Reactive Intuitionistic Tableaux

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Abstract

We introduce reactive Kripke models for intuitionistic logic and show that the reactive semantics is stronger than the ordinary semantics. We develop Beth tableaux for the reactive semantics.

1 Introduction

In [1] we introduced the idea of reactivity and studied reactive Kripke models for modal logics. In many subsequent papers we studied other reactive systems such as reactive argumentation frames, reactive automata, reactive grammars, reactive preferential logics, reactive contrary to duties, reactive inheritance networks, and many more.

The purpose of this paper is to introduce reactive intuitionistic frames (Kripke frames and Beth frames) and study their expressive power and properties.

We begin by briefly introducing the idea of reactive networks (including reactive Kripke models). Consider the network of Figure 1.

Let us first ignore the double arrow in Figure 1. Without the double arrow, Figure 1 is a three point Kripke model for intuitionistic logic. So for example if we want to evaluate $a \nvDash p \to q$, we must check whether there exists a higher point s to a (including a itself) with $x \vDash p$ but $x \nvDash q$.

The above definition is set-theoretical. The notion $x \vDash A$ is defined inductively, and the graph of Figure 1 without the double arrow is just a graph suggesting a Kripke model with a reflexive and transitive binary relation.

To be more precise, Figure 1 (without the double arrow) suggests a set $S = \{a, b, c\}$, a relation $R = \{(a, b), (b, c)\}$ and the reflexive and transitive closure of R being $R^* = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$. If we use an explicit formula for R^*, we get: xR^*y iff $x = y$ or xRy or for some $k \geq 1$ and some $t_1, \ldots, t_k$, we have $xRt_1 \wedge t_1Rt_2 \wedge \ldots \wedge t_{k-1}Rt_k \wedge t_kRy$.

The assignment h to the atoms is also indicated in the Figure.

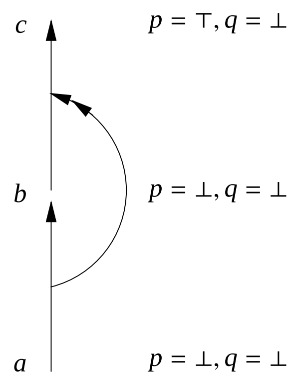

Figure 1: E1

So the Kripke model is (S, R^*, a, h), with $h(p) = \{c\}$ and $h(q) = \varnothing$.

So to check whether $a \nvDash p \to q$ we simply ask set-theoretically whether $\exists x(aR^*x$ and $x \vDash p$ and $x \nvDash q)$.

To introduce the reactive approach we envisage ourselves walking along the arrows of the graph from point a onwards and at each point x that we pass, we evaluate $x \vDash p$ and $x \vDash q$ and compare. This is an actual walk and search along the graph.

Of course, the end result is the same. If there is an x such that aR^*x and $x \vDash p$ and $x \nvDash q$ then we will walk into it sooner or later and vice versa.

Now given this 'walk along the graph' point of view, the reactive double arrow makes sense. What it does is the following: As we cross from a to b, the double arrow gets activated and disconnects the path from b to c. So we do not get to the point c where $c \vDash p$ and $c \nvDash q$. Without getting to c, we will report that $a \vDash p \to q$ holds, beause we cannot get to the counterexample, etc. So in the reactive model of Figure 1 with the double arrow, we have $a \vDash p \to q$.

We now sum up. We introduced two ideas here.

1. Evaluation in Kripke models is done by 'walk along the arrows and check and report' policy.

2. Double arrows along the way can disconnect connections and control where we can go.[1]

Consider now Figure 2. In this Figure, when we walk along $a \to b \to c$ we cannot continue from d to e, because $d \to e$ gets disconnected. However, when we walk along $a \to c \to d$, we can continue to e because there is no double arrow along the path.

[1] There are more complex options for reactivity:

(a) Double arrows can switch arrows on and off.

(b) Double arrows can emanate from other double arrows.

(c) We can have an inductively iterated version of the above.

In this paper we are keeping the reactivity simple.

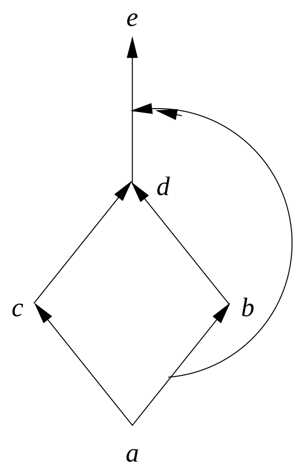

Figure 2: E2

Section 2 gives the formal definitions involved and introduces the reactive models. We also show that we get a richer semantics than ordinary Kripke models.

The idea of reactivity is a general one and can apply to Beth models as well. Beth models are like Kripke models except the inductive truth definition is different. We need the notion of an Belt anti-chain of points. Given (S, R^*, a, h), and $t \in S$ then a set $T \subseteq S$ is a Belt anti-chain for t if all points of T are R^* not comparable and every maximal R^* chain beginning at t must meet the Belt T.

We have $t \vDash A$ iff there exists an antichain T for t such that for all $x \in T, x \vDash A$.

Turning a model reactive is even easier, if we give the correct definition of a reactive path. A reactive path beginning at t is a trace of a walk along the arrows from t onwards, where all double arrows are taken into account. So hopefully we can define reactive Beth models as well.

In Figure 2 there are two maximal reactive paths $a \to b \to d$ and $a \to c \to d \to e$.

2 Reactive Kripke Frames

This section introduces reactive Kripke frames for intuitionistic logic and shows that intuitionistic propositional logic is complete for such frames. We also show that there are intermediate logics which are complete for a class of reactive Kripke frames but are not complete for any class of ordinary Kripke frames.

Thus reactive Kripke frames is a richer and stronger semantics than ordinary frames.

The above also means that we can study a richer class of intermediate logics, e.g. intermediate logics generated by finite reactive frames. We shall see in Remark 3.8 what kind of Heyting like algebras one gets from finite reactive frames.

To appreciate the opportunities opening for us through the notion of reactive Kripke frames for intuitionistic logic, consider a famous beautiful theorem of L. Maksimova.

- There are only seven intermediate logics which have interpolation

Is this still true if we take into account logics generated by reactive frames? The notion of 'logic' may not be the same!

A later section will provide tableaux for logics defined by finite frames.

D21

Definition 2.1 (Ordinary Kripke models for intuitionistic propositional logic) *A Kripke model has the form* $\mathbf{m} = (S, R, R^*, a, h)$ *where* S *is a non-empty set of worlds and* $R \subseteq S \times S$ *is a binary relation on* S. R^* *is the reflexive and transitive closure of* R,[2] $a \in S$ *is the actual world and* h *is an assignment, giving for each atomic* q *a subset* $h(q) \subseteq S$.

The system $\mathbf{F} = (S, R, R^*, a)$ *is called a Kripke frame.*

The following holds

$$t \in h(q) \wedge tR^*s \Rightarrow s \in h(q). \tag{$*$}$$

The satisfaction relation is defined as follows, for $t \in S$ *and a propositinal formula* A.

1. $t \vDash q$ *iff* $t \in h(q)$, *for* q *atomic*
2. $t \vDash A \wedge B$ *iff* $t \vDash A$ *and* $t \vDash b$
   $t \vDash A \vee B$ *iff* $t \vDash A$ *or* $t \vDash B$.
3. $t \vDash A \to B$ *iff for all* s *such that* tR^*s, *if* $s \vDash A$ *then* $s \vDash B$
4. $t \vDash \neg A$ *iff for all* s *such that* tR^*s *we have* $s \nvDash A$.
5. *We say* A *holds in the model iff* $a \vDash A$.
6. *Note we have not used* R *at all, only* R^*.

   This presentation is for later comparison.

T22

Theorem 2.2 *Intuitionistic propositional logic is complete for the sematnics of Definition 2.1.*

Proof. Well known result. ■

D23

Definition 2.3 (Pre-reactive Kripke models) *A pre-reactive Kripke model has the form* (S, S^*, R, a, h) *where* S *is a non-emtpy set,* $a \in S, R \subseteq S \times S$ *is a binary relation (not necessarily reflexive nor transitive) and* S^* *is the set of all* R *increasing sequences* β *of elements from* S *of the form* $\beta = (a_0, a_1, \ldots, a_n)$ *such that* $a_0 = a$ *and for* $i = 0, \ldots, n-1$ *we have* $a_i R a_{i+1}$. *We denote* a_n *by* $|\beta|$. h *is an assignment giving for each atomic* q *a subset* $h(q) \subseteq S$ *such that*

$$t \in h(q) \wedge tRs \Rightarrow s \in h(q).$$

We define satisfaction for β, *a sequence in* S^*, *as follows. (We need the notion of:* β' *is an extension of* β *iff* β *is an initial sequence of* β', *i.e.* $\beta = (a, t_1, \ldots, t_k)$ *and* $\beta' = (a, t_1, \ldots, t_{k+n})$ $n \geq 0$*).*

[2] We have xR^*y iff $x = y$ or xRy or $\exists t_1, \ldots, t_m(xRt_1 \wedge t_1Rt_2 \wedge \ldots t_{m-1}Rt_m \wedge t_mRy)$, for some $m \geq 1$.

1. *$\beta \vDash q$ if $|\beta| \in h(q)$, for q atomic.*

2. *$\beta \vDash A \wedge B$ iff $\beta \vDash A$ and $\beta \vDash B$.*
   $\beta \vDash A \vee B$ iff $\beta \vDash A$ or $\beta \vDash b$.

3. *$\beta \vDash A \to B$ iff for every extension β' of β, we have that if $\beta' \vDash A$ then $\beta' \vDash B$.*

4. *$\beta \vDash \neg A$ iff for all extension β' of β we have $\beta' \nvDash A$.*

5. *A holds in the model if $a \vDash A$.*

L24

Lemma 2.4 *Let $\mathbf{m} = (S, S^*, R, a, h)$ be a model. Let R^* be the reflexive and transtivie closure of R. Consider the model $\mathbf{n} = (S, R, R^*, a, h)$. Then we have for every $\beta \in S^*$*

$$\beta \vDash A \text{ in } \mathbf{m} \text{ iff } |\beta| \vDash A \text{ in } \mathbf{n}.$$

Proof. By induction on A. The crucial point is $A \to B$.

1. Assume $\beta \vDash A \to B$. Then for all β' extending β we have that $\beta' \vDash A$ implies $\beta' \vDash B$.

   By the induction hypothesis we have if $|\beta'| \vDash A$ then $|\beta'| \vDash B$. We now show $|\beta| \vDash A \to B$.

   Let s be such that $|\beta|R^*s$, then $|\beta| = s$ or $sR|\beta|$ or there exist $s_0, s_1, \ldots, s_m$ such that $s_0 = |\beta|$ and $s_m = s$ and for $i = 0, \ldots, m-1$ we have s_iRs_{i+1}. Hence $\beta' = \beta * (s_1, \ldots, s_m)$ is an extension of β with $|\beta'| = s_m = s$. Therefore if $s \vDash A$ then $|\beta'| \vDash A$, hence $\beta' \vDash A$ by the induction hypothesis, hence $\beta' \vDash B$, hence $|\beta'| \vDash B$, i.e. $s \vDash B$.

2. Now assume $|\beta| \vDash A \to B$. Let β' extend β. Hence $|\beta|R^*|\beta'|$. So if $\beta' \vDash A$ then $|\beta'| \vDash A$ hence $|\beta'| \vDash B$ hence $\beta' \vDash B$.

The proof for $\neg A$ is similar. ■

R25

Remark 2.5 *The second type of model is easier to turn reactive. In this new type of model, we view the evaluation of $A \to B$ at a node t as 'going along the relation R and at whatever point t' we reach, if $t' \vDash A$ then $t' \vDash B$.' So in this definition we actually have to traverse the arcs of the model.*

Note that R needs not be reflexive nor transitive. We get these properties from the evaluation process. So consider Figures 3 and 4.

Figure 3 gives $S_1 = \{t, s\}, R_1 = \{(t, s)\}$. Figure 4 gives $S_2 = \{t, s\}, R_2 = \{(t, t), (t, s)\}$. We have $S_1^* = \{(t), (t, s)\}$ and $S_2^* = \{(t), (t, \underbrace{\ldots}_{m \text{ times}}, t, s) | m = 0, 1, 2, \ldots\}$. S_1^* corresponds to the ordinary Kripke model with two linear points $1 < 2$ (as in Figure 3) and S_2^* corresponds to the ordinary Kripke model with the infinite comb of Figure 5 such that the assignments to the points in $\{0, 1, \ldots\}$ are all indentical (representing the point t and also the assignment to the points in $\{w_0, w_1, w_2, \ldots, w_n \ldots\}$ are all identical, (representing the point s).

Figure 3: A1

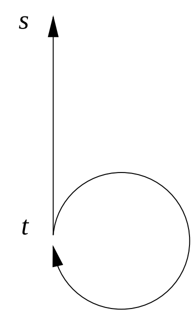

Figure 4: A2

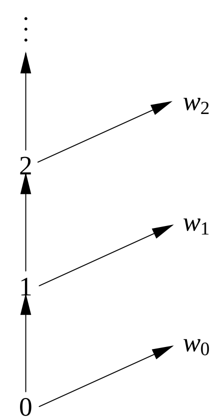

Figure 5: A3

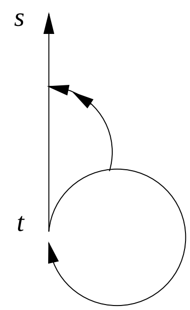

Figure 6: A5

D26

Definition 2.6 (Reactive intuitionistic Kripke frame) *A reactive intuitionistic Kripke frame has the form* $(S, \mathbb{R}, a)$*, where* $a \in S$ *and* $\mathbb{R}$ *is a set of pairs of the form*

1. $(x, y) \in S \times S$ *called arrows*

2. $((x, y), (u, v)) \in S^2 \times S^2$ *called reactive double arrows.*

E27

Example 2.7 *A reactive frame, see Figure 6.*
We have $\mathbb{R} = \{(t, s), (t, t), ((t, t), (t, s))\}$ *and* $a = t$

D28

Definition 2.8 (How reactivity operates)

1. *Let* $\mathbf{m} = (S, \mathbb{R}, a)$ *be a frame. Let* $(t, s) \in \mathbb{R}$ *be an arrow. Define* $\mathbb{R}_{(t,s)}$ *to be* $\mathbb{R}_{(t,s)} = \mathbb{R} - \{(x, y) \mid ((t, s), (x, y)) \in \mathbb{R}\}$*.* $\mathbb{R}_{(t,s)}$ *is the result of traversing the arc* $t \to s$ *and cancelling all connections as indicated by double arrows emanating from* $t \to s$*.*

2. *Let* $\beta = (a, t_1, \ldots, t_k)$*. We now define* $\mathbb{R}_\beta$ *by induction on* k*. For* $k = 1$*, we let* $\mathbb{R}_\beta = \mathbb{R}_{(a,t_1)}$*, provided* $(a, t_1) \in \mathbb{R}$*.*

   Assume $\mathbb{R}_\beta$ *has been defined for* $\beta = (a, t_1, \ldots, t_k)$ *and assume* $(t_k, t_{k+1}) \in \mathbb{R}_\beta$*. Define* $\mathbb{R}_{\beta'}$ *for* $\beta' = (a, t_1, \ldots, t_{k+1})$ *to be* $\mathbb{R}_{\beta'} = (\mathbb{R}_\beta)_{(t_k, t_{k+1})}$*.*

3. *Let* $\mathbb{R}_\beta$ *be a reactive relation as defined in (2) where* $\beta = (a, t_1, \ldots, t_k)$*.*

   Let

   $$\begin{aligned} \beta_1 &= (a, t_1, \ldots, t_{k+1}) \\ &\vdots \\ \beta_n &= (a, t_1, \ldots, t_{k+n}) \end{aligned}$$

   We say β_n *is a legitimate extension of* β *iff the following holds.*

   - $(t_k, t_{k+1}) \in \mathbb{R}_\beta$
   - $\mathbb{R}_{\beta_1}$ *is obtained from* $\mathbb{R}_\beta$ *as in (1) above*

     $\vdots$
   - $(t_{k+n-1}, t_{k+n}) \in \mathbb{R}_{\beta_{n-1}}$

- *$\mathbb{R}_{\beta_n}$ is obtained from $\mathbb{R}_{\beta_{n-1}}$ as in (1) above.*

L29

Lemma 2.9 *Let $(S, \mathbb{R}_{(a)}, a)$ be a reactive Kripke frame. Let β be a legitimate extension of (a). Write $\beta = (a, t_1, \ldots, t_k)$. Then we have $aRt_1, t_1Rt_2, \ldots, t_{k-1}Rt_k$, where $R = \{(x,y)|(x,y) \in S \times S \cap \mathbb{R}\}$.*

Proof. By induction on k. For $k = 1$ we do have aRt_1.

We see from the construction of any $\mathbb{R}_\beta$ that we have $\mathbb{R}_\beta \subseteq \mathbb{R}$.

Hence if $\beta' = \beta * (t_{k+1})$ with $|\beta| = t_k$ and $(t_k, t_{k+1}) \in \mathbb{R}_\beta$, then we have t_kRt_{k+1}. ■

D210

Definition 2.10 (Satisfaction in a reactive model)

1. *Let $(S, \mathbb{R}, a)$ be a reactive frame. Let $R = \{(x,y)|x, y \in S$ and $(x,y) \in \mathbb{R}\}$. Let h be an assignment such that*
   $t \in h(q)$ and tRs implies $s \in h(q)$.

   Let β be a legitimate extension of (a). Let $\mathbb{R}_\beta$ be the corresponding relation. Let $\mathbf{m}_\beta = (S, \mathbb{R}_\beta, |\beta|, h)$. Note that if $\gamma = (|\beta|, t_1, \ldots, t_m)$ is a legitimate extension of $|\beta|$ in $\mathbf{m}_\beta$, then $\beta^1 = \beta * (t_1, \ldots, t_m)$ is a legitimate extension of (a) in $(S, \mathbb{R}, a)$.

2. *We define satisfaction as follows:*

   - *$\beta \vDash q$ iff $|\beta| \in h(q)$, for q atomic*
   - *$\beta \vDash A \wedge B$ iff $\beta \vDash A$ and $\beta \vDash B$*
   - *$\beta \vDash A \vee B$ iff $\beta \vDash A$ or $\beta \vDash B$*

3. *$\beta \vDash A \to B$ in $\mathbf{m}_\beta$ iff for every legitimate extension β' of β, if $\beta' \vDash A$ in $\mathbf{m}_{\beta'}$ then $\beta' \vDash B$ in $\mathbf{m}_{\beta'}$.*

4. *$\beta \vDash \neg A$ in $\mathbf{m}_\beta$ iff for all legitimate extensions β' of β we have $\beta' \nvDash A$ in $\mathbf{m}_{\beta'}$.*

5. *We say $(S, \mathbb{R}, a, h) \vDash A$ iff $(a) \vDash A$.*

L211

Lemma 2.11 *Let $(S, \mathbb{R}, a, h)$ be a model and assume that $\beta \vDash A$ in $\mathbf{m}_\beta$ and that β' is a legitimate extension of β, then $\beta' \vDash A$ in $\mathbf{m}_{\beta'}$.*

Proof. By induction on A.

1. For A atomic, this follows from a previous Lemma 2.9.

2. The cases of $\wedge$ and $\vee$ are immediate.

3. Assume $\beta \vDash A \to B$, then for any legitimate extension β' of β, if $\beta' \vDash A$ in $\mathbf{m}_{\beta'}$ then $\beta' \vDash B$ in $\mathbf{m}_{\beta'}$.

   But now since any legitimate extension β'' of β' is also a legitimate extension of β we get that $\beta' \vDash A \to B$ in $\mathbf{m}_{\beta'}$.

4. The case of $\neg$ is similar.

■

E212

Example 2.12 (Satisfaction in the frame of Figure 6) *Let q be atomic. Let $h(q)$ be $\{s\}$. Then*

1. $(t) \nvDash q$
2. $(t) \nvDash \neg q$
3. $(t) \nvDash \neg q \to q$.

(1) and (2) are clear. To show (3), note that $\beta = (t, t)$ is a legitimate extension of (t) and $\mathbb{R}_{(t,t)}$ is

$$\{(t, t), ((t, t), (t, s)\}.$$

In $(S, \mathbb{R}_{(t,t)}, (t, t)$ we have $(t, t) \vDash \neg q$ but $(t, t) \nvDash q$.

L213

Lemma 2.13 *The logic defined by reactive satisfaction is intuitionistic logic.*

Proof.

1. Since ordinary pre-reactive models are reactive models, (by Definition 2.3 and Lemma 2.4) the logic is not stronger than intuitionistic logic.
2. From Lemma 2.11 we see the logic is not weaker either.

■

T214

Theorem 2.14 *The logic of the frame Figure 6 is not complete for any class of ordinary intuitionistic Kripke frames.*

Proof. The proof has four parts, (A)–(D).

(A) The following formula holds in the frame of Figure 6, under any h,

1. $x \vee (x \to (q \vee \neg q)), x, q$ atomic
2. $(x \to y) \vee (y \to x), x, y$ atomic.

We check (1).
If $x = \bot$ at (t) then to falsify $x \to (q \vee \neg q)$ we need to go to (t, s) where x can hold.
We cannot go to (t, t) because $(t, t) \nvDash x$. At (t, s) clearly $q \vee \neg q$ holds.
We check (2).
Assume $x \to y$ is false at (t). Then we have either
$(t, \underbrace{\ldots}_{m \text{ times}}, t) \vDash x$ and $(t, \underbrace{\ldots}_{m \text{ times}}, t) \nvDash y$ or
$(t, \underbrace{\ldots}_{n \text{ times}}, t, s) \vDash x$ and $(t, \underbrace{\ldots}_{n \text{ times}}, t, s) \nvDash y, m, n \geq 0$.
In the first case, we have $(t) \vDash x$ and hence $(t) \vDash y \to x$.

In the second case we have $(t, s) \vDash x, (t, s) \nvDash y$. Hence $(t) \vDash \neg y$ and so $(t) \vDash y \to x$.

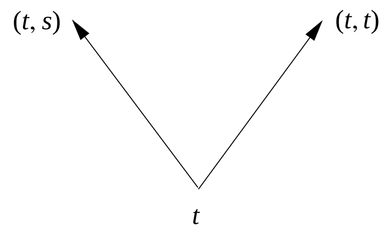

Figure 7: A6

(B) We now show that any ordinary frame which satisfies (1) and (2) under any h must be either a single point or the frame of Figure 3.

This is well known because otherwise either (1) or (2) can be falsified. (1) is falsified by a 3 point chain and (2) by a two point anti-chain. So we can have neither.

(C) We now show that in Figure 3 or in a single point (3) must hold.

3. $x \vee \neg x \vee (\neg x \to x)$.

To falsify $x \vee \neg x$ we need Figure 3 with $t \nvDash x$ and $s \vDash x$ but from the latter it follows that $t \vDash \neg\neg x$ holds and hence $t \vDash \neg x \to x$.

(D) Our proof is concluded because Example 2.12 shows that (3) can be falsified in the frame of Figure 6.

■

R215

Remark 2.15 *It is helpful to have another view of Figure 6. The frame has two paths, as in Figure 7*

We can view Figure 7 as an ordinary 3 point Kripke model with the understanding that the assignment at t and (t,t) is the same, i.e. for every q $t \in h(q)$ iff $(t,t) \in h(q)$.

This is common to reactive models, that they can be 'unfolded' as models of paths with restrictions on the assignments. [2] studies such models. We examine this notion in the next section.

3 Folding reactive frames

We saw in the last section that the reactive frame of Figure 6 can be unfolded into the ordinary frame of Figure 7, with the added understanding that the points (t) and (t,t) must give the same values to the atoms. This unfolding process can be done in a systematic manner, and it seems to have significance for developing Beth tableaux for reactive intuitionistic logics. So in this section we study it in detail. We are going to unfold and then fold again.

Let $(S, \mathbb{R}, a)$ be a reactive frame. Let β be a legitimate path of the form $\beta = (a, t_1, \ldots, t_k)$. We saw that we can calculate $\mathbb{R}_\beta \subseteq \mathbb{R}$. Let $\mathbf{F}_\beta = (S, \mathbb{R}_\beta, |\beta|)$.

If S is finite and β ranges over all legitimate paths, we get only a finite number of different frames $\mathbf{F}_\beta$. Let us take advantage of this.

Definition 3.1 (Path equivalence relation) *Let* $\mathbf{m} = (S, \mathbb{R}, a, h)$ *be a reactive model. Define an equivalence relaton on the paths of the model as follows:* D31

- $\beta \equiv \gamma$ *iff* $|\beta| = |\gamma|$ *and* $\mathbb{R}_\beta = \mathbb{R}_\gamma$.

Let Ω *be the set of equivalence classes,* $\{\beta| \equiv |\beta$ *a legitimate path extending* $(a)\}$.

Then Ω *is finite.*

Define ρ *on* Ω *as follows:*

- $\beta| \equiv \rho\gamma| \equiv$ *iff for some* $\beta_1 \equiv \beta$ *and* $\gamma_1 \equiv \gamma$ *we have* γ_1 *is a legitimate extension of* β_1.

Lemma 3.2 ρ *is reflexive and transitive.* L32

Proof. Reflexivity is not a problem. We show transitivity.

Claim: If β_1 is a legitimate extension of α_1 and $\beta_1 \equiv \beta_2$ and γ_2 is a legitimate extension of β_2, then there exists a $\gamma_1 \equiv \gamma_2$ such that γ_1 is a legitimate extension of α_1.

We now prove the claim:

1. We have $\beta_1 = \alpha_1 * (t_1, \ldots, t_k)$ where $|\alpha_1| R t_1 \wedge t_1 R t_2 \wedge \ldots t_{k-1} R t_k$ and $|\beta_1| = t_k$.

2. Since $\beta_1 \equiv \beta_2$ we get that $\mathbb{R}_{\beta_1} = \mathbb{R}_{\beta_2}$.

3. We also have $(s_1, \ldots, s_m)$ such that $\gamma_2 = \beta_2 * (s_1, \ldots, s_m)$ and $|\beta_2| R s_1 \wedge s_1 R s_2 \wedge \ldots s_{m-1} R s_m$ and $|\gamma_2| = s_m$.

   Consider the path
   $$\gamma_1 = \alpha_1 * (t_1, \ldots, t_k) * (s_1, \ldots, s_m).$$
   It is clear that $|\gamma_1| = |\gamma_2|$.

   We want to show that
   $$\mathbb{R}_{\gamma_1} = \mathbb{R}_{\gamma_2}$$
   Observe that $\mathbb{R}_{\beta_1} = \mathbb{R}_{\beta_2}$. Since γ_1 is an extension of β_1 along the sequence of nodes $(s_1, \ldots, s_m)$ and γ_2 is the extension of β_2 along the sequence $(s_1, \ldots, s_m)$ (same sequence) and they both start at $|\beta_1| = |\beta_2|$ with $\mathbb{R}_{\beta_1} = \mathbb{R}_{\beta_2}$, then they end up at the same relation, namely $\mathbb{R}_{\gamma_1} = \mathbb{R}_{\gamma_2}$. Hence $\gamma_1 \equiv \gamma_2$.

   We now finish the proof of the theorem:

4. Since $\gamma_1 \equiv \gamma_2$ and γ_1 extends α_1, we get $\alpha_1 / \equiv \rho\ \gamma_2 / \equiv$

■

Lemma 3.3 *Let* $\mathbf{m} = (S, \mathbb{R}, a, h)$ *and let* $\equiv, \Omega, \rho$ *be as in Definition 3.1. Consider* $\mu = (\Omega, \rho, \mathbf{h})$*, as an ordinary Kripke model, where* $\mathbf{h}$ *is defined by* L33

$$\alpha / \equiv \in \mathbf{h}(q) \text{ iff } |\alpha| \in h(q).$$

Then for any A we have:

$$\alpha \vDash A \text{ in } \mathbf{m} \text{ iff } \alpha / \equiv\ \vDash A \text{ in } \mu.$$

Proof. By induction on A.

1. For q atomic this holds by the definition of $\equiv$.

2. The key case is that of $\rightarrow$.

Assume $\alpha \nvDash A \rightarrow B$, then for some β which is a legitimate extension of α we have $\beta \vDash A$ and $\beta \nvDash B$. But we also have in this case that $\alpha/\equiv\ \rho\beta/\equiv$ and by the iduction hypotheis, $\beta/\equiv\ \vDash A$ and $\beta/\equiv\ \nvDash B$.

Now assume $\alpha/\equiv\ \nvDash A \rightarrow B$. Then for some $\gamma/\equiv$ we have $\alpha/\equiv\ \rho\ \gamma/\equiv$ and $\gamma/\equiv\ \vDash A$ and $\gamma/\equiv \nvDash B$. Therefore for some $\alpha_1 \equiv \alpha$ and $\gamma_1 \equiv \gamma$ we have $\gamma_1 = \alpha_1 * (t_1, \ldots, t_k)$ and γ_1 is a legitimate extension of α_1. Hence since $\gamma_1 \equiv \gamma$, we get $\gamma_1/\equiv\ \vDash A$ and $\gamma_1/\equiv\ \nvDash B$. By the induction hypothesis we have $\gamma_1 \vDash A$ and $\gamma_1 \nvDash B$.

Now look at the two models $\mathbf{m}_\alpha = (S, \mathbb{R}_\alpha, |\alpha|)$ and $\mathbf{m}_{\alpha_1} = (S, \mathbb{R}_{\alpha_1}, |\alpha_1|)$. Since $\alpha_1 \equiv \alpha$, these two models are the same. So having $\gamma_1 = \alpha_1 * (t_1, \ldots, t_k)$ with $\gamma_1 \vDash A$ and $\gamma_1 \nvDash B$ in $\mathbf{m}_{\gamma_1}$ implies that for $\delta = \alpha * (t_1, \ldots, t_k)$ we also have $\delta \vDash A$ adn $\delta \nvDash B$ in $\mathbf{m}_\alpha$. Hence $\alpha \nvDash A \rightarrow B$ in $\mathbf{m}$. ■

R34

Remark 3.4 (Folding reactive frames) *We started with a reactive model* $\mathbf{m} = (S, \mathbb{R}, a, h)$ *and converted it to a special model* $\mu = (\Omega, \rho, a/\equiv, \mathbf{h})$. *This model is special and we want to highlight some of its properties.*

The elements of Ω *are equivalence classes of legitimate sequences* β *of* $\mathbf{m}$. *We have*

- $\alpha \equiv \beta$ *iff* $|\alpha| = |\beta|$ *and* $\mathbb{R}_\alpha = \mathbb{R}_\beta$.

Consider the new relation $\approx$ *on* Ω

- $\alpha/\equiv\ \approx \beta/\equiv$ *iff* $|\alpha| = |\beta|$

We can have that many different $\equiv$ *classes* $\alpha/\equiv, \beta/\equiv$ *are* $\approx$ *equivalent. This is because to be in the same* $\equiv$ *class we also need* $\mathbb{R}_\alpha = \mathbb{R}_\beta$.

Formally we now have a Kripke model $(\Omega, \rho, \approx, \mathbf{h})$ *with an equivalence relation* $\approx$ *and the property*

$$x \approx y \rightarrow x \in \mathbf{h}(q) \textit{ iff } y \in \mathbf{h}(q) \qquad (*)$$

What other properties does it have? How does $\approx$ *relate to* ρ*?*

Let us check.

Suppose $\alpha/\equiv\ \rho\ \beta_i/\equiv, i = 1, 2$ *and* $\beta_1/\equiv$ *is* $\approx$ *equivalent to* $\beta_2/\equiv$ *but they are different points. This means that* $\mathbb{R}_{\beta_1} \neq \mathbb{R}_{\beta_2}$. *This means that there are two different sequences of points* $(t_1, \ldots, t_k), (s_1, \ldots, s_m)$ *such that* $\mathbb{R}_{\alpha*(t_1,\ldots,t_k)} \neq \mathbb{R}_{\alpha*(s_1,\ldots,s_m)}$ *and* $\alpha * (t_1, \ldots, t_k) \equiv \beta_1$ *and* $\alpha * (s_1, \ldots, s_m) \equiv \beta_2$.

This means that the following holds from the point of view of $(\Omega, \rho, \approx, a/\equiv, \mathbf{h})$.

> *(**)* $[x\rho y_1 \wedge x\rho y_2 \wedge y_1 \approx y_2] \Rightarrow$ *[there are two different sequences* $u_1, \ldots, u_k, v_1, \ldots, v_m$ *such that* $x\rho u_1 \wedge u_1\rho u_2 \wedge \ldots u_k\rho y_1$ *and* $x\rho v_1 \wedge v_1\rho v_2 \wedge \ldots \wedge v_1 v_m \rho y_2]$.

D35

Definition 3.5 (Folded Kripke models) *1. A folded Kripke model has the form* $(S\rho, \approx, a, h)$. *It is a Kripke model* $(S\rho, a, h)$ *with an equivalence relation* $\approx$ *satisfying condition (*) of Remark 3.4.*

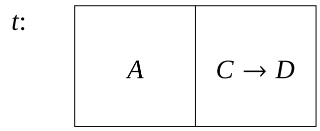

Figure 8: tab1

2. *A folded Kripke model is reactive if it satisfies also (**) of Remark 3.4. Such models are the results of folding a reactive Kripke model.*

C36

Conjecture 3.6 *Every reactive folded Kripke model can be obtained from a reactive Kripke model by the process described above.*

Proof. We shall not prove this now. ■

E37

Example 3.7 (Figure 6) *We saw that the reactive Kripke frame of Figure 6 can be presented as the folded Kripke frame of Figure 7. We have in Figure 7:*

$$\begin{aligned} \Omega &= \{t, s, (t,t)\} \\ \rho &= \{(t,t), (t,s), (s,s), (t,(t,t)), ((t,t),(t,t))\} \end{aligned}$$

and we have $t \approx (t,t)$.

R38

Remark 3.8 *Let us see what is the status of folded Kripke fames in terms of Heyting algebras. An ordinary Kripke frame $\mathbf{m} = (\Omega, \rho)$ gives rise to a free Heyting algebra $\mathbf{H_m}$. When we add an eqivalence relation $\approx$ to form $\mu = (\Omega, \rho, \approx)$ we are adding some equalities among the free generators of $\mathbf{H_m}$. These equalities generate a congruence relation $\cong$ on $\mathbf{H_m}$. If we let $\mathbf{H}_\mu = \mathbf{H_m}/\cong$ then we get the algebra corresponding to μ. It is not a free algebra.*

4 Reactive Tableaux

We begin by explaining the intuitive idea of tableaux for reactive logics. Consider the tableau of Figure 8

The label of the tableau is t. This is usually the name of the possible world we are dealing with. A is on the left and so we want to make $t \vDash A$ and $C \to D$ is on the right hand side of the tableau, so we want to make $t \nvDash C \to D$.

To do the latter we need an accessible world s. such that tR^*s and $s \vDash A$ and $s \nvDash B$.

This means that we move into the following tableau in Figure 9

A carries on into s and in s, we put C on the left and D on the right.

This is the usual tableau process for intuitionistic logic. If we have $\wedge$ and $\vee$ in the language, we might get different alternatives (tableau splitting). Let us assume our language contains only $\to$ so that we can concentrate on the differences between ordinary tableaux and reactive tableaux, without the complexity generated by the presence of $\wedge$ and $\vee$.

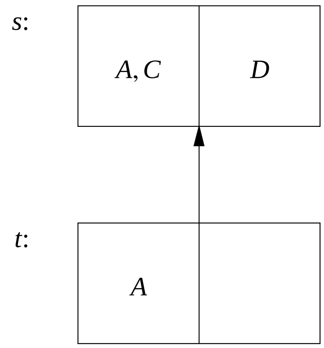

Figure 9: tab2

In the case of reactive semantics the tableau will have labels α, β which are paths. So $t = \alpha$ and $s = \beta$. This is not an essential difference. The difference is essential in the reactive case because we must require that β is a legitimate extension of α. To do that we must record what $\mathbb{R}_\beta$ is.

So to simplify even further and allow us to present the essential ideas of the reactive tableau let us assume our logic has a fixed finite reactive frame, $(S, \mathbb{R}, a)$.

In this case we get the following simplifications:

(S1.) Since the frame is finite and for any legitimate sequence β, $\mathbb{R}_\beta$ is smaller than $\mathbb{R}$, there is only a finite number of frames $(S, \mathbb{R}_i)$, $i = 1, \ldots, n$, that are at play,

(S2.) We can move to the finite folded Kripke frame $\mu = (\Omega, \rho, \approx, a)$ and do our tableaux on μ. This is significantly simpler because μ is like an ordinary Kripke frame with the additional simple condition (simple from the tableaux point of view) imposed by $\approx$.

The next definition gives the notion of Beth tableaux for the implicational fragment. Note three facts:

Fact 1

Every wff can be put in the form

$$E = [A_1 \to (\ldots \to (A_n \to q)\ldots)]$$

where q is atomic and each A_i is of the same form as E.

Fact 2

We need only two tableaux rules:

- To make E false (E on the right) at world t, find a world s such that $t\rho s$ and put all A_i on the left and q on the right.
- To make E true on the left at t when q is on the right at t, we must move one of A_i to the right of t to make it false.

Fact 3

To accommodate $\approx$ we make some adjustments to the usual intuitionistic tableaux rules.

D41

Definition 4.1 (Beth tableaux for folded Kripke frames) *Let $\mu = (\Omega, \rho, \approx)$ be a folded frame.*

1. *A tableau for μ is a pair of functions (τ^+, τ^-) from Ω into the set of wffs satisfying the following conditions:*

   (a) *If $t\rho s$ holds in μ then $\tau^+(t) \subseteq \tau^+(s)$*

   (b) *If $t \approx s$ holds then $\tau^\pm(t)$ and $\tau^\pm(s)$ contain the same atoms (respectively)*

2. *Let $\mathbf{T}$ be a family of tableaux τ as in (1). Let $\tau \in \mathbf{T}$. We define an operation which will split τ into several alternatives and we will replace τ in $\mathbf{T}$ by these alternatives to obtain a new family $\mathbf{T}'$.*

   Right hand operation
   Choose $t \in \Omega$ and choose E in $\tau^-(t)$. Get the following alternatives $\tau_s^\pm$ to replace τ by $\{\tau_s | s \in \Omega$ and $tRs\}$ where

$$\begin{aligned}
\tau_s^-(x) &= \begin{cases} \tau^-(x), \text{ for } x \not\approx s \\ t^-(x) \cup \{q\} \text{ for } x \approx s, x \neq t \end{cases} \\
\tau_s^-(t) &= \tau^-(t) - \{E\} \cup \{q\} \text{ if } x = t \\
\tau_s^+(x) &= \begin{cases} \tau^+(x) \text{ if } x \not\approx s \\ \tau^+(x) \cup \{A_i | A_i \text{ atomic}\} \text{ if } x \approx s \text{ and } x \neq s \\ \tau^+(t) \cup \{A - i\} \text{ if } x = t \end{cases}
\end{aligned}$$

   Left hand operation
   Choose $t \in \Omega$ and choose E in $\tau^+(t)$ such that $q \in \tau^-(t)$.

   Recall that $E = [A_1 \to (\ldots \to (A_n \to q))]$.

   For each A_i form the following tableau τ_i.

$$\begin{aligned}
\tau_i^+(x) &= \tau^+(x), x \in \Omega \\
\tau^-(x) &= \begin{cases} \tau^-(x) \cup \{A_i\}, \text{ if } A_i \text{ is atomic and } x \approx t. \\ \tau^-(x) \text{ if } x \not\approx t \\ t^-(t) \cup \{A_i\} \text{ if } x = t. \end{cases}
\end{aligned}$$

3. *A tableau τ is closed if for some $t \in \Omega$ we have $\tau^+(t) \cap \tau^-(t) \neq \emptyset$.*

4. *$\mathbf{T}$ is closed iff all of its alternatives are closed.*

T412

Theorem 4.2 (Completeness) *The above procedure is complete for the logic defined by the fame μ.*

Proof. Modify a proof for the case of an ordinary intuitionistic frame. ∎

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