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Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL.

Dedicated to the memory of Willem Johannes Blok

Abstract. Substructural logics have received a lot of attention in recent years from the communities of both logic and algebra. We discuss the algebraization of substructural logics over the full Lambek calculus and their connections to residuated lattices, and establish a weak form of the deduction theorem that is known as parametrized local deduction theorem. Finally, we study certain interpolation properties and explain how they imply the amalgamation property for certain varieties of residuated lattices.

Keywords: Substructural logic, pointed residuated lattice, algebraic semantics, parametrized local deduction theorem, interpolation.

1. Introduction

The Gentzen system **FL** of full Lambek calculus is obtained from the Gentzen system **LJ** for intuitionistic propositional logic by removing three structural rules: the rules of exchange, weakening and contraction. The study of *substructural logics* (over **FL**) has been developed extensively in the last decade and close relations between substructural logics over **FL** and subvarieties of the variety $\mathcal{FL}$ of *pointed residuated lattices* have been observed. In fact, every substructural logic **L** over **FL** determines a subvariety $\mathcal{V}(\mathbf{L})$ and, conversely, every subvariety $\mathcal{V}$ of $\mathcal{FL}$ determines a substructural logic $\mathbf{L}(\mathcal{V})$. Moreover, the maps $\mathcal{V}$ and $\mathbf{L}$ are mutually inverse dual isomorphisms between the lattice of substructural logics over **FL** and the lattice of subvarieties of the variety $\mathcal{FL}$.

Although the etymology of the name "substructural logics" refers to sequent calculi that lack some of the structural rules, the first concrete definition was given in [40] by the second author and includes logics that are axiomatic extensions of such systems. In particular, substructural logics include classical, intuitionistic, multi-valued, basic, relevant and (fragments

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of) linear logic. Extending ideas in [40], we introduce the deducibility relation $\vdash_{\mathbf{L}}$ associated with a substructural logic $\mathbf{L}$, which turns out to be a *finitary, substitution invariant consequence relation* in the sense of algebraic logic. Moreover, we show that, for every substructural logic $\mathbf{L}$, the relation $\vdash_{\mathbf{L}}$ is *algebraizable* in the sense of Blok and Pigozzi [7] and the subvariety $\mathcal{V}(\mathbf{L})$ of $\mathcal{FL}$ is the *equivalent algebraic semantics* for $\vdash_{\mathbf{L}}$.

Connections of congruences of residuated lattices to their *convex normal subalgebras* have been studied by Blount and Tsinakis [12] and also by Jipsen and Tsinakis [28]. Additionally, Blok and Pigozzi [7] explore the connection between *deductive filters* of algebraizable consequence relations and congruence relations of the equivalent algebraic semantics. Our investigation is based on these two studies. In particular, we give a number of characterizations of deductive filters and describe their generation process.

As an application of our analysis, we obtain a concrete form of a *parametrized local deduction theorem* in the sense of [15] for all substructural logics, and a *local deduction theorem* for commutative substructural logics, i.e. logics that have the exchange rule.

We use the algebraization result and the local deduction theorem to derive a number of important logical consequences. In the last section, we establish some basic results on various forms of the interpolation property for certain substructural logics, and the amalgamation property for some varieties of residuated lattices, by applying our version of the local deduction theorem to the work of Czelakowski and Pigozzi [16] on abstract algebraic logic.

The above results suggest that the study of substructural logics is quite fertile and promising when developed in close connection with universal algebra and algebraic logic. In fact, we expand our research in this direction in our second paper on substructural logics [21], where we develop a comprehensive study of Glivenko-type theorems for substructural logics.

2. Pointed residuated lattices

For a language $\mathcal{L}$ of connectives, or operation symbols, (and constants) we identify *formulas* and *terms* over $\mathcal{L}$ and denote them by letters like ϕ, ψ, χ, σ or t, s, u, v depending on whether they are used in a logical or algebraic context; we denote the set of all formulas over $\mathcal{L}$ by $Fm_{\mathcal{L}}$. Note that the formulas over $\mathcal{L}$ form an algebra $\mathbf{Fm}_{\mathcal{L}}$, which is known as the *absolutely free algebra of type $\mathcal{L}$*.

We say that an algebra $\mathbf{A}$ over $\mathcal{L}$, or $\mathcal{L}$-algebra for brevity, *satisfies* the *equation* $s \approx t$, in symbols $\mathbf{A} \models s \approx t$, if $\mathbf{A}$ satisfies $(\forall \bar{x})(s(\bar{x}) \approx t(\bar{x}))$, where

$\bar{x}$ is the sequence of variables in the terms s and t. If E is a set of equations, $\mathrm{Mod}(E)$ denotes the class of algebras that satisfy all equations in E. For every term t, the *term operation* induced by t on $\mathbf{A}$ is denoted by $t^{\mathbf{A}}$. For basic definitions and results in universal algebra, see [11].

A *residuated lattice-ordered monoid*, or *residuated lattice*, is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and multiplication is residuated with respect to the order by the division operations $\backslash$, $/$; i.e., for all $a, b, c \in A$,

$$a \cdot b \leq c \;\Leftrightarrow\; a \leq c/b \;\Leftrightarrow\; b \leq a\backslash c.$$

A *pointed residuated lattice*, or *FL-algebra*, $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is an algebra such that $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ is a residuated lattice and 0 is an arbitrary element of A.

Residuated lattices were introduced in the late 30's by M. Ward and R. P. Dilworth, see [46], in a more restrictive sense than the one we consider here. These algebras, in their restricted form and in the augmented type of pointed residuated lattices have been studied extensively in the context of logic under the same name; see, for example, [26], [34]. For an introduction to connections of pointed residuated lattices and related structures to substructural logic, see [40].

The structure theory of residuated lattices, in the general sense that we consider here, was studied only recently by K. Blount and C. Tsinakis [12]. For a survey on residuated lattices, motivation and further references, see [28]; for additional results and a list of examples, see also [18]. A growing literature in the subject includes [30], [4], [19], [45], [9], [10], [23], [5], [24], [20].

We adopt the convention that in a (pointed) residuated-lattice term multiplication has priority over the division operations, which have priority over the lattice operations. So, for example, we write $x/yz \wedge u\backslash v$ for $[x/(yz)] \wedge (u\backslash v)$. We use $t \leq s$ to denote both the equality $t = t \wedge s$ and the equation $t \approx t \wedge s$, depending on whether t, s are elements of a residuated lattice or terms. It is easy to see that the equality $s = t$ holds in a residuated lattice iff the inequality $1 \leq s\backslash t \wedge t\backslash s$ holds.

The *opposite* t^{op} of a (pointed) residuated lattice term t is defined inductively on the complexity of t. For all terms s, t, we define $1^{\mathrm{op}} = 1$, $0^{\mathrm{op}} = 0$, $(s \cdot t)^{\mathrm{op}} = t \cdot s$, $(s\backslash t)^{\mathrm{op}} = t/s$, $(t/s)^{\mathrm{op}} = s\backslash t$, $(s \wedge t)^{\mathrm{op}} = t \wedge s$, and $(s \vee t)^{\mathrm{op}} = t \vee s$. Essentially, the opposite of a term is its "mirror image". We extend the definition to equations, by $(s \approx t)^{\mathrm{op}} = (t^{\mathrm{op}} \approx s^{\mathrm{op}})$, and to metalogical statements in the obvious way. Note that $(s \leq t)^{\mathrm{op}} = (t^{\mathrm{op}} \geq s^{\mathrm{op}})$. Examples of mutually opposite equations can be seen in each statement of the following lemma.

Lemma 2.1. *Residuated lattices satisfy the following identities:*

1. $x(y \vee z) \approx xy \vee xz$ *and* $(y \vee z)x \approx yx \vee zx$;
2. $x\backslash(y \wedge z) \approx (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x \approx (y/x) \wedge (z/x)$;
3. $x/(y \vee z) \approx (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x \approx (y\backslash x) \wedge (z\backslash x)$;
4. $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$;
5. $x(y/z) \leq (xy)/z$ *and* $(z\backslash y)x \leq z\backslash(yx)$;
6. $(x/y)/z \approx x/(zy)$ *and* $z\backslash(y\backslash x) \approx (yz)\backslash x$;
7. $x\backslash(y/z) \approx (x\backslash y)/z$;
8. $x/1 \approx x \approx 1\backslash x$;
9. $1 \leq x/x$ *and* $1 \leq x\backslash x$;
10. $x \leq y/(x\backslash y)$ *and* $x \leq (y/x)\backslash y$;
11. $y/((y/x)\backslash y) \approx y/x$ *and* $(y/(x\backslash y))\backslash y \approx x\backslash y$;
12. $x/(x\backslash x) \approx x$ *and* $(x/x)\backslash x \approx x$;
13. $(z/y)(y/x) \leq z/x$ *and* $(x\backslash y)(y\backslash z) \leq x\backslash z$.

Multiplication is order preserving, and the division operations are order preserving in the numerator and order reversing in the denominator. Moreover, if a residuated lattice has a least element $\bot$, then it has a greatest element $\top$, as well, and $\top = \bot/\bot = \bot\backslash\bot$.

The proofs of statements (1)-(9) and of the remark concerning the bounds can be found in [12]. The proof of the remaining statements is left to the reader.

Proposition 2.2. *[28], [12] An algebra of the appropriate type is a (pointed) residuated lattice iff it satisfies the lattice equations, the monoid equations and the following equations*

$$\begin{array}{ll} x(x\backslash z \wedge y) \leq z, & (y \wedge z/x)x \leq z \\ y \leq x\backslash(xy \vee z), & y \leq (z \vee yx)/x. \end{array}$$

Consequently, the class $\mathcal{RL}$ of residuated lattices and the class $\mathcal{FL}$ of pointed residuated lattices are varieties. We denote their subvariety lattices by $\mathbf{S}(\mathcal{RL})$ and $\mathbf{S}(\mathcal{FL})$, respectively.

A (pointed) residuated lattice is called *commutative*, if its monoid reduct is commutative; i.e., if it satisfies the identity $xy \approx yx$. It is called *integral*, if its lattice reduct has a top element and the latter coincides with the multiplicative identity 1; i.e., if it satisfies $x \leq 1$. Finally, it is called *contractive*, if it satisfies the identity $x \leq x^2$. It is easy to see that in a residuated lattice commutativity is equivalent to $x/y \approx y\backslash x$; in this context we write $x \to y$ for $x\backslash y$.

3. Substructural logics and algebraization

In this section, we define substructural logics and prove that varieties of pointed residuated lattices constitute equivalent algebraic semantics for their deducibility relations in the sense of [7]. Substructural logics and their deducibility relations have not been explicitly defined before, except for a brief discussion in [40], despite the fact that the term has been used in many places in an informal way.

3.1. Substructural logics and their deducibility relations

Let $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1, 0\}$ be the language of pointed residuated lattices. By **FL** we denote both the full Lambek sequent calculus over $\mathcal{L}$, given by the rules and axioms in Figure 1, as well as the set of formulas provable in it; see below. The definition of **FL** appeared for the first time in [36].

A particular *instance* of each of the following rules is obtained by replacing the lower case letters of the (Greek) alphabet by formulas and the upper case letters by finite sequences of formulas. Note that for every *sequent* $\Gamma \Rightarrow \alpha$ the right-hand side α consists of a single formula and the left-hand side Γ is a finite, possibly empty, sequence of formulas.

Note that (instances of) sequents $\Gamma \Rightarrow \alpha$ can be identified with pairs (Γ, α) and (instances of) rules of inference can be identified with pairs (S, s), where $S \cup \{s\}$ is a set of sequents. Nevertheless, we follow the standard notation with the separator $\Rightarrow$ and the fraction notation.

As usual, a *proof* in **FL** of a sequent s from a set of sequents S is a

$$\frac{}{\alpha \Rightarrow \alpha}\ (\mathrm{id}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Pi \Rightarrow \delta}{\Sigma, \Gamma, \Pi \Rightarrow \delta}\ (\mathrm{cut})$$

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \alpha \cdot \beta, \Sigma \Rightarrow \delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \beta/\alpha, \Gamma, \Sigma \Rightarrow \delta}\ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \Gamma, \alpha\backslash\beta, \Sigma \Rightarrow \delta}\ (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \delta}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \delta}\ (\wedge 1 \Rightarrow) \qquad \frac{\Gamma, \beta, \Sigma \Rightarrow \delta}{\Gamma, \alpha \wedge \beta, \Sigma \Rightarrow \delta}\ (\wedge 2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge) \qquad \frac{\Gamma, \alpha, \Sigma \Rightarrow \delta \quad \Gamma, \beta, \Sigma \Rightarrow \delta}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \delta}\ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\Gamma, \Sigma \Rightarrow \delta}{\Gamma, 1, \Sigma \Rightarrow \delta}\ (1 \Rightarrow) \qquad \frac{}{\Rightarrow 1}\ (\Rightarrow 1)$$

Figure 1. The Gentzen system **FL**.

finite sequence of sequents $s_1, \ldots, s_n = s$ such that, for every i, s_i is in S, or it is an instance of an axiom of **FL**, or there is a subset S_0 of the set $\{s_1, \ldots, s_{i-1}\}$ such that (S_0, s_i) is an instance of one of the rules of **FL**. Note that substitution instances of elements of S that are not already in S cannot be used in the proof. We usually present proofs in a tree arrangement, called a *proof tree*, where the nodes are sequents labeled by rules. The tree in Figure 2 demonstrates a proof of $\alpha(\beta \vee \gamma) \Rightarrow \alpha\beta \vee \alpha\gamma$ from the empty set.

If there is a proof in **FL** of s from S, we say that s is *deducible*, or *provable*, in **FL** from S and we write $S \vdash^{\mathbf{FL}} s$. If $\Phi \cup \{\psi\}$ is a set of formulas, we write $\Phi \vdash_{\mathbf{FL}} \psi$ for $\{(\ \Rightarrow \phi) \mid \phi \in \Phi\} \vdash^{\mathbf{FL}} (\ \Rightarrow \psi)$; note the position of '**FL**' as a superscript or subscript in $\vdash^{\mathbf{FL}}$ and $\vdash_{\mathbf{FL}}$. The relation $\vdash_{\mathbf{FL}}$ is called the *external consequence relation associated with* $\vdash^{\mathbf{FL}}$, see [1].

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\alpha \Rightarrow \alpha$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\beta \Rightarrow \beta$}
\RightLabel{$(\cdot \Rightarrow)$}
\BinaryInfC{$\alpha, \beta \Rightarrow \alpha\beta$}
\RightLabel{$(\Rightarrow \vee 1)$}
\UnaryInfC{$\alpha, \beta \Rightarrow \alpha\beta \vee \alpha\gamma$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\alpha \Rightarrow \alpha$}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\gamma \Rightarrow \gamma$}
\RightLabel{$(\cdot \Rightarrow)$}
\BinaryInfC{$\alpha, \gamma \Rightarrow \alpha\gamma$}
\RightLabel{$(\Rightarrow \vee 2)$}
\UnaryInfC{$\alpha, \gamma \Rightarrow \alpha\beta \vee \alpha\gamma$}
\RightLabel{$(\vee \Rightarrow)$}
\BinaryInfC{$\alpha, \beta \vee \gamma \Rightarrow \alpha\beta \vee \alpha\gamma$}
\RightLabel{$(\cdot \Rightarrow)$}
\UnaryInfC{$\alpha(\beta \vee \gamma) \Rightarrow \alpha\beta \vee \alpha\gamma$}
\end{prooftree}
$$

Figure 2. A proof in **FL**.

A *consequence relation* $\vdash$ on a set A is a subset of $\mathcal{P}(A) \times A$ such that, for all subsets $\Phi \cup \Psi \cup \{\phi, \psi, \chi\}$ of A,

- if $\phi \in \Phi$, then $\Phi \vdash \phi$ (we use infix notation for $\vdash$) and
- if $\Phi \vdash \psi$, for all $\psi \in \Psi$, and $\Psi \vdash \chi$, then $\Phi \vdash \chi$.

The set $Thm(\vdash) = \{\phi \in A \mid \emptyset \vdash \phi\}$ is called the set of *theorems* of $\vdash$. A consequence relation on a set A is called *finitary*, if for all subsets $\Phi \cup \{\phi\}$ of A, if $\Phi \vdash \phi$, then there exists a finite subset Φ_0 of Φ such that $\Phi_0 \vdash \phi$. A consequence relation on $\mathbf{Fm}_{\mathcal{L}}$ is called *substitution invariant*, if for all sets $\Phi \cup \{\phi\}$ of formulas and all substitutions σ over $\mathbf{Fm}_{\mathcal{L}}$, if $\Phi \vdash \phi$, then $\sigma[\Phi_0] \vdash \sigma(\phi)$. The notion of substitution invariance can be extended to consequence relations over $\mathbf{Fm}_{\mathcal{L}}^2$ in a natural way.

It is easy to see that $\vdash_{\mathbf{FL}}$ is a finitary and substitution invariant consequence relation on $\mathbf{Fm}_{\mathcal{L}}$. Moreover, $\vdash^{\mathbf{FL}}$ is a consequence relation on the set of sequents over $\mathbf{Fm}_{\mathcal{L}}$; the relation $\vdash^{\mathbf{FL}}$ will be discussed in detail in [22].

A subset F of an $\mathcal{L}$-algebra $\mathbf{A}$ is said to be a *deductive filter* of $\mathbf{A}$ with respect to a consequence relation $\vdash$ on $\mathbf{Fm}_{\mathcal{L}}$, if for every set of formulas $\Phi \cup \{\phi\}$ such that $\Phi \vdash \phi$ and for every homomorphism $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $f[\Phi] \subseteq F$ implies $f(\phi) \in F$. The deductive filters of $\mathbf{Fm}_{\mathcal{L}}$ with respect to $\vdash$ are called *theories* of $\vdash$. For more on consequence relations and deductive filters, see [17] and [7].

A *substructural logic (over* **FL***)* is a theory of $\vdash_{\mathbf{FL}}$, i.e. a set of formulas closed under $\vdash_{\mathbf{FL}}$, that is closed under substitution. Equivalently, a substructural logic is the set of theorems of an axiomatic extension of $\vdash_{\mathbf{FL}}$. In Section 4 we provide alternative descriptions of deductive filters and, consequently, of substructural logics; see Corollary 4.5.

We say that a substructural logic $\mathbf{L}$ is *axiomatized* over **FL** by a set of formulas Φ, if $\mathbf{L}$ is the smallest substructural logic containing Φ. It is easy to see that the set of all substructural logics forms a lattice, which we denote by **SL**.

For a set of formulas $\Phi \cup \{\phi\}$, we write $\Phi \vdash_{\mathbf{L}} \phi$ for $\Phi \cup \mathbf{L} \vdash_{\mathbf{FL}} \phi$.

LEMMA 3.1. *Let* $\mathbf{L}$ *be a substructural logic and* $\Phi \cup \Psi \cup \{\psi\}$ *a set of formulas.*

1. *The relation* $\vdash_{\mathbf{L}}$ *is a finitary, substitution invariant consequence relation.*
2. *If* Φ *is finite, then* $\Phi, \Psi \vdash_{\mathbf{L}} \psi$ *iff* $\bigwedge \Phi, \Psi \vdash_{\mathbf{L}} \psi$*, where* $\bigwedge \Phi$ *denotes the conjunction of all the (finitely many) formulas of* Φ*.*
3. *If* Φ *is finite, then* $\Psi \vdash_{\mathbf{L}} \phi$*, for all* $\phi \in \Phi$*, iff* $\Psi \vdash_{\mathbf{L}} \bigwedge \Phi$*.*

PROOF. (1) follows easily from the definition.

Note that if Φ is a finite set of formulas, then by repeated applications of $(\wedge 1 \Rightarrow)$ and $(\wedge 2 \Rightarrow)$ we obtain $\vdash^{\mathbf{FL}} \bigwedge \Phi \Rightarrow \phi$, for all $\phi \in \Phi$. By the cut rule, we have $\bigwedge \Phi \vdash_{\mathbf{FL}} \phi$, so $\bigwedge \Phi \vdash_{\mathbf{L}} \phi$, for all $\phi \in \Phi$.

Moreover, it is clear by $(\Rightarrow \wedge)$ that, for all formulas ϕ, ψ, we have $\{\phi, \psi\} \vdash_{\mathbf{FL}} \phi \wedge \psi$. By repeated applications of this fact we obtain $\Phi \vdash_{\mathbf{L}} \bigwedge \Phi$.

(2) and (3) follow easily from the above facts and (1). ■

If a consequence relation satisfies condition (2) of the previous lemma, it is called *conjunctive*.

The relation $\vdash_{\mathbf{L}}$ is called the *deducibility relation of the substructural logic* $\mathbf{L}$. Note that $\mathbf{L}$ and $\vdash_{\mathbf{L}}$ are mutually definable, since $\mathbf{L} = Thm(\vdash_{\mathbf{L}})$ and $\Phi \vdash_{\mathbf{L}} \phi$ iff $\Phi \cup \mathbf{L} \vdash_{\mathbf{FL}} \phi$. The relations of the form $\vdash_{\mathbf{L}}$, where $\mathbf{L}$ is a substructural logic, can be abstractly characterized as the axiomatic extensions of $\vdash_{\mathbf{FL}}$, i.e. the substitution invariant consequence relations that are minimal with respect to containing $\vdash_{\mathbf{FL}}$ and $\{\emptyset\} \times \Phi$, for some set of formulas Φ.

In our definition of a (substructural) logic we deviate from the notion of logic in the setting of Abstract Algebraic Logic (AAL). In AAL a *logic* or *deductive system* is a pair $\mathcal{S} = (\mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}})$, where $\mathcal{L}$ is an algebraic language, $\mathbf{Fm}_{\mathcal{L}}$ is the algebra of formulas over $\mathcal{L}$ and $\vdash_{\mathcal{S}}$ is a substitution invariant consequence relation on $\mathbf{Fm}_{\mathcal{L}}$; see [7] or [17]. Nevertheless, the difference between the two conflicting definitions is not essential, because substructural logics $\mathbf{L}$ (in our sense) are in bijective correspondence to deductive systems, or logics (in the sense of AAL), $\mathcal{S} = (\mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}})$, where $\vdash_{\mathcal{S}}$ is an axiomatic extension of $\vdash_{\mathbf{FL}}$, via the maps $\mathbf{L} \mapsto \mathcal{S}_{\mathbf{L}} = (\mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathbf{L}})$ and $\mathcal{S} \mapsto Thm(\vdash_{\mathcal{S}})$.

In other words we can think of a substructural logic over **FL** either as a set of formulas or as a consequence relation/deductive system.

Note that, if $\mathbf{L}$ is a substructural logic, then $\vdash_{\mathbf{L}}$ is different than the *internal relation* $\vdash^{\mathrm{I}}_{\mathbf{L}}$ associated with $\mathbf{L}$, which is defined by $\psi_1, \psi_2, \ldots, \psi_n \vdash^{\mathrm{I}}_{\mathbf{L}} \phi$ iff $\{(\Rightarrow \chi) \mid \chi \in \mathbf{L}\} \vdash^{\mathbf{FL}} (\psi_1, \psi_2, \ldots, \psi_n \Rightarrow \phi)$ or, equivalently, if the sequent $\psi_1, \psi_2, \ldots, \psi_n \Rightarrow \phi$ is provable in the system obtained from **FL** by adding the sequents of the form $\Rightarrow \chi$, for all $\chi \in \mathbf{L}$; the relation $\vdash^{\mathrm{I}}_{\mathbf{L}}$ was introduced in [1] and was also considered in [42]. Note that $\psi_1, \psi_2, \ldots, \psi_n \vdash^{\mathrm{I}}_{\mathbf{L}} \phi$ iff $\vdash_{\mathbf{L}} \psi_1\psi_2\cdots\psi_n\backslash\phi$; hence the relations $\vdash_{\mathbf{L}}$ and $\vdash^{\mathrm{I}}_{\mathbf{L}}$ are different (for example, $\{\phi, \psi\} \vdash_{\mathbf{L}} \phi \wedge \psi$ and $\{\phi, \psi\} \not\vdash^{\mathrm{I}}_{\mathbf{L}} \phi \wedge \psi$), but they have the same theorems, i.e. $\vdash_{\mathbf{L}} \phi$ iff $\vdash^{\mathrm{I}}_{\mathbf{L}} \phi$. Observe that $\vdash^{\mathrm{I}}_{\mathbf{L}}$ is not a consequence relation. The relation $\vdash^{w}_{\mathbf{L}}$, also introduced in [42], coincides with $\vdash_{\mathbf{L}}$ in the special cases considered in [42]. It follows from our analysis in the remainder of the paper that $\vdash_{\mathbf{L}}$ is the appropriate relation for the study of the connections between substructural logics and residuated lattices.

Consider the following *structural* rules and axiom. They are called *exchange*, *contraction*, *weakening* and 0-*weakening*, respectively.

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \delta}\ (e) \qquad \frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \delta}{\Gamma, \alpha, \Sigma \Rightarrow \delta}\ (c) \qquad \frac{\Gamma, \Sigma \Rightarrow \delta}{\Gamma, \alpha, \Sigma \Rightarrow \delta}\ (\ell w) \qquad \frac{}{0 \Rightarrow \delta}\ (0w)$$

We denote the combination of (ℓw) and $(0w)$ by (w) and we write (i) for (ℓw). We denote the sequent calculi obtained from **FL** by adding one or more of the rules (e), (c), (w) and (i), by attaching corresponding subscripts e, c, w, i to **FL**. For example $\mathbf{FL_{ew}}$ is obtained by adding (e), (ℓw) and $(0w)$ to **FL**.

A substructural logic $\mathbf{L}$ is called *integral*, if $\phi\backslash(1\wedge\phi) \in \mathbf{L}$, for every ϕ; it is called *contractive*, if $\phi\backslash\phi^2 \in \mathbf{L}$, for every ϕ; finally, it is called *commutative*, if $\phi\psi\backslash\psi\phi \in \mathbf{L}$, for every ϕ, ψ. If a logic is integral and $0\backslash\phi \in \mathbf{L}$, for every ϕ, we say that $\mathbf{L}$ *has weakening*. It is easy to see that a logic is integral, contractive, commutative or has weakening, iff it contains the logic $\mathbf{FL_i}$, $\mathbf{FL_c}$, $\mathbf{FL_e}$ or $\mathbf{FL_w}$, respectively.

Various substructural logics have been investigated, most of which are integral and commutative. Classical propositional logic, intuitionistic logic, Łukasiewicz many-valued logic, Hájek basic logic, relevant logic and the multiplicative additive fragment of linear logic are examples of commutative substructural logics over **FL**.

3.2. Algebraization

For every class $\mathcal{K}$ of pointed residuated lattices and for every set K of formulas over L, let $\mathbf{L}(\mathcal{K}) = \{\phi \in Fm_{\mathcal{L}} \mid \mathcal{K} \models 1 \leq \phi\}$ and $\mathcal{V}(K) = \mathcal{FL} \cap \mathrm{Mod}(\{1 \leq \phi \mid \phi \in K\})$; recall that $1 \leq \phi$ is short for $1 \approx 1 \wedge \phi$. Moreover, if Φ is a set of formulas over L and E is a set of equations over L, we define the set of equations $Eq(\Phi) = \{1 \leq \phi \mid \phi \in \Phi\}$, and the set of formulas $Fm(E) = \{t \backslash s \wedge s \backslash t \mid (t \approx s) \in E\}$. Let $s \approx t$ and $s_i \approx t_i$, $i \in I$, be equations in the language of $\mathcal{FL}$, $\bar{x}$ the sequence of variables in them and $\mathcal{K}$ a subclass of $\mathcal{FL}$. Following [7], we say that $s \approx t$ is a $\mathcal{K}$-*consequence* of $E = \{s_i \approx t_i \mid i \in I\}$, in symbols $E \models_{\mathcal{K}} s \approx t$, iff, for all $\mathbf{A} \in \mathcal{K}$ and every valuation $\bar{a}$ of $\bar{x}$ in $\mathbf{A}$, if $\mathbf{A} \models s_i(\bar{a}) = t_i(\bar{a})$, for all $i \in I$, then $\mathbf{A} \models s(\bar{a}) = t(\bar{a})$. In particular, when E is finite,

$$E \models_{\mathcal{K}} s \approx t \text{ iff } \mathcal{K} \models (\forall \bar{x})(\textstyle\bigwedge_{i \in I} s_i(\bar{x}) = t_i(\bar{x}) \ \Rightarrow\ s(\bar{x}) = t(\bar{x})).$$

It is easy to see that $\models_{\mathcal{K}}$ is a finitary, substitution preserving consequence relation on $\mathbf{Fm}^2_{\mathcal{L}}$.

The notion of *equivalence* of two consequence relations is defined in [6]; see also [25]. We omit the general definition of equivalence, but we note that a finitary and substitution invariant consequence relation $\vdash$ on $\mathbf{Fm}_{\mathcal{L}}$ is equivalent to a finitary $\models_{\mathcal{K}}$ for some class $\mathcal{K}$ of algebras iff there is a finite set $\{\delta_i(x) \approx \epsilon_i(x) \mid i \in I\}$ of unary equations, called the *defining equations*, and a finite set $\{\Delta_j(x, y) \mid j \in J\}$ of binary connectives, called the *equivalence formulas*, such that for all sets of formulas $\Phi \cup \{\phi, \psi\}$,

- $\Phi \vdash \psi$ iff $\{\delta_i(\phi) \approx \epsilon_i(\phi) \mid \phi \in \Phi, i \in I\} \models_{\mathcal{K}} \delta_k(\psi) \approx \epsilon_k(\psi)$, for all $k \in I$, and
- $\phi \approx \psi \mathrel{=\!\mid\models_{\mathcal{K}}} \{\delta_i(\Delta_j(\phi, \psi)) \approx \epsilon_i(\Delta_j(\phi, \psi)) \mid i \in I, j \in J\}$

If $\vdash$ is equivalent to $\models_{\mathcal{K}}$ for some class $\mathcal{K}$ of algebras, then $\vdash$ is called *algebraizable* and $\mathcal{K}$ is called an *equivalent algebraic semantics* for $\vdash$.

THEOREM 3.2. *The consequence relation $\vdash_{\mathbf{FL}}$ is algebraizable with defining equation $1 \approx x \wedge 1$ and equivalence formula $x \backslash y \wedge y \backslash x$. An equivalent algebraic semantics for $\vdash_{\mathbf{FL}}$ is the variety $\mathcal{FL}$ of pointed residuated lattices.*

PROOF. We assume familiarity with the terminology and the results in [7]. To show that $\vdash_{\mathbf{FL}}$ is algebraizable, given that it is finitary and substitution invariant, it suffices to check the conditions of Theorem 4.7 of [7]. It is easy to see that for $\phi \,\Delta\, \psi = \phi \backslash \psi \wedge \psi \backslash \phi$ we have $\vdash_{\mathbf{FL}} \phi \,\Delta\, \phi$; $\phi \,\Delta\, \psi \vdash_{\mathbf{FL}} \psi \,\Delta\, \phi$;

$\phi \,\Delta\, \psi, \psi \,\Delta\, \chi \vdash_{\mathbf{FL}} \phi \,\Delta\, \chi$; and $\phi_1 \,\Delta\, \phi_2, \psi_1 \,\Delta\, \psi_2 \vdash_{\mathbf{FL}} (\phi_1 \star \psi_1)\,\Delta(\phi_2 \star \psi_2)$, for all $\star \in \{\wedge, \vee, \cdot, \backslash, /\}$ and all $\phi, \psi, \chi \in Fm_{\mathcal{L}}$. Moreover, for $\delta(x) = 1$ and $\varepsilon(x) = x \wedge 1$, we have $\phi \dashv\vdash_{\mathbf{FL}} \delta(\phi)\,\Delta\,\varepsilon(\phi)$. So, $\vdash_{\mathbf{FL}}$ is algebraizable with *defining equation* $1 \approx x \wedge 1$ and *equivalence formula* $x\backslash y \wedge y\backslash x$.

Theorem 2.17 of [7] provides an axiomatization for the equivalent algebraic semantics of an algebraizable deductive system. The axiomatization in the case of $\vdash_{\mathbf{FL}}$ consists of the following quasiequations: $1 \approx 1 \wedge x\backslash x \wedge x\backslash x$; $1 \approx 1 \wedge \phi_1, \ldots, 1 \approx 1 \wedge \phi_n$ implies $1 \approx 1 \wedge \phi$, for all rules of inference $\langle\{\phi_1, \ldots \phi_n\}, \phi\rangle$; and $1 \approx 1 \wedge x\backslash y \wedge y\backslash x$ implies $x \approx y$.

It is easy to check that all these quasiequations are true in $\mathcal{FL}$. Conversely, one can show that all axioms of $\mathcal{FL}$ follow from this list. To see this observe that, for all terms $s, t \in Fm_{\mathcal{L}}$, if $s \Rightarrow t$ and $t \Rightarrow s$ are provable in **FL**, then $\Rightarrow s\backslash t \wedge t\backslash s$ is also provable, so $\vdash_{\mathbf{FL}} s\backslash t \wedge t\backslash s$. Thus, the equivalent algebraic semantics satisfies $1 \approx 1 \wedge s\backslash t \wedge t\backslash s$ and, by the last quasiequation in the list, it satisfies $s \approx t$, as well. So, the proof amounts to checking that $s \Rightarrow t$ and $t \Rightarrow s$ are provable in **FL**, for every axiom $s \approx t$ of $\mathcal{FL}$. ■

Statements 3 and 4 of the following theorem can be proved via a standard Tarski-Lindembaum algebra construction argument and they provide a standard and inverse completeness theorem for substructural logics. They also follow immediately from Theorem 3.4, which is a standard and inverse strong completeness theorem, in the sense that it deals with deductions rather than theorems.

THEOREM 3.3.

1. *For every $\mathcal{K} \subseteq \mathcal{FL}$, $\mathbf{L}(\mathcal{K})$ is a substructural logic and for every $K \subseteq Fm_{\mathcal{L}}$, $\mathcal{V}(K)$ is a subvariety of $\mathcal{FL}$.*
2. *The maps $\mathbf{L} : \mathbf{S}(\mathcal{FL}) \to \mathbf{SL}$ and $\mathcal{V} : \mathbf{SL} \to \mathbf{S}(\mathcal{FL})$ are mutually inverse, dual lattice isomorphisms.*
3. *If a substructural logic **L** is axiomatized relative to **FL** by a set of formulas K, then the variety $\mathcal{V}(\mathbf{L})$ is axiomatized relative to $\mathcal{FL}$ by the set of equations $Eq(K)$.*
4. *If a subvariety $\mathcal{V}$ of $\mathcal{FL}$ is axiomatized relative to $\mathcal{FL}$ by a set of equations E, then the substructural logic $\mathbf{L}(\mathcal{V})$ is axiomatized relative to **FL** by the set of formulas $Fm(E)$.*
5. *A substructural logic is commutative, integral or contractive iff the corresponding variety is.*

THEOREM 3.4. *For every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL}$*, its deducibility relation* $\vdash_{\mathbf{L}}$ *is algebraizable and* $\mathcal{V}(\mathbf{L})$ *is an equivalent algebraic semantics for it. In particular,*

1. *if* $\Sigma \cup \{\phi\}$ *is a subset of* $Fm_{\mathcal{L}}$ *and* $\mathbf{L}$ *is a substructural logic, then*
$$\Sigma \vdash_{\mathbf{L}} \phi \textit{ iff } Eq(\Sigma) \models_{\mathcal{V}(\mathbf{L})} 1 \leq \phi, \textit{ and}$$
2. *if* $E \cup \{t \approx s\}$ *is a set of equations in* $\mathcal{L}$ *and* $\mathcal{V}$ *is a subvariety of* $\mathcal{FL}$*, then*
$$E \models_{\mathcal{V}} t \approx s \textit{ iff } Fm(E) \vdash_{\mathbf{L}(\mathcal{V})} t\backslash s \wedge s\backslash t.$$

PROOF. By Corollary 4.9 of [7], every extension of $\vdash_{\mathbf{FL}}$ is also algebraizable with the same equivalence formula and defining equation. By Corollary 2.17 of [7], the equivalent algebraic semantics of a substructural logic $\mathbf{L}$ axiomatized relative to $\mathbf{FL}$ by a set Σ of formulas is axiomatized relative to $\mathcal{FL}$ by the equations $Eq(\Sigma)$. The translation is obviously onto the subvarieties of $\mathcal{FL}$. Then, (2) follows from Corollary 2.9 of [7]. ■

4. Filter generation and the deduction theorem

In this section we prove some of the properties of deductive filters and explore the connections between them and congruence relations, convex normal subalgebras and convex normal submonoids. Moreover, we describe the generation process of deductive filters and use it to obtain a form of the deduction theorem, called *parametrized local deduction theorem*, that holds for all substructural logics.

4.1. Deductive filters

Let $\mathbf{A}$ be an algebra of the type of pointed residuated lattices and F a subset of A. Recall that F is deductive filter of $\mathbf{A}$ relative to $\vdash_{\mathbf{FL}}$ if, for all homomorphisms $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ and for all subsets $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ such that $\Phi \vdash_{\mathbf{FL}} \phi$, $f[\Sigma] \subseteq F$ implies $f(\phi) \in F$. Also, recall that a theory of $\vdash_{\mathbf{FL}}$ is a deductive filter of $\mathbf{Fm}_{\mathcal{L}}$ relative to $\vdash_{\mathbf{FL}}$. By $\mathbf{FL}$ we denote the Gentzen system introduced in the previous section, as well as the set of theorems of $\vdash_{\mathbf{FL}}$.

If $\mathbf{A}$ is an algebra of the type of (pointed) residuated lattices, we set $\mathbf{FL}(\mathbf{A}) = \{t(\bar{a}) \mid t \in \mathbf{FL}, \bar{a} \in A^{\alpha(t)}\}$, where $\alpha(t)$ is the arity of the term t.

LEMMA 4.1. *Let* $\mathbf{A}$ *be an algebra of the type of (pointed) residuated lattices. If a subset* F *of* A *is a deductive filter of* $\mathbf{A}$ *relative to* $\vdash_{\mathbf{FL}}$ *and* $x, y \in A$, *then*

(*fl*)	$\mathbf{FL}(\mathbf{A}) \subseteq F$.
(mp_ℓ)	*If* $x, x\backslash y \in F$, *then* $y \in F$.
(adj_u)	*If* $x \in F$, *then* $x \wedge 1 \in F$.
(pn)	*If* $x \in F$, *then* $y\backslash xy, yx/y \in F$.

PROOF. (*fl*) is clear from the definition of a deductive filter. For (mp_ℓ), it suffices to show that $\phi, \phi\backslash\psi \vdash_{\mathbf{FL}} \psi$, for all $\phi, \psi \in Fm_{\mathcal{L}}$; i.e. that there is a proof of the sequent $\Rightarrow \psi$ in **FL** from the sequents $\Rightarrow \phi$ and $\Rightarrow \phi\backslash\psi$.

$$\dfrac{\Rightarrow \phi \qquad \dfrac{\Rightarrow \phi\backslash\psi \qquad \dfrac{\dfrac{}{\phi \Rightarrow \phi}\,(\mathrm{id}) \qquad \dfrac{}{\psi \Rightarrow \psi}\,(\mathrm{id})}{\phi, \phi\backslash\psi \Rightarrow \psi}\,(\backslash\Rightarrow)}{\phi \Rightarrow \psi}\,(\mathrm{cut})}{\Rightarrow \psi}\,(\mathrm{cut})$$

Likewise, we verify the rules (adj_u) and (pn). ■

Note that theories of $\vdash_{\mathbf{FL}}$ are deductive filters of $\mathbf{Fm}_{\mathcal{L}}$, so theories are closed under the rules (*fl*), (mp_ℓ), (adj_u) and (pn). Conversely, to establish that a rule is satisfied by all deductive filters, it is enough to establish that the rule is satisfied by all theories; i.e. it is enough to consider only one algebra of the type of residuated lattices: the absolutely free algebra $\mathbf{Fm}_{\mathcal{L}}$ and its deductive filters (namely the theories). We choose to state the results in the more general setting of algebras of the type of (pointed) residuated lattices and deductive filters, rather than for $\mathbf{Fm}_{\mathcal{L}}$ and theories, because we wish to apply them to residuated lattices.

LEMMA 4.2. *Let* $\mathbf{A}$ *be an algebra of the type of (pointed) residuated lattices and* F *a deductive filter of* A *relative to a consequence relation that satisfies* (*fl*), (mp_ℓ), (adj_u) *and* (pn). *Then,* F *is closed under the following rules, where* $x, y, z \in A$.

(tr_ℓ)	*If* $x\backslash y, y\backslash z \in F$*, then* $x\backslash z \in F$*.*
(n)	*If* $x \in F$ *and* $y \in A$*, then* $(x\backslash y)\backslash y, y/(y/x) \in F$*.*
(np_ℓ)	*If* $x \in F$ *and* $y\backslash(x\backslash z) \in F$*, then* $y\backslash z \in F$*.*
(np_r)	*If* $x \in F$ *and* $(z/x)/y \in F$*, then* $z/y \in F$*.*
(sym)	*For all* $x, y \in A$*,* $x/y \in F$ *iff* $y\backslash x \in F$*.*
(p)	*If* $x, y \in F$*, then* $xy \in F$*.*
(adj)	*If* $x, y \in F$*, then* $x \wedge y \in F$*.*
(pn_u)	*If* $x \in F$ *and* $z \in A$*, then* $z\backslash xz \wedge 1, zx/z \wedge 1 \in F$*.*
(up_m)	*If* $x \wedge y \in F$*, then* $x, y \in F$*.*
(mp_r)	*If* $x, y/x \in F$*, then* $y \in F$*.*
(n_u)	*If* $x \in F$ *and* $y \in A$*, then* $(x\backslash y)\backslash y \wedge 1, y/(y/x) \wedge 1 \in F$*.*
(u)	$1 \in F$*.*

Proof. (tr_ℓ): Note that $(y\backslash z)\backslash[(x\backslash y)\backslash(x\backslash z)] \in \mathbf{FL}(\mathbf{A}) \subseteq F$. If $x\backslash y, y\backslash z \in F$, then two applications of (mp_ℓ) yield $x\backslash z \in F$.

(n): If $x \in F$, then, $(x\backslash y)\backslash[x(x\backslash y)] \in F$, by (pn). Moreover, we have $\{(x\backslash y)\backslash[x(x\backslash y)]\}\backslash[(x\backslash y)\backslash y] \in \mathbf{FL}(\mathbf{A})$, so $(x\backslash y)\backslash y \in F$, by (mp_ℓ). The opposite formula is in F by a similar argument.

(np_ℓ): If $x \in F$, then $(x\backslash z)\backslash z \in F$, by (n). Consequently, we have $[y\backslash(x\backslash z)]\backslash(y\backslash z) \in F$, by (mp_ℓ), since $[(x\backslash z)\backslash z]\backslash\{[y\backslash(x\backslash z)]\backslash(y\backslash z)\} \in \mathbf{FL}(\mathbf{A})$. On the other hand, we have $\{[y\backslash(x\backslash z)]\backslash(y\backslash z)\}\backslash(y\backslash z) \in F$, by (n), since $y\backslash(x\backslash z) \in F$. Thus, we obtain $y\backslash z \in F$, by (mp_ℓ). The proof of (np_r) is similar.

(sym): Note that $(y/(x\backslash y))/x \in \mathbf{FL}(\mathbf{A})$, so, if $x\backslash y \in F$, then $y/x \in F$, by (np_r). For the converse direction we use (np_ℓ) and $x\backslash[(y/x)\backslash y] \in F$.

(p): If $x \in F$, then $y\backslash xy \in F$, by (pn). If, moreover, $y \in F$, then $xy \in F$, by (mp_ℓ).

(adj): If $x, y \in F$, then $x \wedge 1, y \wedge 1 \in F$, by (adj_u). So, $(x \wedge 1)(y \wedge 1) \in F$, by (p). Since $(x \wedge 1)(y \wedge 1)\backslash(x \wedge y) \in \mathbf{FL}(\mathbf{A})$, we have $x \wedge y \in F$, by (mp_ℓ).

(pn_u) follows from (pn) and (adj_u). (up_m) follows from (mp_ℓ) and the fact that $(x \wedge y)\backslash y \in \mathbf{FL}(\mathbf{A})$. (mp_r) follows from (mp_ℓ) and (sym). (n_u) follows from (n) and (adj_u). Finally, (u) follows from (fl). ■

Theorem 4.3. *Let* $\mathbf{A}$ *be an algebra of the type of (pointed) residuated lattices. A subset* F *of* A *is a deductive filter of* $\mathbf{A}$ *relative to* $\vdash_{\mathbf{FL}}$ *iff it is closed under the rules* (fl)*,* (mp_ℓ)*,* (adj_u) *and* (pn)*.*

Proof. For every sequent $s = (\psi_1, \psi_2, \ldots, \psi_n \Rightarrow \phi)$, we define $fm(s) = (\psi_1\psi_2 \cdots \psi_n)\backslash\phi$. In our forthcoming paper [22] it is shown that for every set of sequents $S \cup \{s\}$, we have $S \vdash^{\mathbf{FL}} s$ iff $fm[S] \vdash_{\mathbf{FL}} fm(s)$; actually we show that the two consequence relations $\vdash^{\mathbf{FL}}$ and $\vdash_{\mathbf{FL}}$ are equivalent. The proof of the theorem is an easy consequence of this result.

Here we give a different proof. We show that any finitary and substitution invariant consequence relation $\vdash$ on $\mathbf{Fm}_{\mathcal{L}}$ that satisfies the rules (*fl*), (mp_ℓ), (adj_u) and (pn) is algebraizable with defining equation $1 \approx x \wedge 1$ and equivalence formula $x\backslash y \wedge y\backslash x$, and that an equivalent algebraic semantics for $\vdash$ is the variety $\mathcal{FL}$ of pointed residuated lattices. It follows then, by Theorem 3.2, that $\vdash = \vdash_{\mathbf{FL}}$.

As in the proof of Theorem 3.2, we need to show that for $\phi \,\Delta\, \psi = \phi\backslash\psi \wedge \psi\backslash\phi$ and for all $\phi, \psi, \chi \in Fm_{\mathcal{L}}$ we have $\vdash \phi \,\Delta\, \phi$; $\phi \,\Delta\, \psi \vdash \psi \,\Delta\, \phi$; $\phi \,\Delta\, \psi, \psi \,\Delta\, \chi \vdash \phi \,\Delta\, \chi$; and $\phi_1 \,\Delta\, \phi_2, \psi_1 \,\Delta\, \psi_2 \vdash (\phi_1 \star \psi_1) \,\Delta (\phi_2 \star \psi_2)$, for all $\star \in \{\wedge, \vee, \cdot, \backslash, /\}$. Moreover, we need to show that, for $\delta(x) = 1$ and $\varepsilon(x) = x \wedge 1$, we have $\phi \dashv\vdash \delta(\phi) \,\Delta\, \varepsilon(\phi)$.

In the following, we often refer to formulas and claim that they are provable in **FL**, i.e. they are theorems of $\vdash_{\mathbf{FL}}$. To show the provability of ϕ it is enough to construct a proof of $\Rightarrow \phi$ in **FL** as in Figure 2.

We assume that $\vdash$ satisfies the following rules:

(*fl*)	If $\vdash_{\mathbf{FL}} \phi$, then $\vdash \phi$.	(mp_ℓ)	$\phi, \phi\backslash\psi \vdash \psi$.
(adj_u)	$\phi \vdash \phi \wedge 1$.	(pn)	$\phi \vdash \psi\backslash\phi\psi$ and $\phi \vdash \psi\phi/\psi$.

By Lemma 4.2, $\vdash$ satisfies all the rules stated in it. Note that $\vdash_{\mathbf{FL}} \phi\backslash\phi \wedge \phi\backslash\phi$, so $\vdash \phi\backslash\phi \wedge \phi\backslash\phi$, by (*fl*).

Moreover, $\vdash (\phi\backslash\psi\wedge\psi\backslash\phi)\backslash(\psi\backslash\phi\wedge\phi\backslash\psi)$, by (*fl*), so $\phi\backslash\psi\wedge\psi\backslash\phi \vdash \psi\backslash\phi\wedge\phi\backslash\psi$, by (mp_ℓ).

We have $\phi\backslash\psi \wedge \psi\backslash\phi \vdash \phi\backslash\psi$, and $\psi\backslash\chi \wedge \chi\backslash\psi \vdash \psi\backslash\chi$, by (up_m). So, $\phi\backslash\psi\wedge\psi\backslash\phi, \psi\backslash\chi\wedge\chi\backslash\psi \vdash \phi\backslash\chi$, by (tr_ℓ). Likewise, $\phi\backslash\psi\wedge\psi\backslash\phi, \psi\backslash\chi\wedge\chi\backslash\psi \vdash \chi\backslash\phi$, so $\phi\backslash\psi \wedge \psi\backslash\phi, \psi\backslash\chi \wedge \chi\backslash\psi \vdash \phi\backslash\chi \wedge \chi\backslash\phi$, by (adj).

Note that $\vdash_{\mathbf{FL}} [1\backslash(\phi\wedge 1) \wedge (\phi\wedge 1)\backslash 1]\backslash(\phi\wedge 1)$ and $\vdash_{\mathbf{FL}} (\phi\wedge 1)\backslash[1\backslash(\phi\wedge 1) \wedge (\phi\wedge 1)\backslash 1]$. So, $1\backslash(\phi\wedge 1)\wedge(\phi\wedge 1)\backslash 1 \dashv\vdash \phi\wedge 1$, by (*fl*) and (mp_ℓ). Moreover, $\phi \dashv\vdash \phi\wedge 1$, by (adj), (up_m) and the fact that $\vdash 1$. Thus, $1\backslash(\phi\wedge 1)\wedge(\phi\wedge 1)\backslash 1 \dashv\vdash \phi$.

Observe that if $\phi_1, \psi_1 \vdash \chi_1$ and $\phi_2, \psi_2 \vdash \chi_2$, then $\phi_1\wedge\phi_2, \psi_1\wedge\psi_2 \vdash \chi_1\wedge\chi_2$, by (adj) and (up_m). We will use this fact to reduce the work in half in the following proofs.

For $\star = \wedge$, we have $\vdash (\phi_1\backslash\phi_2)\backslash[(\phi_1\wedge\psi_1)\backslash\phi_2]$ and $\vdash (\phi_1\backslash\phi_2)\backslash[(\phi_1\wedge\psi_1)\backslash\psi_1]$, by (*fl*), so $\phi_1\backslash\phi_2 \vdash (\phi_1 \wedge \psi_1)\backslash\phi_2$ and $\phi_1\backslash\phi_2 \vdash (\phi_1 \wedge \psi_1)\backslash\psi_1$, by (mp_ℓ), and $\phi_1\backslash\phi_2 \vdash (\phi_1 \wedge \psi_1)\backslash\phi_2 \wedge (\phi_1 \wedge \psi_1)\backslash\psi_1$, by (adj). On the other hand, $\vdash [(\phi_1\wedge\psi_1)\backslash\phi_2\wedge(\phi_1\wedge\psi_1)\backslash\psi_1]\backslash[(\phi_1\wedge\psi_1)\backslash(\phi_2\wedge\psi_1)]$, by (*fl*), so $(\phi_1\wedge\psi_1)\backslash\phi_2\wedge(\phi_1\wedge\psi_1)\backslash\psi_1 \vdash (\phi_1\wedge\psi_1)\backslash(\phi_2\wedge\psi_1)$, by (mp_ℓ). Thus, $\phi_1\backslash\phi_2 \vdash (\phi_1\wedge\psi_1)\backslash(\phi_2\wedge\psi_1)$. A similar argument shows that $\psi_1\backslash\psi_2 \vdash (\phi_2\wedge\psi_1)\backslash(\phi_2\wedge\psi_2)$. So, $\phi_1\backslash\phi_2, \psi_1\backslash\psi_2 \vdash$

$(\phi_1 \wedge \psi_1)\backslash(\phi_2 \wedge \psi_2)$, by (tr_ℓ). By the argument of the previous paragraph, we get $\phi_1\backslash\phi_2 \wedge \phi_2\backslash\phi_1, \psi_1\backslash\psi_2 \wedge \psi_2\backslash\psi_1 \vdash (\phi_1 \wedge \psi_1)\backslash(\phi_2 \wedge \psi_2) \wedge (\phi_2 \wedge \psi_2)\backslash(\phi_1 \wedge \psi_1)$.

For $\star = \cdot$, by (*fl*), we have $\vdash (\phi_1\backslash\phi_2)\backslash(\psi_1\phi_1\backslash\psi_1\phi_2)$, so, by (mp_ℓ), we obtain $\phi_1\backslash\phi_2 \vdash \psi_1\phi_1\backslash\psi_1\phi_2$. On the other hand $\vdash (\psi_2/\psi_1)\backslash(\psi_2\phi_2/\psi_1\phi_2)$, by (*fl*), so $\psi_2/\psi_1 \vdash \psi_2\phi_2/\psi_1\phi_2$, by (mp_ℓ). By (*sym*), we have $\psi_1\backslash\psi_2 \vdash \psi_1\phi_2\backslash\psi_2\phi_2$; hence, $\phi_1\backslash\phi_2, \psi_1\backslash\psi_2 \vdash \psi_1\phi_1\backslash\psi_2\phi_2$, by (tr_ℓ).

For $\star = \backslash$, we have $\vdash (\phi_1\backslash\phi_2)\backslash[(\psi_1\backslash\phi_1)\backslash(\psi_1\backslash\phi_2)]$, by (*fl*), so $\phi_1\backslash\phi_2 \vdash (\psi_1\backslash\phi_1)\backslash(\psi_1\backslash\phi_2)$, by (mp_ℓ). Moreover, $\vdash (\psi_2\backslash\psi_1)\backslash[(\psi_2\backslash\phi_2)/(\psi_1\backslash\phi_2)]$, by (*fl*), so using (mp_ℓ) we have $\psi_2\backslash\psi_1 \vdash (\psi_2\backslash\phi_2)/(\psi_1\backslash\phi_2)$ and, by (*sym*), $\psi_2\backslash\psi_1 \vdash (\psi_1\backslash\phi_2)\backslash(\psi_2\backslash\phi_2)$. Thus, $\phi_1\backslash\phi_2, \psi_2\backslash\psi_1 \vdash (\psi_1\backslash\phi_1)\backslash(\psi_2\backslash\phi_2)$, by (tr_ℓ).

We leave the verification of the cases $\star = \vee$ and $\star = /$ to the reader.

So, $\vdash$ is algebraizable with *defining equation* $1 \approx x \wedge 1$ and *equivalence formula* $x\backslash y \wedge y\backslash x$. To show that $\mathcal{FL}$ is the equivalent algebraic semantics of $\vdash$, we proceed exactly as in the last two paragraphs of the proof of Theorem 3.2. ∎

Corollary 4.4. *A subset F of $\mathbf{Fm}_\mathcal{L}$ is a theory of $\vdash_{\mathbf{FL}}$ iff*

- (*fl*) $\mathbf{FL} \subseteq F$.
- (mp_ℓ) *If $\phi, \phi\backslash\psi \in F$, then $\psi \in F$.*
- (adj_u) *If $\phi \in F$, then $\phi \wedge 1 \in F$.*
- (pn) *If $\phi \in F$, then $\psi\backslash\phi\psi, \psi\phi/\psi \in F$.*

It is possible to reduce **FL** in (*fl*) to a finite list of axioms (axiom schemata rather). This provides a Hilbert-style presentation of the consequence relation $\vdash_{\mathbf{FL}}$; substructural logics are axiomatic extensions of this Hilbert system. In [22], the Hilbert system is further refined into a system that enjoys the strong separation property.

Consequently, we have the following description of substructural logics (c.f. [40]):

Corollary 4.5. *A set of formulas $\mathbf{L}$ is a substructural logic iff it is closed under substitution and the rules (fl), (mp_ℓ), (adj_u) and (pn).*

A deductive filter F of an algebra $\mathbf{A}$ of the type of (pointed) residuated lattices is called *integral*, if $x\backslash(1 \wedge x) \in F$, for every $x \in A$; it is called *contractive*, if $x\backslash x^2 \in F$, for every $x \in A$; finally, it is called *commutative*, if $xy\backslash yx \in F$, for every $x, y \in A$. Note that this definition agrees with the one for substructural logics on page 9.

Lemma 4.6. *Let $\mathbf{A}$ be an algebra of the type of (pointed) residuated lattices. Condition (pn) for a subset F of A to be a deductive filter can be replaced* *by either* (n)*, or* (sym)*, or both of* (np_ℓ) *and* (np_r)*. Moreover, the combination of conditions* (adj_u) *and* (pn) *can be replaced by* (pn_u)*. Among the conditions in the definition of an deductive filter,* (pn) *is redundant, if* F *is commutative, and* (adj_u) *is redundant, if* F *is integral.*

PROOF. In the proof of Lemma 4.2 we showed that, under the assumption of (fl), (mp_ℓ) and (adj_u), condition (sym) follows from the conjunction of (np_ℓ) and (np_r), that conditions (np_ℓ) and (np_r) follow from (n), and that condition (n) follows from (pn). We will show that, under the same assumptions, (pn) follows from (sym). Since $x\backslash(xy/y) \in \mathbf{FL}(\mathbf{A}) \subseteq F$, if $x \in F$ then $xy/y \in F$ by (mp_ℓ). By (sym), we have $y\backslash xy \in F$. Likewise, we show that $yx/y \in F$, when $x \in F$.

Assume that conditions (fl), (mp_ℓ) and (pn_u) hold. We will show that conditions (adj_u) and (pn) hold as well. If $x \in F$, then $z\backslash xz \wedge 1 \in F$, by (pn_u). Moreover, $(z\backslash xz \wedge 1)\backslash(z\backslash xz) \in \mathbf{FL}(\mathbf{A})$, so, $(z\backslash xz) \in F$, by (mp_ℓ). Additionally, if $x \in F$, then $1\backslash x1 \wedge 1 \in F$, by (pn_u). Since $(1\backslash x1\wedge 1)\backslash(x\wedge 1) \in \mathbf{FL}(\mathbf{A})$, we have $x \wedge 1 \in F$, by (mp_ℓ).

If F is commutative, then we have $zx\backslash xz \in F$. So, $x\backslash(z\backslash xz) \in F$ by (mp_ℓ), since $(zx\backslash xz)\backslash[x\backslash(z\backslash xz)] \in \mathbf{FL}(\mathbf{A}) \subseteq F$. If $x \in F$, then $z\backslash xz \in F$, by (mp_ℓ). Thus F satisfies (pn).

If F is integral, then $x\backslash(x \wedge 1) \in F$. So, $x \wedge 1 \in F$, by (mp_ℓ). Thus, F satisfies (adj_u). ■

We will now concentrate in the case where $\mathbf{A}$ is a (pointed) residuated lattice. Under this assumption, condition (fl) for a subset F of A to be a deductive filter is equivalent to the stipulation that, for all $x \in A$,

$$(up_u) \quad \text{if } 1 \leq x, \text{ then } x \in F.$$

To see this, recall that $t \in \mathbf{FL}$ iff $1 \leq t$ holds in all pointed residuated lattices. Also, let t be a term such that $1 \leq t$ holds in all (pointed) residuated lattices, and $\bar{a}$ an element of an appropriate power of A. Then, $1 \leq t(\bar{a})$ is true in $\mathbf{A}$; hence, if we assume (up_u), then $t(\bar{a}) \in F$. Conversely, assume (fl) holds and $1 \leq a$, for some $a \in A$. For $t = (1 \wedge x)\backslash x$, the equation $1 \leq t$ holds in all (pointed) residuated lattices and $t(a) = a$. So, $a = t(a) \in \mathbf{FL}(\mathbf{A}) \subseteq F$. In [44], C. van Alten discusses deductive filters of integral residuated lattices.

Additionally, (up_m) takes the form

$$(up) \quad \text{if } x \in F \text{ and } x \leq y, \text{ then } y \in F.$$

Note that in every (pointed) residuated lattice $\mathbf{A}$, its positive part $A^+ = \{a \in A \mid 1 \leq a\}$ is the least deductive filter of $\mathbf{A}$. Moreover, a (pointed) residuated lattice is commutative, integral or contractive iff its positive part is iff every deductive filter of $\mathbf{A}$ is.

Lemma 4.7. *If $\mathbf{A}$ is a (pointed) residuated lattice and F a subset of A, then F is a deductive filter of $\mathbf{A}$ iff it satisfies one of the following sets of conditions.*

1. *Conditions (u), (up), (mp_ℓ) and (pn_u),*
2. *Conditions (u), (up), (mp_ℓ), (adj_u) and (n),*
3. *Conditions (u), (up), (p) and (pn_u),*
4. *Conditions (u), (up), (p), (adj_u) and (pn).*

Among the last list of conditions, (pn) is redundant, if $\mathbf{A}$ is commutative; (adj_u) is redundant, if $\mathbf{A}$ is integral; (p) is redundant, if $\mathbf{A}$ is contractive.

Proof. To show that (up_u) is equivalent to the combination of (u) and (up), it suffices to show that (up), follows from (up_u), under the assumption of (mp_ℓ). If we assume that $x \in F$ and $x \leq y$, then $1 \leq x\backslash y$; hence $x\backslash y \in F$, by (up_u). By (mp_ℓ), we obtain $y \in F$. Consequently, in the setting of (pointed) residuated lattices, (fl) is equivalent to the combination of (u) and (up), under the assumption of (mp_ℓ).

By Lemma 4.6, (pn_u) is equivalent to the combination of (adj_u) and (pn), so a deductive filter is defined by conditions (u), (up), (mp_ℓ) and (pn_u). By the same lemma, (pn_u) is equivalent to the combination of (adj_u) and (n). We will show that under the assumption of (u), (up) and (pn_u), conditions (mp_ℓ) and (p) are equivalent. If $x, x\backslash y \in F$, then $x(x\backslash y) \in F$, by (p). Since $x(x\backslash y) \leq y$, we have $y \in F$, by (up). Conversely, it was shown in the proof of Lemma 4.6, that (p) follows from (mp_ℓ) and (pn). Since (pn) follows from (pn_u), by the same lemma, we obtain the converse implication.

Moreover, under the assumption of (u), (up) and (p), condition (pn_u) is equivalent to the combination of (adj_u) and (pn). Indeed, the backward direction requires no assumptions. Moreover, note that, by Lemma 4.6, the combination of (adj_u) and (pn) follows from (p), under the assumption of (mp_ℓ) and of (fl). We have seen that, under our hypothesis, (mp_ℓ) follows from (p) and (fl) follows from (u) and (up).

Note that if $\mathbf{A}$ is commutative, integral or contractive, then every deductive filter of $\mathbf{A}$ has these properties. Therefore, the last part of the lemma follows from the corresponding part of Lemma 4.6. ∎

4.2. Convex normal subalgebras and submonoids, congruences and deductive filters

Let $\mathbf{A}$ be a (pointed) residuated lattice. For $a, x \in A$, we define the *left conjugate* $\lambda_a(x) = a\backslash xa \wedge 1$ and the *right conjugate* $\rho_a(x) = ax/a \wedge 1$ of x with respect to a. An *iterated conjugate* of x is a composition $\gamma_{a_1}(\gamma_{a_2}(\cdots\gamma_{a_n}(x)))$, where n is a positive integer, $a_1, a_2, \ldots, a_n \in A$ and $\gamma_{a_i} \in \{\lambda_{a_i}, \rho_{a_i}\}$, for all $i \in \{1, 2, \ldots, n\}$. We denote the set of all iterated conjugates of elements of $X \subseteq A$ by $\Gamma(X)$. A subset X of A is called *normal*, if for all $x \in X$ and $a \in A$, $\lambda_a(x), \rho_a(x) \in X$. It is called *convex*, if for all $x, z \in X$ and $y \in A$, $x \leq y \leq z$ implies $y \in X$. A set X is said to be an *order filter* of $\mathbf{A}$, if $X = \uparrow X$, where $\uparrow X = \{a \in A \mid x \leq a$, for some $x \in X\}$. For every subset X of A, we define the sets

$$\begin{aligned}
&X \wedge 1 = \{x \wedge 1 \mid x \in X\},\\
&\Delta(X) = \{x \wedge (x\backslash 1) \wedge 1 \mid x \in X\},\\
&\Pi(X) = \{x_1 x_2 \cdots x_n \mid n \geq 1, x_i \in X\} \cup \{1\},\\
&\Xi(X) = \{a \in A \mid x \leq a \leq x\backslash 1, \text{ for some } x \in X\} \text{ and}\\
&\Xi^-(X) = \{a \in A \mid x \leq a \leq 1, \text{ for some } x \in X\}.
\end{aligned}$$

Note that the *negative part* $A^- = \{a \in A \mid a \leq 1\}$ of A is closed under multiplication and it contains 1, so it is a submonoid of $\mathbf{A}$. If $\mathbf{A}$ is a pointed residuated lattice, we denote its 0-free residuated lattice reduct by $\mathbf{A}_r$. If $\mathbf{A}$ is a residuated lattice, we set $\mathbf{A}_r = \mathbf{A}$.

THEOREM 4.8. *For every (pointed) residuated lattice* $\mathbf{A}$*, the following properties hold.*

1. *If S is a convex normal subalgebra of $\mathbf{A}_r$, M a convex normal in $\mathbf{A}$ submonoid of A^-, θ a congruence on $\mathbf{A}$ and F a deductive filter of $\mathbf{A}$, then*
   - (a) *$S_m(M) = \Xi(M)$, $S_c(\theta) = [1]_\theta$ and $S_f(F) = \Xi(F^-)$ are convex normal subalgebras of $\mathbf{A}_r$,*
   - (b) *$M_s(S) = S^-$, $M_c(\theta) = [1]_\theta^-$ and $M_f(F) = F^-$ are convex, normal in $\mathbf{A}$ submonoids of A^-,*
   - (c) *$\Theta_s(S) = \{(a, b) \mid a\backslash b \wedge 1, b\backslash a \wedge 1 \in S\}$, $\Theta_m(M) = \{(a, b) \mid a\backslash b \wedge 1, b/a \wedge 1 \in M\}$ and $\Theta_f(F) = \{(a, b) \mid a\backslash b, b\backslash a \in F\}$ are congruences on $\mathbf{A}$.*
   - (d) *$F_s(S) = \uparrow S$, $F_m(M) = \uparrow M$, and $F_c(\theta) = \uparrow [1]_\theta$ are deductive filters of $\mathbf{A}$.*

Moreover, the deductive filters, as well as the congruence relations and the convex normal submonoids, of $\mathbf{A}$ *and* $\mathbf{A}_r$ *are identical.*

2. *The convex, normal subalgebras of* $\mathbf{A}_r$*, the convex, normal in* $\mathbf{A}$ *submonoids of* A^- *and the deductive filters of* $\mathbf{A}$ *form lattices, denoted by* $\mathbf{CNS}(\mathbf{A}_r)$, $\mathbf{CNM}(\mathbf{A})$ *and* $\mathbf{Fil}(\mathbf{A})$*, respectively. All these lattices are isomorphic to the congruence lattice* $\mathbf{Con}(\mathbf{A})$ *of* $\mathbf{A}$ *via the appropriate pairs of maps defined above.*

3. *If* X *is a subset of* A^- *and* Y *is a subset of* A*, then*

   (a) *the convex, normal in* $\mathbf{A}$ *submonoid* $M(X)$ *of* A^- *generated by* X *is equal to* $\Xi^-\Pi\Gamma(X)$.

   (b) *The convex, normal subalgebra* $S(Y)$ *of* $\mathbf{A}$ *generated by* Y *is equal to* $\Xi\Pi\Gamma\Delta(Y)$.

   (c) *The deductive filter* $F(Y)$ *of* $\mathbf{A}$ *generated by* Y *is equal to* $\uparrow\Pi\Gamma(Y\wedge 1) = \uparrow\Pi\Gamma(Y)$.

   (d) *The congruence* $\Theta(P)$ *on* $\mathbf{A}$ *generated by a set of pairs* P *is equal to* $\Theta_m(M(P'))$*, where* $P' = \{a\backslash b \wedge b\backslash a \wedge 1 \mid (a,b) \in P\}$.

Proof. The parts of (1) and (2) that do not refer to deductive filters, as well as (3a) and (3b) are shown in [12]; see also [28]. Moreover, it follows from Theorem 3.4 and from Theorem 5.1 and Lemma 5.2 of [7] that for every (pointed) residuated lattice $\mathbf{A}$, the maps $F \mapsto \Theta_f(F)$ and $\theta \mapsto F_\theta = \{a \in A \mid 1\ \theta\ 1 \wedge a\}$ are mutually inverse lattice isomorphisms between the lattices $\mathbf{Fil}(\mathbf{A})$ and $\mathbf{Con}(\mathbf{A})$. It is immediate that $F_\theta = F_c(\theta)$. Therefore, the remaining parts of (1) and (2) will follow, if we show that the proposed maps are indeed the compositions of the isomorphisms already established.

We will show that $S_f(F) = S_c(\Theta_f(F))$. If $a \in S_c(\Theta_f(F))$ then $a\ \Theta_f(F)\ 1$, so $a\backslash 1, 1\backslash a \in F$. Hence $a, 1/a \in F$, by (*sym*). Since $1 \in F$, we get $x = a \wedge 1/a \wedge 1 \in F^-$, by (*adj*). Obviously, $x \leq a$; also $a \leq (1/a)\backslash 1 \leq x\backslash 1$. Thus, $a \in S_f(F)$. Conversely, if $a \in S_f(F)$, then $x \leq a \leq x\backslash 1$, for some $x \in F^-$. So, $a \in F$, by (*up*), and $1/(x\backslash 1) \leq 1/a$. Since, $x \leq 1/(x\backslash 1)$, we have $x \leq 1/a$ and $1/a \in F$, by (*up*). Thus both $a\backslash 1$ and $1\backslash a$ are in F. Hence, $a \in [1]_{\Theta_f(F)}$.

$M_f(F) = M_s(S_f(F))$ follows from the fact that $F^- = (\Xi(F^-))^-$. Indeed, $a \in (\Xi(F^-))^-$, iff $a \leq 1$ and $x \leq a \leq 1/x$ for some $x \in F^-$, iff $a \in F^-$.

To show that $F_s(S) = F_c(\Theta_s(S))$, note that $a \in F_c(\Theta_s(S))$ iff $1\ \Theta_s(S)\ 1 \wedge a$ iff $1\backslash(1 \wedge a) \wedge 1, (1 \wedge a)\backslash 1 \wedge 1 \in S$ iff $1 \wedge a \in S$ iff $a \in \uparrow S$. An identical argument shows that $F_m(M) = F_c(\Theta_m(M))$.

Also, note that congruence relations do not change under the expansion of the language by a constant. So, the congruences of a pointed residuated lattice coincide with the congruences of its 0-free residuated-lattice reduct.

(3c) We will first show that $F(Y) = F(Y \wedge 1)$. If $y \in Y$, then $y \wedge 1 \leq y$, so $y \in \uparrow(Y \wedge 1) \subseteq F(Y \wedge 1)$. Hence, $F(Y) \subseteq F(Y \wedge 1)$. Conversely, if $y \in Y$, then $y \wedge 1 \in F(Y)$, so $Y \wedge 1 \subseteq F(Y)$. Thus, $F(Y \wedge 1) \subseteq F(Y)$.

By (1b), $(F(Y \wedge 1))^-$ is a convex, normal in $\mathbf{A}$ submonoid of A^- and contains $Y \wedge 1$. So, it contains $M(Y \wedge 1)$; hence $M(Y \wedge 1) \subseteq F(Y \wedge 1)$. Since $F(Y \wedge 1)$ is increasing, we have $\uparrow M(Y \wedge 1) \subseteq F(Y \wedge 1)$. On the other hand, $\uparrow M(Y \wedge 1)$ is a deductive filter, by (1d), and it contains $Y \wedge 1$, so it contains $F(Y \wedge 1)$. Thus, $F(Y \wedge 1) = \uparrow M(Y \wedge 1) = \uparrow \Xi^- \Pi\Gamma(Y \wedge 1) = \uparrow \Pi\Gamma(Y \wedge 1)$, by (3a). Consequently, $F(Y) = \uparrow \Pi\Gamma(Y \wedge 1)$.

If $x \in \uparrow \Pi\Gamma(Y)$, then it is greater or equal to a product of conjugates of elements y_i of Y. This product is in turn greater or equal to the product of the same conjugates of the elements $y_i \wedge 1$; so, $x \in \uparrow \Pi\Gamma(Y \wedge 1)$. Thus, $\Pi\Gamma(Y) \subseteq \uparrow \Pi\Gamma(Y \wedge 1)$. Conversely, if $x \in \uparrow \Pi\Gamma(Y \wedge 1)$, then it is greater or equal to a product of conjugates of elements $y_i \wedge 1$, where $y_i \in Y$. Note that $y_i \wedge 1$ is the left conjugate of y_i by 1, so x is greater or equal to a product of conjugates of the elements y_i; i.e., $x \in \uparrow \Pi\Gamma(Y)$. Consequently, $\uparrow \Pi\Gamma(Y) = \uparrow \Pi\Gamma(Y \wedge 1)$.

(3d) First we will show that $\Theta(P) = \Theta(P' \times \{1\})$. For every congruence θ on $\mathbf{A}$, and every $a, b \in A$, if $a\ \theta\ b$, then $(a\backslash b \wedge 1)\ \theta\ (b\backslash b \wedge 1) = 1$. Likewise, $b\backslash a \wedge 1\ \theta\ 1$, so $a\backslash b \wedge b\backslash a \wedge 1\ \theta\ 1$. Conversely, if $a\backslash b \wedge b\backslash a \wedge 1\ \theta\ 1$, then since $a\backslash b \wedge b\backslash a \wedge 1 \leq a\backslash b \wedge 1 \leq 1$ and the congruence blocks of θ are convex (θ is a lattice congruence), we have $a\backslash b \wedge 1\ \theta\ 1$; so, $a(a\backslash b \wedge 1)\ \theta\ a$. Since $a(a\backslash b \wedge 1) \leq a(a\backslash b) \wedge a \leq b \wedge a \leq a$, we get $a \wedge b\ \theta\ a$. Likewise, we have $a \wedge b\ \theta\ b$, so $a\ \theta\ b$. Thus, for every congruence θ on $\mathbf{A}$, and every $a, b \in A$, $(a, b) \in \theta$ iff $(a\backslash b \wedge b\backslash a \wedge 1, 1) \in \theta$. Consequently, for every $(a, b) \in P$, we have $(a\backslash b \wedge b\backslash a \wedge 1, 1) \in \Theta(P)$. So, $P' \times \{1\} \subseteq \Theta(P)$; hence $\Theta(P' \times \{1\}) \subseteq \Theta(P)$. Conversely, for every $(a, b) \in P$, we have $(a, b) \in \Theta(P' \times \{1\})$, since $(a\backslash b \wedge b\backslash a \wedge 1, 1) \in \Theta(P' \times \{1\})$. So, $P \subseteq \Theta(P' \times \{1\})$; hence $\Theta(P) \subseteq \Theta(P' \times \{1\})$.

Finally, we will prove that for every subset X of A^-, $\Theta(X \times \{1\}) = \Theta_m(M(X))$. If $x \in X$, then $x/1 \wedge 1 = x, 1/x \wedge 1 = 1 \in M(X)$, so $(x, 1) \in \Theta_m(M(X))$. Consequently, $X \times \{1\} \subseteq \Theta_m(M(X))$, hence $\Theta(X \times \{1\}) \subseteq \Theta_m(M(X))$. Conversely, if $(a, b) \in \Theta_m(M(X))$, then $a/b \wedge 1, b/a \wedge 1 \in M(X) = \Xi^- \Pi\Gamma(X)$, so $p \leq a/b$ and $q \leq b/a$, for some $p, q \in \Pi\Gamma(X)$. Thus, $pb \leq a$, $qa \leq b$, so $pqa \leq pb \leq a$. On the other hand since every element of X is congruent to 1 modulo $\Theta(X \times \{1\})$, every conjugate and every product of conjugates of elements of X is congruent to 1. In particular, p, q and pq

are congruent to 1. So, $(pqa, a), (pb, b) \in \Theta(X \times \{1\})$. Since $pqa \leq pb \leq a$, we have $(a, b) \in \Theta(X \times \{1\})$. ■

It is worth mentioning that the the relations established between pointed residuated lattices and **FL** can be transferred verbatim to relations between residuated lattices and $\mathbf{FL}^+$, the system obtained from **FL** by removing the constant 0 from the language.

4.3. Parametrized local deduction theorem

By combining the algebraization result of the previous section with the result on filter generation, we provide a specific form of a parametrized local deduction theorem for substructural logics. This theorem, together with the algebraization, connects algebraic arguments with logical derivations, and thus has many important ramifications, as it can be seen in the next section as well as in our second paper on substructural logics [21]. In the following theorem, we give preference to $\backslash$ over $/$; nevertheless, the result can be stated also in terms of the other division connective.

THEOREM 4.9. *If $\Sigma \cup \Delta \cup \{\phi\}$ is a subset of $Fm_{\mathcal{L}}$ and* **L** *is a substructural logic, then*

$$\Sigma, \Delta \vdash_{\mathbf{L}} \phi \quad \textit{iff} \quad \Sigma \vdash_{\mathbf{L}} (\textstyle\prod_{i=1}^{n} \gamma_i(\psi_i))\backslash\phi,$$

for some non-negative integer n, iterated conjugates γ_i and $\psi_i \in \Delta$, $i < n$.

If **L** *is commutative, then*

$$\Sigma, \Delta \vdash_{\mathbf{L}} \phi \quad \textit{iff} \quad \Sigma \vdash_{\mathbf{L}} (\textstyle\prod_{i=1}^{n} (\psi_i \wedge 1)) \to \phi,$$

for some non-negative integer n,and $\psi_i \in \Delta$, $i < n$.

PROOF. Assume that $\Sigma = \{\psi_j \mid j \in J\}$, $\Delta = \{\psi_i \mid i \in I\}$ and $K = I \cup J$. Then, $\Sigma, \Delta \vdash_{\mathbf{L}} \phi$ iff $\{1 \approx \psi_k \wedge 1 \mid k \in K\} \models_{\mathcal{V}(\mathbf{L})} 1 \approx \phi \wedge 1$, by Theorem 3.4(1). Let $F_{\mathcal{V}(\mathbf{L})}$ be the free algebra in $\mathcal{V}(\mathbf{L})$. By general Universal Algebra considerations, the last statement is equivalent to the fact that $(1, \phi \wedge 1)$ is in the congruence of $F_{\mathcal{V}(\mathbf{L})}$ generated by the set $\{(1, \psi_k \wedge 1) \mid k \in K\}$; here we identify terms with their equivalence classes in the free algebra. Using Theorem 4.8(3d), this is the case if and only if $(1, \phi \wedge 1) \in \Theta_m(M(\{\psi_k \wedge 1 \mid k \in K\}))$, which is the case, by Theorem 4.8(1c), if and only if $\phi \wedge 1$ is in the convex normal submonoid of $F_{\mathcal{V}(\mathbf{L})}$ generated by $\{\psi_k \wedge 1 \mid k \in K\}$. By Theorem 4.8(3a) this is equivalent to the fact that in $F_{\mathcal{V}(\mathbf{L})}$ we have $\prod_{m=1}^{n} \gamma_m(\psi_{k_m}) \leq \phi$, for some non-negative integer n, some iterated conjugates γ_m, $1 \leq m \leq n$, and some $k_m \in K$, where $1 \leq m \leq n$.

Note that if a, b are elements of a residuated lattice, then $b(b\backslash ab \wedge 1) \leq ab$; i.e. $b\lambda_b(a) \leq ab$. This allows us to rearrange the order of the factors in a product, at the cost of replacing some factors by iterated conjugates of them, to obtain a product less than or equal to the original. In the present case the insertion of additional iterated conjugates and the fact that the resulting product is less than or equal to the original one do not create any problem. Therefore, continuing the sequence of equivalences, we obtain $F_{\mathcal{V}(\mathbf{L})} \models \prod_{m=1}^{n_1} \gamma'_m(\psi_{i_m}) \prod_{l=1}^{n_2} \gamma''_l(\psi_{j_l}) \leq \phi$, for some non-negative integers n_1, n_2, some iterated conjugates γ'_m, $1 \leq m \leq n_1$, γ''_l, $1 \leq l \leq n_2$, some i_m, $1 \leq m \leq n_1$, in some finite subset I_0 of I and some j_l, $1 \leq l \leq n_2$, in some finite subset J_0 of J. Using the fact that $\prod_{m=1}^{n_1} \gamma'_m(\psi_{i_m}) \prod_{l=1}^{n_2} \gamma''_l(\psi_{j_l}) \leq \phi$ is equivalent to $\prod_{l=1}^{n_2} \gamma''_l(\psi_{j_l}) \leq (\prod_{m=1}^{n_1} \gamma'_m(\psi_{i_m}))\backslash\phi$ and working as before we obtain $\Sigma \vdash_{\mathbf{L}} (\prod_{m=1}^{n_1} \gamma'_m(\psi_{i_m}))\backslash\phi$.

For the commutative case, note that if a, b are elements of a residuated lattice, then $a \wedge 1 \leq b\backslash ab \wedge 1 = \lambda_b(a)$ and $a \wedge 1 \leq \rho_b(a)$; hence, $a \wedge 1 \leq \gamma(a)$, for every iterated conjugate γ. Therefore, any product of conjugates is bounded from below by a product of simple conjugates of the form $\lambda_1(a) = a \wedge 1$. ■

If we set $\Delta = \{\psi\}$ in the preceding theorem, we see that substructural logics have a *parametrized local deduction theorem* (PLDT), in the sense of Czelakowski and Dziobiak. These two authors prove that *protoalgebraic logics*, and hence algebraizable logics also, have a PLDT; therefore, substructural logics have a PLDT, by Theorem 3.4. The proof for protoalgebraic logics is not constructive, but Theorem 4.9 shows a concrete form of PLDT for substructural logics.

It is noted in Theorem 4.9 that under the assumption of commutativity a *local deduction theorem* (LDT), that is a form of the PLDT without parameters, can be shown to hold. Recall that a consequence relation $\vdash$ over a set of formulas Fm has a LDT iff there exists a family $\mathcal{P}$ of sets $P(x, y)$ of formulas over two variables x, y, such that for all $T \cup \{\phi, \psi\} \subseteq Fm$, $T \cup \{\phi\} \vdash \psi$ iff $T \vdash P(\phi, \psi)$, for some $P(x, y) \in \mathcal{P}$. Actually, a LDT in the form given in Theorem 4.9 holds under conditions weaker that commutativity; for example it holds for the logic corresponding to the variety axiomatized relative to $\mathcal{FL}$ by the equation $(x \wedge 1)^n y \approx y(x \wedge 1)^n$, for any fixed positive integer n; for further remarks on the LDT for substructural logics, see Corollary 5.7 and [18]. For more on the PLDT and LDT, see [15], [14] and [8].

In particular, if $\mathbf{L}$ is commutative and integral, then we obtain the familiar form:

$$\Sigma, \psi \vdash_{\mathbf{L}} \phi \text{ iff } \Sigma \vdash_{\mathbf{L}} \psi^n \to \phi, \text{ for some non-negative integer } n.$$

We also note that if $\mathbf{L}$ is commutative and contractive, then the $\psi_i \in \Delta$ in the LDT of Theorem 4.9, can be taken to be distinct.

Theorem 4.9 is a generalization of various LDT's for particular substructural logics; see chapter 2, section 3.2 of [42] and section 4.3 of [43].

5. Interpolation

In this section, we discuss the interpolation property for substructural logics. We consider various types of interpolation properties and we describe their connections. As a corollary we obtain the amalgamation property for some varieties of residuated lattices.

We say that a substructural logic $\mathbf{L}$ has the *interpolation property*, or just IP for short, if for all formulas ϕ, ψ, whenever $\vdash_{\mathbf{L}} \phi \backslash \psi$, there exists a formula σ such that

1. $\vdash_{\mathbf{L}} \phi \backslash \sigma$ and $\vdash_{\mathbf{L}} \sigma \backslash \psi$, and
2. $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$,

where $Var(\phi)$ denotes the set of propositional variables in ϕ. A formula σ that satisfies conditions (1) and (2) is called an *interpolant* of $\phi \backslash \psi$.

A substructural logic $\mathbf{L}$ has the *deductive interpolation property*, or just DIP for short, if for all formulas ϕ, ψ, whenever $\phi \vdash_{\mathbf{L}} \psi$, there exists a formula σ such that

1. $\phi \vdash_{\mathbf{L}} \sigma$ and $\sigma \vdash_{\mathbf{L}} \psi$, and
2. $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$.

The DIP is called *turnstile interpolation property*, by Madarasz [31] and *interpolation property for deducibility*, by Maksimova [32].

Note that if a substructural logic satisfies the standard deduction theorem, that is if $\phi \vdash_{\mathbf{L}} \psi$ iff $\vdash_{\mathbf{L}} \phi \backslash \psi$, for all ϕ, ψ, then DIP is equivalent to IP. For example, superintuitionistic logics fall in this class.

A substructural logic $\mathbf{L}$ has the *strong deductive interpolation property*, or just SDIP for short, if for all sets of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, then there exists a set of formulas Δ such that

1. $\Gamma \vdash_{\mathbf{L}} \delta$, for all $\delta \in \Delta$ and $\Delta, \Sigma \vdash_{\mathbf{L}} \psi$, and
2. $Var(\Delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\psi\})$.

The SDIP is called *Maehara interpolation property* in [16] and GINT in [35]. Note that the *interpolant set* Δ in the SDIP can be chosen to be a singleton in the case of substructural logics, because $\vdash_{\mathbf{L}}$ is finitary and conjunctive; see Lemma 3.1. More generally, each modification of the SDIP, obtained by stipulating that some of the sets Γ, Σ or Δ are finite or singletons, is equivalent to the SDIP. Moreover, any of the above modifications that assumes that Σ is empty is equivalent to DIP.

LEMMA 5.1.

1. *If a substructural logic has the SDIP then it has the DIP, as well.*
2. *For commutative substructural logics the converse is also true. In general, the converse holds for any substructural logic with a LDT.*

PROOF. (1) In view of the preceding remark, DIP is a special case of SDIP for $\Gamma = \{\phi\}$ and $\Sigma = \emptyset$.

(2) Let $\mathbf{L}$ be a commutative substructural logic that has the DIP and let $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$. By the local deduction theorem, we have $\Gamma \vdash_{\mathbf{L}} (\prod_{i=1}^{n}(\sigma_i \wedge 1)) \to \psi$, for some $\sigma_i \in \Sigma$. By Lemma 3.1(1) and (2), there is a finite subset Γ_0 of Γ such that $\bigwedge \Gamma_0 \vdash_{\mathbf{L}} (\prod_{i=1}^{n}(\sigma_i \wedge 1)) \to \psi$. By the DIP, there exists a formula σ such that $\bigwedge \Gamma_0 \vdash_{\mathbf{L}} \sigma$; $\sigma \vdash_{\mathbf{L}} (\prod_{i=1}^{n}(\sigma_i \wedge 1)) \to \psi$; and $Var(\sigma) \subseteq Var(\bigwedge \Gamma_0) \cap Var(\prod_{i=1}^{n}(\sigma_i \wedge 1)) \to \psi)$. So, we have $\Gamma \vdash_{\mathbf{L}} \sigma$; $\sigma, \Sigma \vdash_{\mathbf{L}} \psi$; and $Var(\sigma) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\psi\})$.

It is clear from the proof above that (1) holds if $\vdash_{\mathbf{L}}$ is a finitary and conjunctive consequence relation, and that (2) holds under the additional assumption that $\vdash_{\mathbf{L}}$ admits a LDT; see also [16], p. 211. ∎

LEMMA 5.2. *If a commutative substructural logic has the IP, then it has the SDIP.*

PROOF. Assume that $\mathbf{L}$ is a commutative substructural logic that has the IP; by the preceding lemma, it suffices to show that $\mathbf{L}$ has the DIP. If $\phi \vdash_{\mathbf{L}} \psi$, then, by the local deduction theorem (Theorem 4.9), there exists a non-negative integer n such that $\vdash_{\mathbf{L}} (\phi \wedge 1)^n \to \psi$. So, by the IP, there exists a σ such that $\vdash_{\mathbf{L}} (\phi \wedge 1)^n \to \sigma$, $\vdash_{\mathbf{L}} \sigma \to \psi$ and $Var(\sigma) \subseteq Var((\phi \wedge 1)^n) \cap Var(\psi)$. Consequently, there exists a σ such that $\phi \vdash_{\mathbf{L}} \sigma$, $\sigma \vdash_{\mathbf{L}} \psi$ and $Var(\sigma) \subseteq Var(\phi) \cap Var(\psi)$. ∎

A subvariety $\mathcal{V}$ of $\mathcal{FL}$ has the *congruence extension property*, CEP for brevity, if, for every $\mathbf{A} \in \mathcal{V}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$ and for every congruence θ on $\mathbf{B}$, there exists a congruence θ' on $\mathbf{A}$ such that $\theta' \cap B^2 = \theta$.

Recall that a variety $\mathcal{V}$ has the *amalgamation property*, AP for brevity, if whenever $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are in $\mathcal{V}$ and $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ are embeddings, then there exists an algebra $\mathbf{D}$ in $\mathcal{V}$ and embeddings $f' : \mathbf{B} \to \mathbf{D}$, $g' : \mathbf{C} \to \mathbf{D}$ such that $f' \circ f = g' \circ g$.

A subvariety $\mathcal{V}$ of $\mathcal{FL}$ has the *equational interpolation property*, EqIP for brevity, if for every set of equations $G \cup S \cup \{\varepsilon\}$, whenever $G, S \models_{\mathcal{V}} \varepsilon$, there exists a set of equations D such that

1. $G \models_{\mathcal{V}} \delta$, for all $\delta \in D$ and $D, S \models_{\mathcal{V}} \varepsilon$, and
2. $Var(D) \subseteq Var(G) \cap Var(S \cup \{\varepsilon\})$.

Our definition of EqIP is essentially the same as the one considered in [47]. The additional assumption that the set of terms on the common variables of G and $S \cup \{\varepsilon\}$ is non-empty is satisfied in our case, because of the existence of constants in the language, and is omitted from our definition. Note that [29] and [3] consider a different notion, which is equivalent to the algebraic formulation of our DIP.

Lemma 5.3. *[47] A class of algebras has the EqIP iff it has the AP and the CEP.*

The following is a direct consequence of the algebraization theorem and the definitions.

Lemma 5.4. *A substructural logic* $\mathbf{L}$ *has the SDIP iff* $\mathcal{V}(\mathbf{L})$ *has the EqIP.*

Corollary 5.5. *A substructural logic* $\mathbf{L}$ *has the SDIP iff* $\mathcal{V}(\mathbf{L})$ *has the CEP and the AP.*

In [35] two properties, ROB* and limited GINT, both weaker than EqIP are introduced. The ROB* property is shown to be equivalent to AP and limited GINT is shown to be equivalent to CEP. In view of the algebraization theorem, it is natural to introduce the following two properties for logics, RP and ExIP, that correspond to ROB* and limited GINT.

A substructural logic $\mathbf{L}$ has the *Robinson property*, RP for brevity, if for all sets of formulas $\Gamma \cup \Sigma \cup \{\phi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \phi$ implies $\Sigma \vdash_{\mathbf{L}} \phi$, under the assumption that $\Gamma \vdash_{\mathbf{L}} \psi$ iff $\Sigma \vdash_{\mathbf{L}} \psi$, for all ψ such that $Var(\psi) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\phi\})$. The RP is called *ordinary interpolation property* OIP in [16].

A substructural logic **L** has the *extension interpolation property*, ExIP for brevity, if for all sets of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, then there exists a set Δ of formulas such that

1. $\Gamma \vdash_{\mathbf{L}} \delta$, for all $\delta \in \Delta$ and $\Delta, \Sigma \vdash_{\mathbf{L}} \psi$, and
2. $Var(\Delta) \subseteq Var(\Sigma \cup \{\psi\})$.

The property ExIP is introduced in [16]. Note that the corresponding property, limited GINT, given in [35] demands that $Var(\Sigma \cup \{\psi\}) \subseteq Var(\Gamma)$. Nevertheless, this assumption can be omitted, since we can add 'dummy' assumptions of the form $p\backslash p$ to Γ, for all $p \in Var(\Sigma \cup \{\psi\}) - Var(\Gamma)$, and obtain a set Γ' that can substitute Γ, and that satisfies $Var(\Sigma \cup \{\psi\}) \subseteq Var(\Gamma')$ and $\Gamma', \Sigma \vdash_{\mathbf{L}} \psi$.

The following is a consequence of Lemma 5.3, Lemma 5.4 and results in [35].

COROLLARY 5.6. *A substructural logic* **L** *has the SDIP iff it has both the RP and the ExIP.*

From [8] and [16] we have the following.

COROLLARY 5.7. *A substructural logic* **L** *has the ExIP iff* **L** *has a LDT iff* $\mathcal{V}(\mathbf{L})$ *has the CEP.*

We can summarize the relations, modulo algebraization, of the various properties we discussed as follows.

$$\begin{gathered} \text{EqIP} \Leftrightarrow \text{AP} + \text{CEP} \Leftrightarrow \text{SDIP} \Leftrightarrow \text{DIP} + \text{LDT} \\ \text{AP} \Leftrightarrow \text{ROB*} \Leftrightarrow \text{RP} \\ \text{CEP} \Leftrightarrow \text{ExIP} \Leftrightarrow \text{LDT} \end{gathered}$$

THEOREM 5.8. *If* **L** *is a commutative substructural logic, then the following conditions are equivalent.*

1. **L** *has the SDIP.*
2. **L** *has the DIP.*
3. **L** *has the RP.*
4. $\mathcal{V}(\mathbf{L})$ *has the AP.*

Consequently, if **L** *is commutative and has the IP, then* $\mathcal{V}(\mathbf{L})$ *has the AP.*

Note that the preceding theorem holds for all algebraizable, conjunctive consequence relations that have a LDT.

If $\mathbf{b}$ is any subset of $\{\mathbf{e}, \mathbf{c}, \mathbf{w}, \mathbf{i}\}$, the logic $\mathbf{FL_b}$ is called *basic substructural logic*. For example, $\mathbf{FL_w}$, $\mathbf{FL_{ec}}$ and $\mathbf{FL}$ are basic substructural logics. There is a total of ten such logics because $\mathbf{FL_{ecw}} = \mathbf{FL_{cw}}$, $\mathbf{FL_{eci}} = \mathbf{FL_{ci}}$ and $\mathbf{FL_i} \subseteq \mathbf{FL_w}$. If $\mathbf{L}$ is any of the basic substructural logics, the positive fragment $\mathbf{L}^+$ of $\mathbf{L}$ is defined as the logic on the 0-free fragment of the language axiomatized by the axioms and rules of the Gentzen system of $\mathbf{L}$ that do not involve 0 (essentially, it applies only to $(0w)$). Note that $\mathbf{FL_w}^+ = \mathbf{FL_i}^+$.

We attach the prefixes $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{K}$ to varieties of (pointed) residuated lattices to denote their subvarieties axiomatized by $x \leq 1$, $xy \approx yx$ and $x \leq x^2$, respectively. For example, $\mathcal{KCRL} = \mathcal{RL} \cap \mathrm{Mod}(x \leq x^2, xy \approx yx) = \mathcal{V}(\mathbf{FL}^+_{\mathbf{ec}})$.

Theorem 5.9. *Every basic substructural logic and every positive fragment of such a logic, except for* $\mathbf{FL_c}$ *and* $\mathbf{FL_c}^+$*, has the IP. Consequently, the varieties* $\mathcal{CFL}$*,* $\mathcal{CRL}$*,* $\mathcal{KCRL}$ *and* $\mathcal{CIRL}$ *have the AP.*

Proof. The fact that the logics have the IP can be shown by Maehara's method; see [38] for more details. ∎

T. Kowalski obtained the above result for $\mathbf{FL_{ew}}$ and $\mathcal{CIRL}$ in his unpublished draft, and H. Takamura extended the technique and obtained the result for $\mathbf{FL_e}$ and $\mathcal{CFL}$.

Proposition 5.10. *The properties IP and SDIP are independent.*

Proof. We first show that the IP does not imply the SDIP. It follows from Theorem 5.9 that $\mathbf{FL_w}$ has the IP. On the other hand, $\mathcal{V}(\mathbf{FL_w}) = \mathcal{IFL}$ does not have the CEP; [18] contains a 5-element integral pointed residuated lattice that does not have the CEP. By Lemma 5.3 and Lemma 5.4, $\mathbf{FL_w}$ does not have the SDIP.

To see that DIP does not imply the IP, let $\mathbf{MV}$ be Łukasiewicz' infinite-valued logic; see [13]. Recall that $\mathcal{V}(\mathbf{MV})$ is generated by the commutative pointed residuated lattice with underlying set the interval of real numbers $[0, 1]$, under the usual order, where $ab = max\{0, a + b - 1\}$ and $a \to b = min\{1, 1 - a + b\}$. It is easy to see that $\vdash_{\mathbf{MV}} (p \wedge \neg p) \to (q \vee \neg q)$, where p, q are distinct propositional variables and $\neg p = p \to 0$, but the formula has no interpolants. On the other hand, $\mathbf{MV}$ has the AP, see [33], so it has the DIP. In particular, for the example under discussion, i.e., for $p \wedge \neg p \vdash_{\mathbf{MV}} q \vee \neg q$ we have $p \wedge \neg p \vdash_{\mathbf{MV}} 0$, because $\vdash_{\mathbf{MV}} (p \wedge \neg p)^2 \to 0$, and $0 \vdash_{\mathbf{MV}} q \vee \neg q$. ∎

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