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Strong measure zero and infinite games

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Abstract We show that strong measure zero sets (in a σ-totally bounded metric space) can be characterized by the nonexistence of a winning strategy in a certain infinite game. We use this characterization to give a proof of the well known fact, originally conjectured by K. Prikry, that every dense G_δ subset of the real line contains a translate of every strong measure zero set. We also derive a related result which answers a question of J. Fickett.

Keywords Strong measure zero · Infinite game · First category

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1 Introduction

A metric space X is said to have *strong measure zero* if, for every sequence $(\varepsilon_n : n \in \mathbb{N})$ of positive numbers, there is a partition $X = \bigcup_{n\in\mathbb{N}} X_n$ with $\operatorname{diam}(X_n) \leq \varepsilon_n$ for each n. This notion was introduced by Borel [2], who conjectured that every strong measure zero set of real numbers is countable. In fact, by a celebrated result of Laver [6], as extended by Carlson [3], it is consistent with ZFC that every metric space of strong measure zero is countable. On the other hand, as noted by Sierpiński [10], the existence of an uncountable strong measure zero set of real numbers is a consequence of the

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continuum hypothesis. (Some properties related to strong measure zero are discussed in the papers of Duda and Telgársky [4] and Lelek [7].)

In an unpublished manuscript ("Solutions of some games", dated February 1970), Mycielski and Solovay introduced a natural "gamification" of strong measure zero. In Sect. 2 we define the Mycielski–Solovay game $G(X)$, as well as a technical variant $\hat{G}(X, \mathcal{F})$, and study conditions under which one player or the other has a winning strategy. We show that, among σ-totally bounded metric spaces, the strong measure zero spaces are characterized by the nonexistence of a winning strategy for White in these games.

Karel Prikry observed that a set X of real numbers has strong measure zero if every dense open subset of the real line contains a translate of X; and he conjectured that, conversely, every dense open subset and even every dense G_δ subset of the real line contains a translate of every strong measure zero set. In Sects. 3 and 4 we use our game-theoretic characterization of strong measure zero to give a proof of Prikry's conjecture. In fact, a concise, elegant, game-free proof of Prikry's conjecture is already available in Miller's [8] survey. Our proof is not shorter or simpler than Miller's, and from one point of view it may be regarded as merely an obfuscated presentation of Miller's argument. However, we think the motivating intuition from game theory may be of some independent interest. Also, our main result (Theorem 3) can be viewed as a generalization of Prikry's conjecture.

James Fickett asked whether there is a similar characterization of the sets X of real numbers such that every dense open set, or every dense G_δ set, contains a homothetic copy of X. The answer to Fickett's question for dense G_δ sets is that X has strong measure zero; this is proved in Sect. 4. For dense open sets the answer is that X is the union of a bounded set and a strong measure zero set; this is proved in Sect. 5.

Most of the results in this paper, except for Theorem 3 (about Cartesian products), were announced in an abstract [5].

2 Strong measure zero games

Given a metric space X, we define an infinitely long game $G(X)$ between two players, White and Black. At move n, first White chooses a real number $\varepsilon_n > 0$, and then Black chooses a set $B_n \subseteq X$ with $\operatorname{diam}(B_n) \leq \varepsilon_n$. Black wins a play $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots)$ of this game if $\bigcup_{n \in \mathbb{N}} B_n = X$, otherwise White wins. We say that the game $G(X)$ is a win for White (Black) if White (Black) has a winning strategy.

Theorem 1 *For any metric space X, the game $G(X)$ is a win for Black if and only if X is countable.*

Proof First, suppose X is countable, say $X = \{x_n : n \in \mathbb{N}\}$. Then Black has an obvious winning strategy: at move n, choose $B_n = \{x_n\}$.

Now suppose σ is a winning strategy for Black. For each point $x \in X$, choose a finite sequence $s(x) = (r_1^x, \ldots, r_{n(x)}^x)$ of positive rational numbers such that $x \in \sigma(r_1^x, \ldots, r_{n(x)}^x, r)$ for every rational number $r > 0$. (Such a sequence must exist, for otherwise White could defeat Black's strategy σ by choosing an infinite sequence of rational numbers so that Black never covers x.) It is easy to see that the map $x \mapsto s(x)$ is injective, and so X is countable. $\square$

Let X be a σ-totally bounded metric space, and let $\mathcal{F} = (F_n : n \in \mathbb{N})$ be a sequence of totally bounded subsets of X, such that $\bigcup_{n\in\mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each n. In this setting, besides the game $G(X)$, we define another game $\hat{G}(X, \mathcal{F})$ which is more difficult for Black. Namely, at move n, first White chooses $\varepsilon_n > 0$, and then Black chooses $B_n \subseteq F_n$ with $\operatorname{diam}(B_n) \le \varepsilon_n$; Black wins if $\limsup_n B_n = X$, otherwise White wins.

Lemma 1 *Given a totally bounded metric space F and a number $\delta > 0$, we can find a nonempty finite collection $\mathcal{B}$ of subsets of F, each of diameter at most δ, such that every subset of F of diameter at most $\frac{1}{3}\delta$ is contained in some member of $\mathcal{B}$.*

Proof Since F is totally bounded, for some $m \in \mathbb{N}$ we can write $F = U_1 \cup \cdots \cup U_m$ with $\operatorname{diam}(U_j) \le \frac{1}{3}\delta$ for each $j \in [m]$. Let $\mathcal{B} = \{W_1, \ldots, W_m\}$, where $W_j = \{x \in F : d(x, u) \le \frac{1}{3}\delta$ for some $u \in U_j\}$. $\square$

Theorem 2 *If X is a σ-totally bounded metric space, and if $\mathcal{F} = (F_n : n \in \mathbb{N})$ is a sequence of totally bounded subsets of X such that $\bigcup_{n\in\mathbb{N}} F_n = X$ and $F_n \subseteq F_{n+1}$ for each n, then the following statements are equivalent:*

(1) X does not have strong measure zero;
(2) $G(X)$ is a win for White;
(3) $\hat{G}(X, \mathcal{F})$ is a win for White.

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear; we will prove $\neg$(1) $\Rightarrow$ $\neg$(3). Assume that X has strong measure zero; we will show that $\hat{G}(X, \mathcal{F})$ is not a win for White. Let σ be any strategy for White in $\hat{G}(X, \mathcal{F})$.

Using Lemma 1, we can recursively define $\delta_n, \mathcal{B}_n$ ($n \in \mathbb{N}$) so that:

(i) $\delta_n = \min\{\sigma(B_1, B_2, \ldots, B_{n-1}) : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, \ldots, B_{n-1} \in \mathcal{B}_{n-1}\}$;
(ii) $\mathcal{B}_n$ is a nonempty finite collection of subsets of F_n of diameter at most δ_n;
(iii) every subset of F_n of diameter at most $\frac{1}{3}\delta_n$ is contained in some member of $\mathcal{B}_n$.

Partition $\mathbb{N}$ into disjoint sets M_n ($n \in \mathbb{N}$) so that $\min(M_n) \ge n$. Let $\alpha_k = \frac{1}{3}\delta_k$. Since each F_n has strong measure zero, we can choose sets A_k ($k \in \mathbb{N}$), with $\operatorname{diam}(A_k) \le \alpha_k$, so that for each $n \in \mathbb{N}$ we have $F_n = \bigcup_{k\in M_n} A_k$.

If $k \in \mathbb{N}$, then $k \in M_n$ for some $n \in \mathbb{N}$, and so $A_k \subseteq F_n \subseteq F_k$; since $\operatorname{diam}(A_k) \le \alpha_k = \frac{1}{3}\delta_k$, we can choose $B_k \in \mathcal{B}_k$ so that $A_k \subseteq B_k$. Finally, for each $n \in \mathbb{N}$ let $\varepsilon_n = \sigma(B_1, B_2, \ldots, B_{n-1})$. Since $B_n \subseteq F_n$ and $\operatorname{diam}(B_n) \le \delta_n \le \varepsilon_n$, the infinite sequence $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots, \varepsilon_n, B_n, \ldots)$ is a σ-play of the game $\hat{G}(X, \mathcal{F})$.

We claim that $\limsup_n B_n = X$. To see this, consider any point $x \in X$. Then $x \in F_n = \bigcup_{k\in M_n} A_k$ for all sufficiently large n; therefore, for each sufficiently large n, we have $x \in A_k \subseteq B_k$ for some $k \in M_n$. Since the sets M_n ($n \in \mathbb{N}$) are disjoint, it follows that the set $\{k : x \in B_k\}$ is infinite, whence $x \in \limsup_n B_n$.

Thus Black wins the σ-play $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots, \varepsilon_n, B_n, \ldots)$ of $\hat{G}(X, \mathcal{F})$. As White's strategy σ was arbitrary, it follows that the game $\hat{G}(X, \mathcal{F})$ is not a win for White. $\square$

3 A theorem on Cartesian products

For metric spaces X and Y, a set $A \subseteq X \times Y$ is *vertically dense* if, for each $x \in X$, the set $\{y \in Y : (x, y) \in A\}$ is dense in Y.

Lemma 2 *Let X, Y be metric spaces. Given a compact set $K \subseteq X$, a nonempty open set $W \subseteq Y$, and a vertically dense open set $A \subseteq X \times Y$, we can find a number $\varepsilon > 0$ such that, for any set $B \subseteq K$ with $\operatorname{diam}(B) \leq \varepsilon$, there is a nonempty open set $V \subseteq W$ with $B \times V \subseteq A$.*

Proof The collection

$$\mathcal{S} = \{S \subseteq X : S \text{ is open, and } S \times V \subseteq A \text{ for some nonempty open set } V \subseteq W\}$$

is an open cover of X. By Lebesgue's covering lemma, we can find a number $\varepsilon > 0$ such that, for any set $B \subseteq K$ with $\operatorname{diam}(B) \leq \varepsilon$, we have $B \subseteq S$ for some $S \in \mathcal{S}$, and so $B \times V \subseteq A$ for some nonempty open set $V \subseteq W$. $\square$

Theorem 3 *If X is a σ-compact metric space, Y is a complete metric space with no isolated points, $Z \subseteq X$ is a strong measure zero set, $A \subseteq X \times Y$ is a vertically dense G_δ set, $U \subseteq Y$ is a nonempty open set, and $D \subseteq Y$ is a dense G_δ set, then there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z \times P \subseteq A$.*

Proof Let $A = \bigcap_{n \in \mathbb{N}} A_n$, where A_n is open and $A_n \supseteq A_{n+1}$ for each n; let $D = \bigcap_{n \in \mathbb{N}} D_n$, where D_n is open and $D_n \supseteq D_{n+1}$ for each n; and let $X = \bigcup_{n \in \mathbb{N}} K_n$, where K_n is compact and $K_n \subseteq K_{n+1}$ for each n. Let $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ where $F_n = Z \cap K_n$; thus $Z = \bigcup_{n \in \mathbb{N}} F_n$, each F_n is totally bounded, and $F_n \subseteq F_{n+1}$ for each n.

By Theorem 2, White has no winning strategy in $\hat{G}(Z, \mathcal{F})$; i.e., if σ is any strategy for White in $\hat{G}(Z, \mathcal{F})$, then there is a σ-play $(\varepsilon_1, B_1, \varepsilon_2, B_2, \ldots)$ of $\hat{G}(Z, \mathcal{F})$ such that $\limsup_n B_n = Z$. To prove Theorem 3, it will suffice to find a strategy for White in $\hat{G}(Z, \mathcal{F})$ which ensures that $(\limsup_n B_n) \times P \subseteq A$ for some nonempty perfect set $P \subseteq U \cap D$.

Let $U_\emptyset = U$. At move n, suppose nonempty open sets $U_s \subseteq Y$ $(s \in \{0, 1\}^{n-1})$ have been defined. White chooses $\varepsilon_n > 0$ so that, for each set $B \subseteq F_n$ with $\operatorname{diam}(B) \leq \varepsilon_n$, and for each $s \in \{0, 1\}^{n-1}$, there is a nonempty open set $V \subseteq U_s \cap D_n$ with $B \times V \subseteq A_n$; this is possible by Lemma 2. After Black chooses $B_n \subseteq F_n$ with $\operatorname{diam}(B_n) \leq \varepsilon_n$, White chooses for each $s \in \{0, 1\}^{n-1}$ a nonempty open set $V_s \subseteq U_s \cap D_n$ with $B_n \times V_s \subseteq A_n$, and two nonempty open sets $U_{s^\frown 0}, U_{s^\frown 1}$ of diameter at most 2^{-n}, with $\overline{U}_{s^\frown 0} \cup \overline{U}_{s^\frown 1} \subseteq V_s$ and $\overline{U}_{s^\frown 0} \cap \overline{U}_{s^\frown 1} = \emptyset$.

For each $s \in \{0,1\}^{\mathbb{N}}$ there is a unique point $f(s) \in Y$ such that $\bigcap_{n\in\mathbb{N}} U_{s(1),s(2),\ldots,s(n)} = \{f(s)\}$. The mapping $f : \{0,1\}^{\mathbb{N}} \to Y$ so defined is continuous and injective, whence the set $P = \{f(s) : s \in \{0,1\}^{\mathbb{N}}\}$ is perfect.

Now $P \subseteq U$ and $P \subseteq D_n$ for all n, so $P \subseteq U \cap D$. Finally, since $B_n \times P \subseteq A_n$ for all n, it follows that $(\limsup_n B_n) \times P \subseteq \limsup_n A_n = \bigcap_{n\in\mathbb{N}} A_n = A$. □

4 Prikry's conjecture

Theorem 4 *For any set $Z \subseteq \mathbb{R}$ the following statements are equivalent:*

(1) Z has strong measure zero;
(2) there is a number $k \in \mathbb{N}$ such that Z can be covered by k translates of every dense open subset of $\mathbb{R}$;
(3) for every dense G_δ set $D \subseteq \mathbb{R}$, there are countable sets $A, B \subseteq \mathbb{R}$ such that $Z \subseteq AD + B$;
(4) for any dense G_δ set $D \subseteq \mathbb{R}$ and any nonempty open set $U \subseteq \mathbb{R}$, there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z + P \subseteq D$.

Proof Clearly (4) ⇒ (2) and (4) ⇒ (3). We will prove (1) ⇒ (4), (2) ⇒ (1), and (3) ⇒ (1).

(1) ⇒ (4): Let a strong measure zero set $Z \subseteq \mathbb{R}$, a dense G_δ set $D \subseteq \mathbb{R}$, and a nonempty open set $U \subseteq \mathbb{R}$ be given. Let $X = Y = \mathbb{R}$, and let $A = \{(x, y) \in \mathbb{R}\times\mathbb{R} : x + y \in D\}$. Since the hypotheses of Theorem 3 are satisfied, there is a nonempty perfect set $P \subseteq U \cap D$ such that $Z \times P \subseteq A$, i.e., $Z + P \subseteq D$.

(2) ⇒ (1): Suppose $Z \subseteq \mathbb{R}$ and $k \in \mathbb{N}$ are such that Z can be covered by k translates of every dense open set; we have to show that Z has strong measure zero. Let positive numbers ε_n $(n \in \mathbb{N})$ be given. Partition $\mathbb{N}$ into disjoint infinite sets $M_1, \ldots, M_k$ and define $f : \mathbb{N} \to [k]$ so that $n \in M_{f(n)}$. For each $j \in [k]$, choose open intervals I_n $(n \in M_j)$ with $\mathrm{diam}(I_n) = \varepsilon_n$ so that the set $D_j = \bigcup_{n\in M_j} I_n$ is dense in $\mathbb{R}$. Since $D = D_1 \cap \cdots \cap D_k$ is a dense open set, there are numbers $t_1, \ldots, t_k \in \mathbb{R}$ such that

$$Z \subseteq (t_1 + D) \cup \cdots \cup (t_k + D) \subseteq (t_1 + D_1) \cup \cdots \cup (t_k + D_k) = \bigcup_{n\in\mathbb{N}} (t_{f(n)} + I_n).$$

This shows that Z has strong measure zero.

(3) ⇒ (1): Suppose $Z \subseteq \mathbb{R}$ is such that, for every dense G_δ set $D \subseteq \mathbb{R}$, there are countable sets $A, B \subseteq \mathbb{R}$ with $Z \subseteq AD + B$. Let positive numbers ε_n $(n \in \mathbb{N})$ be given. Partition $\mathbb{N}$ into disjoint infinite sets M_j $(j \in \mathbb{N})$. For each $j \in \mathbb{N}$, choose open intervals I_n $(n \in M_j)$ with $\mathrm{diam}(I_n) \le \frac{1}{j}\varepsilon_n$, so that the set $D_j = \bigcup_{n\in M_j} I_n$ is dense in $\mathbb{R}$. Thus $D = \bigcap_{j\in\mathbb{N}} D_j$ is a dense G_δ set. Choose countable sets $A, B \subseteq \mathbb{R}$ with $Z \subseteq AD + B$. Choose an injection $j : A \times B \to \mathbb{N}$ with $j(a, b) \ge |a|$. Let

$M = \bigcup_{(a,b)\in A\times B} M_{j(a,b)}$. For each $n \in M$, there is a unique pair $(a_n, b_n) \in A \times B$ such that $n \in M_{j(a_n,b_n)}$. Now we have

$$\begin{aligned} Z \subseteq AD + B &= \bigcup_{(a,b)\in A\times B} (aD + b) \\ &\subseteq \bigcup_{(a,b)\in A\times B} \left(aD_{j(a,b)} + b\right) \\ &= \bigcup_{(a,b)\in A\times B} \bigcup_{n\in M_{j(a,b)}} (aI_n + b) \\ &= \bigcup_{n\in M} (a_n I_n + b_n). \end{aligned}$$

Since, for $n \in M$, we have

$$\operatorname{diam}(a_n I_n + b_n) = |a_n| \operatorname{diam}(I_n) \le \frac{|a_n|\varepsilon_n}{j(a_n, b_n)} \le \varepsilon_n,$$

this shows that Z has strong measure zero. □

Corollary 1 (Prikry's Conjecture) *For any set $X \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) X has strong measure zero;
(2) every dense open set contains a translate of X;
(3) every dense G_δ set contains a translate of X.

Proof Theorem 4. □

This is called Prikry's conjecture after Karel Prikry, who pointed out the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and conjectured that all three statements were equivalent. That statements (2) and (3) hold for every *countable* set X had been proved earlier by Scheeffer [9] and Bagemihl [1], respectively.

Sierpiński [11] asked whether there is an uncountable set X of real numbers which has strong measure zero, is always of the first category, and is such that every translate of X is contained in X except for a countable set of points. The next corollary answers Sierpiński's question in the negative.

Corollary 2 *If $X \subseteq \mathbb{R}$ is a set of the first category, and if $(X + t) \setminus X$ is countable for each $t \in \mathbb{R}$, then every strong measure zero subset of X is countable.*

Proof If Z is a strong measure zero subset of X, then by Corollary 1 we have $Z + t \subseteq \mathbb{R} \setminus X$ for some $t \in \mathbb{R}$, whence $Z + t = (Z + t)\setminus X \subseteq (X + t)\setminus X$ and $|Z| = |Z + t| \le |(X + t)\setminus X| \le \aleph_0$. □

5 Fickett's question

James Fickett asked us which sets X of real numbers have the property that every dense open set, or every dense G_δ set, contains a homothetic copy of X. For dense G_δ sets the answer is strong measure zero sets, as shown by Theorem 4. Fickett's question for dense open sets is answered by the following theorem.

Theorem 5 *For every set $X \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) for every dense open set $D \subseteq \mathbb{R}$ there exist $a, b \in \mathbb{R}$, $a > 0$, with $aX + b \subseteq D$;
(2) X is the union of a bounded set and a strong measure zero set.

Proof (2) $\Rightarrow$ (1): Let $X = K \cup Z$, where K is bounded and Z has strong measure zero, and let D be a dense open set. Choose $a > 0$ and an interval B so that $aK + B \subseteq D$. Choose a countable set M so that $B + M = \mathbb{R}$. Then $G = \bigcap_{m \in M}(D + m)$ is a dense G_δ set. Since aZ is a strong measure zero set, by Corollary 1 we have $aZ + t \subseteq G$ for some $t \in \mathbb{R}$. Write $t = b + m$ where $b \in B$, $m \in M$; then $aZ + b + m \subseteq G \subseteq D + m$, whence $aZ + b \subseteq D$. Since $b \in B$, we also have $aK + b \subseteq D$, and so $aX + b \subseteq D$.

$\neg$(2) $\Rightarrow$ $\neg$(1): Suppose X is *not* the union of a bounded set and a strong measure zero set. It follows that, given $k > 0$, we can find sets $X', X'' \subseteq X$ such that neither X' nor X'' has strong measure zero, and $\mathrm{d}(X', X'') > k$; hence we can find positive numbers ε_n ($n \in \mathbb{N}$) such that X is not covered by any sequence of intervals whose diameters are $k, \varepsilon_1, \varepsilon_2, \ldots$.

Let $\{r_m : m \in \mathbb{N}\}$ be dense in $\mathbb{R}$. For each $m \in \mathbb{N}$ we define J_m, K_m, k_m and $\varepsilon_{m,n}$ ($n \in \mathbb{N}$) satisfying the following conditions:

(i) J_m is a finite open interval containing r_m, and $\mathrm{diam}(J_m) \leq \frac{1}{p}\varepsilon_{p,m-p}$ for each $p \in [m-1]$;
(ii) K_m is a finite interval containing $J_1 \cup \cdots \cup J_m$;
(iii) $k_m = m \cdot \mathrm{diam}(K_m)$;
(iv) $\varepsilon_{m,1}, \varepsilon_{m,2}, \ldots$ are positive numbers such that X is not covered by any sequence of intervals whose diameters are $k_m, \varepsilon_{m,1}, \varepsilon_{m,2}, \ldots$.

Now $D = \bigcup_{m \in \mathbb{N}} J_m$ is a dense open set. Assume for a contradiction that $a, b \in \mathbb{R}$, $a > 0$, and $aX + b \subseteq D$. Define $\hat{S} = \frac{1}{a}(S - b)$ for $S \subseteq \mathbb{R}$. Fix $m \in \mathbb{N}$ with $\frac{1}{a} \leq m$. Then

$$X \subseteq \hat{D} = \hat{J}_1 \cup \cdots \cup \hat{J}_m \cup \hat{J}_{m+1} \cup \hat{J}_{m+2} \cup \cdots \subseteq \hat{K}_m \cup \hat{J}_{m+1} \cup \hat{J}_{m+2} \cup \cdots .$$

Since $\mathrm{diam}(\hat{K}_m) \leq m \cdot \mathrm{diam}(K_m) = k_m$, and $\mathrm{diam}(\hat{J}_{m+n}) \leq m \cdot \mathrm{diam}(J_{m+n}) \leq m \cdot \frac{1}{m}\varepsilon_{m,n} = \varepsilon_{m,n}$ for each $n \in \mathbb{N}$, this contradicts condition (iv). □

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