wa'', b, c)R'(a'', b'', c'')$;
or $a'' = b''$ thus

$$(a, b, c)P'(wc'', b, c)R'(a'', b'', c'');$$

or $a'' = c''$ and

$$(a, b, c)P'(wb'', b, c)R'(a'', b'', c'').$$

- Clearly V' is admissible and that any choice of I picking one element from each P -class and containing $\{(a, a, *) : a \in W^+\}$ works as an initial family.
- Let us check that it is coherent. Let $(a, b, c)R'(a', b', c')$, $(a, b, c)R'(a'', b'', c'')$ and $(a', b', c')P'(a'', b'', c'')$. From $(a', b', c')P'(a'', b'', c'')$ we conclude that $(b', c') = (b'', c'')$. If $c = *$ then $a' = a = a''$. If $c \neq * \neq c'$, then or $a \in W$ and $a' = b' = c' = c'' = b'' = a''$; or $a \notin W$, in which case we have three subcases: or $b' \neq a' \neq c'$ and so $r(a) = a' = a''$; or $a' = b'$, thus $r(a) = c' = c''$ and $a'' = b'' = b' = a'$; or $a' = c'$ and $r(a) = b' = b''$ so $a'' = c'' = c' = a'$. In any case, $(a', b', c') = (a'', b'', c'')$.

– Let $f : (W^+)^3 \rightarrow W$ be defined by $f((a, b, c)) = o(a)$.

It is straightforward to see that f is surjective, let us see that it is actually a p -morphism:

The condition in the valuation is trivial as before;

* f is a p -morphism in R

- $(a, b, c)R'(a', b', c')$ then $f((a, b, c)) = o(a)Ro(a') = f((a', b', c'))$,
- $f((a, b, c))Rw$,
- if $a \in W$ then $(a, b, c)R'(w, w, w)$ otherwise $(a, b, c)R'(w, r(a), w)$;

* f is a p -morphism in P

- $(a, b, c)P'(a', b', c')$ then $f((a, b, c)) = o(a)Po(a') = f((a', b', c'))$,
- $f((a, b, c))Pw$ then $(a, b, c)P'(w, b, c)$;

- Given a PR -commutative cs -frame with initial family I then (I, Δ^I) is a such a reactive frame:

Let $\lambda ww', \lambda'w' \in \Delta^I$, so there is $\gamma, \gamma' \in W^*$ and $s, s', s'' \in W$ s.t. $\gamma ss'$ and $\gamma' s''$ are I -initial paths, $l_I(\gamma ss') = \lambda ww'$ and $l_I(\gamma' s'') = \lambda'w'$. So $sRs'Ps''$, thus there is t s.t. $sPtRs''$. Let γ'' be s.t. $\gamma''t$ is an I -initial path, $\gamma''ts''$ is I -initial too. Furthermore, using lemma 3.7, we conclude that $\lambda w' \sim \lambda''ww' = l_I(\gamma''ts'')$.

■

It is clear that if a reactive frame is static then it is also quasi-static. Furthermore it is interesting to notice that if a frame is static all the variants of transitivity and reflexivity coincide.

3.3 Open problem - symmetry

Our method seems not to be as fruitful with the notions of generalized symmetry. We are unable to prove reactive frame completeness. To prove that it is sound to a certain class of reactive frames and that it is complete to the correspondent class of shattered frames is straightforward as before. However completeness cannot be proved in the same way. We just cannot simply pass from the shattered to cs -frames. Let us look just to the case of strong symmetry.

Conjecture 3.14 Let $L_X^B = L_X \oplus p \rightarrow \square_R \diamond_R p$. L_Π^B is sound and complete with respect to the class of reactive frames that are **strongly symmetric**, i.e. satisfying:

$$\lambda ww' \in \Delta \rightarrow \lambda w \sim \lambda ww' w \in \Delta$$

Still we have not been able to prove it is not complete either. We present the positive result as a conjecture and add a brief explanation on how our attempts failed.

We present a R -reflexive shattered frame that cannot be transformed into a cs -frame using the classic backward truth preserving transformations.

If a shattered frame (W, R, P) with $R \neq \emptyset$ satisfies for all w there is w' s.t. wPw' , there is no w'' satisfying $w'Rw''$ or $w''Rw'$, then it does not admit an initial family.

Let us assume the contrary, let I be its initial family and

$$A = \{w : \text{there is no } w' \text{ s.t. } wRw' \text{ or } w'Rw\}.$$

Since I R -generates W , the isolated points must be in I , so $A \subseteq I$. From the fact that I picks only one element from each P -class and for every w there is $w' \in A$ s.t. wPw' we have that $I = A$. As there is an element in $W - A$ I does not R -generate all W .

It is also clear that a frame that has such a frame as generated subframe cannot have an initial family. Or else the elements of the initial family of the bigger frame, present in the smaller one, would be an initial family to the latest.

Furthermore, the operation of taking pre-images, ultrafilter extensions and ultra-powers preserves this (bad) property.

The following shattered frame $(\{0, 1\}, \{(1, 1)\}, \{0, 1\}^2)$ is R -symmetric and satisfies the bad property. Hence classical ways of generating new models from old preserving modal satisfaction do not allow us to find a general recipe to convert the relevant shattered frames into the correspondent cs -frames. In particular no variation of blow up will work.

This may not be a limitation on the method, instead it may be that the conjecture is not true and that the missing axiom would restrict us to a class of shattered frames that do not have this property. We have not been able to prove either way.

In the presence of (strong) reflexivity this problem disappears and the blow up method works:

Proposition 3.15 *Let $L_X^{TB} = L_X^T \oplus p \rightarrow \Box_R \Diamond_R p = L_X^B \oplus p \rightarrow \Diamond_R p$. L_{Π}^{TB} is sound and complete with respect to the class of strongly symmetric reactive frames that are also strongly reflexive.*

Proof. Soundness is just as before obtained by checking that $p \rightarrow \Box_R \Diamond_R p$ is sound:

Given a strongly symmetric reactive model $\mathcal{M} = (W, \Delta, \nu)$ and $\lambda w \in \Delta$ s.t. $\mathcal{M}, \lambda w \models \varphi$ then given v s.t. $\lambda wv \in \Delta$ then $\lambda wv \sim \lambda wvw \in \Delta$ so, by proposition 2.5, $\mathcal{M}, \lambda wvw \models \varphi$ thus $\mathcal{M}, \lambda wv \models \Diamond_R \varphi$. Hence and $\mathcal{M}, \lambda w \models \Box_R \Diamond_R \varphi$

In this case we are able to prove completeness by applying the blow up method:

Given $M = (W, R, P, V)$ a R -symmetric and R -reflexive shattered frame, let $M' = (W', R', P', V)$ be defined by:

- $W' = \bigcup_{n < \omega} W^{2n+1} \times \bigcup_{n < \omega} W^{2n+1}$
- $(x, y)R'(x', y')$ iff $x = x'$ or
 $(y = y' \ \& \ wRw' \ \& \ (x = z \ \& \ x' = z' \ \text{or} \ x' = z \ \& \ x = z')) \ \& \ z = (w, \bar{v}) \ \& \ z' = (w', w', w, \bar{v})$
- $(x, y)P'(x', y')$ iff $y = y' \ \& \ x = (w, \bar{v}) \ \& \ x' = (w', \bar{v}) \ \& \ wPw'$
- $((x_1, \dots, x_k), w) \in V'(p)$ iff $x_1 \in V(p)$.
where $\bar{v} = v_1, \dots, v_k$.

We have that (W', R', P') is a cs -frame (where R' is clearly symmetric and reflexive) and the M' is a bounded morphic image of M :

- clearly V' is admissible and that any choice of I picking one element from each P -class and containing $\{(x, y) \in W' : x = y\}$ works as an initial family.
- let us check that it is coherent. Given $(x, y), (x', y'), (x'', y'') \in W'$. If $(x', y')P'(x'', y'')$ then $x' = (v, \bar{v}), x'' = (v', \bar{v})$ (so $|x'| = |x''|$) with wPw' and $y' = y''$. Let us assume that we also have $(x, y)R'(x', y')$ and $(x, y)R'(x'', y'')$.
 If $x = x'$ and $x \neq x''$ then $|x'| = |x| \neq |x''|$ which contradicts $(x', y')P'(x'', y'')$, so $x = x''$. The same applies if $x = x''$ and $x \neq x'$.
 If $x \neq x'$ and $x' \neq x''$ then or $x = (w, \bar{v})$ and $x' = (w', w', w, \bar{v}) = x''$, or $x = (w', w', w, \bar{v})$ and $x' = (w, \bar{v}) = x''$.
 In any case: $(x', y') = (x'', y'')$.
- let $f : W' \rightarrow W$ be defined by $f(((w, \bar{x}), y)) = w$.
 It is straightforward to see that f is surjective, let us see that it is actually a p -morphism:
 The condition in the valuation is trivial as before;
 - f is a p -morphism in R
 - * $((w, \bar{x}), y)R'((w', \bar{x}), y')$ then or $w = w'$ or wRw' , in any case $f(((w', \bar{x}), y)) = wRw' = f(((w', \bar{x}), y'))$.
 - * $f(((w, \bar{x}), v))Rw'$ then $((w, \bar{x}), v)R'((w', w', w, \bar{x}), v)$;
 - f is a p -morphism in P
 - * $((w, \bar{x}), y)P'((w', \bar{x}), y')$ then $f(((w', \bar{x}), y)) = wPw' = f(((w', \bar{x}), y'))$.
 - * $f(((w, \bar{x}), v))Pw'$ then $((w, \bar{x}), v)P'((w', w, \bar{x}), v)$.

In proposition 3.8 we have checked that given a R -reflexive cs -frame with initial family I then (I, Δ^I) is a strongly reflexive reactive frame. Let us see now that if the cs -frame is also R -symmetric then (I, Δ^I) is also strongly symmetric:

Let $\lambda ww' \in \Delta^I$, so there is $\gamma \in W^*$ and $s, s' \in W$ s.t. $\gamma ss'$ is an I -initial path and $l_I(\gamma ss') = \lambda ww'$. Since F is R -symmetric we have that $\gamma ss's$ is also an I -initial path and clearly $l_I(\gamma ss's) = \lambda ww'w \in \Delta$. Applying lemma 3.7 we conclude that $\lambda w \sim \lambda ww'w$.

■

This result tells us that if the above conjecture is not true, the formula valid in all strong symmetric reactive frames that is not in L_X^B must be a L_X -consequence of $P \rightarrow \Diamond_{RP}$.

4 Results and final comments

Table 1 summarizes the main results presented in this paper. One can read along its lines the correspondence between logic, reactive frame property (including its dependence on X) and the shattered frame unfolded property.

For example, in the first line we see that the logic L_X corresponds to all shattered frames and to the whole class of (X -models over) reactive frames, hence being the smallest (minimal) “reactive” logic. In general there is a dependence on X , for instance in the sixth line, we see that the completeness of $L_X^T = L \oplus p \rightarrow \diamond p$ with respect to the class of (all X -models over) strong reflexive reactive frames requires that $X = \Pi$, i.e. that all variables have to be fixed. We do not know if this dependence can be avoided by a more inspired choice of axiom as in the case of static reactive frames, see proposition 3.12.

In section 3.3 we stated that the question of axiomatizing the generalized notions of symmetry is still open. The same happens with many other completeness results. It is not clear that the combination of the axioms corresponding to certain reactive properties, will correspond to the combination of those properties. This is the case in the classical setting when we add reflexivity to transitivity or symmetry. It may be that the reactive properties interact and those classes satisfy formulas not captured by the axioms. It certainly does not follow from the results we presented above. For that to happen the (blow up) transformation would have to preserve the combination of all properties in question. The fact is that we have been unable to find a transformation that would work for all cases, in particular when considering generalizations of different properties. In this sense our choice of transformations is very weak. By contrast, in the strong symmetry plus strong reflexivity case we see how gathering properties may also be helpful. The strong symmetry case alone is worse behaved than the combined one.

In this paper we have used classical tools to study completeness on a new interpretation of modal logic. We have been successful in studying some of its subsystems although we are aware that it may fail (see section 3.3) and we should not expect it to work in every case. Furthermore it would be interesting to find some new methods that would allow a more direct way of studying these logics, leading to a reactivation of modal techniques.

logic	X	reactive property	unfolded property
$L = K_R \oplus S_P + X, P \leftrightarrow \Box P P$	any		minimal
$L_X^S = L_X \oplus \Diamond P P \rightarrow P$	Π		
$L_X^S = L_X \oplus \Diamond P \Diamond R P \rightarrow \Diamond R \Diamond P P$	any	$\lambda w, \lambda w' \in \Delta \rightarrow \lambda w \sim \lambda' w$	static
$L_X^S = L_X \oplus \Diamond R \Box P P \rightarrow \Box P \Diamond R P$	any		
$L_X^{qS} = L_X \oplus \Diamond R \Diamond P P \rightarrow \Diamond P \Diamond R P$	Π	$\lambda w w', \lambda' w' \in \Delta \rightarrow \exists \lambda'' \lambda' w' \sim \lambda'' w w' \in \Delta$	q-static
$L_X^r = L \oplus p \rightarrow \Diamond p$	Π	$\lambda w \in \Delta \rightarrow \lambda w \sim \lambda w w \in \Delta$	gener-
$L_X^{l_0} = L \oplus p \rightarrow \Diamond R \Diamond P P$	any	$\lambda w \in \Delta \rightarrow \lambda w w \in \Delta$	alising
$L_X^{l_1} = L \oplus p \rightarrow \Diamond P \Diamond R P$	Π	$\lambda w \in \Delta \rightarrow \exists \lambda' \lambda w \sim \lambda' w w \in \Delta$	reflexe-
$L_X^{l_1} = L \oplus \Box P P \rightarrow \Diamond P \Diamond R P$	any	$w \in W \rightarrow \exists \lambda \lambda w w$	tivity
$L_X^4 = L \oplus \Diamond R \Diamond R P \rightarrow \Diamond R P$	Π	$\lambda w w' w'' \in \Delta \rightarrow \lambda w w' w'' \sim \lambda w w'' \in \Delta$	
$L_X^{4_l} = L \oplus \Diamond R \Diamond R P \rightarrow \Diamond R \Diamond P P$	any	$\lambda w w' w'' \in \Delta \rightarrow \lambda w w'' \in \Delta$	general-
$L_X^{4_m} = L \oplus \Diamond R \Diamond R P \rightarrow \Diamond P \Diamond R \Diamond P P$	any	$\lambda w w' w'' \in \Delta \rightarrow \exists \lambda' \lambda' w w'' \in \Delta$	ising
$L_X^{4_{s'}} = L \oplus \Diamond R \Diamond R P \rightarrow \Diamond P \Diamond R P$	Π	$\lambda w w' w'' \in \Delta \rightarrow \exists \lambda' \lambda w w' w'' \sim \lambda' w w'' \in \Delta$	
$L_X^{4_{sm}} = L \oplus \Diamond R \Diamond P \Diamond R P \rightarrow \Diamond R \Diamond P P$	any	$\lambda w w', \lambda' w' w'' \in \Delta \rightarrow \lambda w w'' \in \Delta$	transi-
$L_X^{4_{sm}} = L \oplus \Diamond R \Diamond P \Diamond R P \rightarrow \Diamond P \Diamond R \Diamond P P$	any	$\lambda w w', \lambda' w' w'' \in \Delta \rightarrow \exists \lambda'' \lambda'' w w'' \in \Delta$	tivity
$L_X^{4_{sr}} = L \oplus \Diamond R \Diamond P \Diamond R P \rightarrow \Diamond P \Diamond R P$	Π	$\lambda w w', \lambda' w' w'' \in \Delta \rightarrow \exists \lambda'' \lambda' w' w'' \sim \lambda'' w w'' \in \Delta$	
$L_X^{4_{sr}} = L \oplus \Diamond R \Diamond P \Diamond R P \rightarrow \Diamond R \Diamond P P$	Π	$\lambda w w' \in \Delta \rightarrow \lambda w \sim \lambda w w \in \Delta$ & $\lambda w \sim \lambda w w' w \in \Delta$	ref.+trans.
$L_X^{4_{sr}} = L \oplus \Diamond R \Diamond P \Diamond R P \rightarrow \Diamond R \Diamond P P$	Π		R-reflexive and R-symmetric

Table 1: Table of results

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