On the Role of Inconsistency in Quantum Foundational Debate and Hilbert Space Formulation

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This article is intended mainly to develop an expository outline of an inherently inconsistent reasoning in the development of quantum mechanics during 1920s, which set up the background of proposing different variants of quantum logic a bit later. We will discuss here two of the quantum logical variants with reference to Hilbert space formulation, based on the proposals of Bohr and Schrödinger as a result of addressing the same kernel of difficulties and will give a relative comparison. Our presentation is fairly informal, as our goal here is to simply sketch the central ideas leaving further details for other occasions.
Quanta 2022; 11: 28–41.

1 Background: Characterizing the foundational debate within an inherent human-centric framework

It is curious to note that the founding fathers were deeply dissatisfied with the non-standard look of quantum mechanics as it was gradually unfolded. In fact, most of them were guided by their underlying commitments to get *as much as possible* to old classical framework of reality, which conditioned their overall responses. We will classify issues chronologically, so that it can help to appreciate the mark of inconsistent reasoning in a wider background, and the outgrowth of two apparently different looking variants of quantum logic from the same embryo of overall feelings of *inconsistency*.

The outcome of the whole historical development of the language of standard quantum mechanics is described by Edwin Thompson Jaynes as follows

> But our present [quantum mechanical] formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature—all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen

how to unscramble. Yet we think that the un-scrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple. [1]

Though this passage seems to create an impression that, both Bohr and Heisenberg shared a unique version of interpretation of quantum mechanics, this was not actually so. It was others who *mixed* their opinions together within a common brand of *Copenhagen interpretation* later. Quantum formalism involves "in part incomplete human information" is more of Heisenberg's approach which was not approved by Bohr.

Though we are not quite in complete agreement with Jaynes, this small passage calls for a clarification of issues centered around the relation between Logic and Reality including the way human language—a very special feature of the Universe, evolved through history hosting often within it unqualified excess baggage. In fact, human languages as well as *our conceptual frameworks of understanding* as a whole arguably owe their structural origin to our everyday experience based on our slowly varying and moderately hot (almost *static* for our practical purposes) environment. And this framework of understanding had been spilled over through ages, characteristically conditioned by different (social) superstructures, in different human-made formal and conceptual artifacts.

Newton da Costa rightly noted

the notion of object, of property and of relation of both Aristotelian logic and of present day mathematical logic came from the static and Euclidean view of reality. [2]

Mathematical language of quantum mechanics is also one of the latest creations by such ordinary language-users living in and grown up within a *slowly varying (nearly permanent) static* and *Euclidean part* of our Universe.

This made quantum mechanics to develop basically within classical mathematical embryo—functional analysis, epistemically extended—inevitably as a *mixture of subjective and objective*, endorsed by features or metaphysical presuppositions characteristic of ordinary language-users. So the *peculiarity* of the "peculiar mixture", Jaynes referred to, is likely to be discussed in relation to this human-centric perspective as a whole.

However, this is a huge topic and in this presentation we will try to get only to the grounds of some of the logico-philosophical aspects of this *mixture* that led to what Jaynes termed as "scrambled omelette", and to understand what possible logical senses *can be made of unscrambling*.

In Section 2 we will describe the challenge posed by Nature as early as late 1920s, in Section 3 we will discuss about the way the challenge was faced, and in Section 4 we will discuss the question of semantic inadequacy of these approaches referring to the comparative merits of the logical variants developed on the basis of Bohr's and Schrödinger's proposals in relation to Hilbert space.

2 The challenge posed by Nature as early as 1920s

It is well known that Planck, the grandfather of quantum revolution in 1900, was quite unhappy with the way he himself introduced the new constant h, named after him, and the energy quantization rule as a consequence: "I was prepared to sacrifice my well established physical concepts". He wrote much later in a letter to Wood: "This was a purely formal assumption, and actually I did not ponder over it too much, being sure that I must obtain the positive result regardless of consequences or of the price to be paid".

The "price paid" was, first of all, famously about the failure of energy *equipartition* principle of classical statistics. So this is quite self-evident that, the price paid or the sacrificed "well established" physical concepts were all about the question of preference for equipartition principle presupposing this as *consistent*. It is really curious to note, that, Nature does not seem to share Planck's worry, and still assures correct results in spite of the so called sacrificed consistency in the method employed by Planck.

Planck's "inconsistent" quantum hypothesis earned a supportive *turn* with Einstein's *light quantum hypothesis* in his famous 1905 paper [3, 4]. Though this hypothesis provisionally served a heuristic role to make sense of photoelectric effect, Einstein himself was quite uncomfortable ever since with the way his proposed light quantum (named *photon* much later in 1926 by Gilbert Lewis [5]) turned out to have deviated from ordinary particle ontology. The deviation was typically manifested in more joint probability than usual or failure of statistical independence in case of photons to occupy any given state. Einstein categorically mentioned about this deviation in many of his correspondences with others during this time. In a letter to Schrödinger, Einstein wrote (28 February 1925)

In the Bose statistics employed by me, the quanta or molecules are not treated as being independent of one another. . . . A complexion is characterized through giving the number of molecules that are present in each individual cell. The number of the complexions so defined should determine the entropy. According to this procedure, the molecules do not appear as being localized independently of one another, but rather they have a preference to sit together with another molecule in the same cell. One can easily picture this in the case of small numbers. [In particular] 2 quanta, 2 cells:

	Bose-statistics			independent molecules	
	1st cell	2nd cell		1st cell	2nd cell
1st case	●●	-	1st case	I II	-
2nd case	●	●	2nd case	I	II
			3rd case	II	I
3rd case	-	●●	4th case	-	I II

According to Bose the molecules stack together relatively more often than according to the hypothesis of the statistical independence of the molecules. [6, p. 65]

So, for Einstein, the phrase "have a preference to sit together" can be understood as an expression of what *he thought* incompatible or *flatly inconsistent* with ordinary particle ontology as well as an intuitive space time picture confirming separability, as quite obviously ordinary particle being spatially exclusive cannot show unlimited preference to sit together in a given state. For this non-separability or indistinguishability at the level of fundamental ontology, Einstein had hardly any adequate conceptual justification at that time.

Concept of something like "pulsating" particle implicit in Louis de Broglie's suggestion of wave-particle duality in 1924 [7, 8] is known to have played subsequently a lead role. This provided at least a provisional justification of the conceptual problem with strange photon ontology (in question of "unlimited preference to sit together") Einstein was long worried about. Quite appreciably, the classical particle ontology, committed to spatial exclusiveness, if replaced with some kind of *waviness* (even if not in a very well defined sense), situation would be more comfortable to make sense of some kind of *preference* to *sit together* as Einstein exclaimed.

The counting scheme pertaining to the Bose–Einstein statistics, developed by Einstein in 1925 [9], was definitely an outgrowth within this heuristic framework of justification of *unlimited preference to sit together* (technically speaking more joint probability)

$$W_{BE} = \prod_i \frac{(n_i + g_i - 1)!}{(g_i - 1)! n_i!} \tag{1}$$

where W_{BE} stands for the number of microstates possible, with n_i particles and g_i number of cells. So $(g_i - 1)$ is the number of fictitious partitions.

This counting scheme clearly violates the norms of *additivity* restricted for categorically identical things. In other words, this counting scheme instantiates *failure* to preserve even numerical identity of the photons. Photons here thus effectively assigned denial of any rigid denotation criteria or marks of distinction or *label* for them.

Thus the chief investigators of photon story like Einstein and Schrödinger, on the basis of purely theoretical considerations, were led almost to the doorway of a land of *anonymity* or ambiguity in the notion of spatio-temporal individuation in standard everyday sense, where one is not actually allowed to preserve numerical identity, create marks of distinction, and consequently to name, distinguish, put them in order—all the activities underlying standard Logic and Set Theory as well as functional analysis based on them in the sense Newton da Costa warned. And consequently *photon*—the strange creature, apparently seems to *fail* to qualify as something which can enter into what we understand as Physical Law in spatio-temporal sense. But it is curious to note at the same time that Compton's scattering (1923) of X-ray photons by electrons endorsed the long expected particle behavior of energy quantum [10]. This was accepted as a final confirmation of physical existence of energy quantum. Physics community so far was confused about the reality of energy quantum proposed heuristically by Einstein in his 1905 paper [3, 4].

So, the challenge posed for the physicists as early as 1924–1925 was precisely to describe something which seems to be utterly dualistic—while it confirms its particle behavior in Compton scattering, it violates at the same

time the *standard norms of spatio-temporal individuality* or describability predominant in our intuitive understanding as well as standard logico-mathematical (functional analytical) artifacts of description. Light quantum let itself to be acknowledged as *entity* with no familiar determinate identity criteria. So the challenge can be stated as about describing a world of anonymity by the physicists though themselves being psycho-linguistically committed to all those descriptive features denied by photon.

2.1 A seemingly different face of challenge in German camp

While Einstein was looking for a conceptual justification for reconciling photon within a viable ontology he can negotiate with, a group of physicists based mainly in Göttingen were struggling ahead under the leadership of Born and Bohr to make as much as possible fruitful sense of Bohr's ailing theory of atom. Statistical weirdness of photon or even the enigmatic duality were not, at least, a direct guiding concern for them to proceed further. Göttingen physicists during the long period of nearly 10 years or so, following Bohr's proposal of atomic model in 1913 [11], were bothered basically to applying the idea of Bohr orbit to make sense of spectral lines of radiation for elements comparatively complex than hydrogen.

3 How the challenge was faced? Founding fathers responded with classical mathematics epistemically extended

In the very first appearance of Schrödinger equation in January 1926 [12, 13], the relativistic link of de Broglie was divested off, and turned out to be accommodable within the framework of mathematics, physicists were long familiar with. In fact Schrödinger equation, not explicitly disclosing its affiliation with de Broglie, displayed just that a *series of numbers*—known as quantum numbers, can be obtained within the framework of well-known mathematical process physicists were long familiar with—the framework of *classical string boundary value problem* developed much earlier by Jacques Charles Sturm (1803-1855) and Joseph Liouville (1809-1882)—having no apparent connection with quantum mechanics. In fact, representation of an arbitrary function by an infinite series of functions from a prescribed set (for example, a set of trigonometric function series) was a long debated issue among the mathematicians.

But question of convergence of the series was a major issue to be settled—under what circumstances *the series*

converges? If it converges at a point *x*, does it necessarily converge to $f(x)$? Moreover, the general concept of function has not been clarified enough. A lengthy debate took place over these issues revolving around the *question of representing arbitrary functions on a bounded interval by such series*, until it was finally settled by Peter Gustav Lejeune Dirichlet (1805–1859) in the middle of the 19th century.

All these developments of functional analysis, almost entirely by the mathematicians, much before and quite independent of quantum mechanics, provided heuristic grounds for Schrödinger to get ahead. It is interesting to note that *Methods of Mathematical Physics* written by two mathematicians David Hilbert and Richard Courant was published in 1924 before the official appearance of quantum mechanics. This book [14, 15] contained in a capsular form practically every mathematical method, trick and special detail required to develop Schrödinger's theory.

However Schrödinger's paper [12, 13] was followed by natural disappointments to know more about the real tricks behind, and it was in a sequel paper in 1926 [16], Schrödinger disclosed how his wave equation happened to be a natural outgrowth of de Broglie and of the outstanding Irish mathematician William Rowan Hamilton (1805–1865). Technically speaking, classical Hamilton–Jacobi equation

$$H + \frac{\partial S}{\partial t} = 0 \qquad (2)$$

is a short wavelength limit of the quantum mechanical version

$$\left[\frac{1}{2m} (\nabla S)^2 + V \right] + \frac{\partial S}{\partial t} = \frac{\iota \hbar}{2m} \nabla^2 S \qquad (3)$$

The expression in the square brackets is the classical Hamiltonian *H* of Hamilton–Jacobi equation for a single particle.

We need not go here into further technical details of the tricks behind. In fact, Hamilton showed *geometrical optics* and *classical mechanics* formally as two aspects of *calculus of variation*. He envisaged a normal particle trajectory as orthogonal to a fictitious wave surface of constant phase. Further details of how Schrödinger extrapolated this formal analogy is not important for us within the scope of purpose of this article (see for details [17, 18]).

3.1 Replacing guesswork by mathematics

In German camp during the same time, young Heisenberg had been partly successful, as Born described later, to "cut the Gordian knot by means of a philosophical principle and replaced guess-work by a mathematical rule" [19].

The Gordian knot was crucially about, as we mentioned, the inadequacy of old quantum theory (based on Bohr's atomic model) to describe time dependent processes such as emission and absorption of radiation.

The "philosophical principle" was all about a decision to take care of things only that are *observables* pertaining to the (visible) spectral lines rather than the empirically inaccessible Bohr orbits inside the atom. This simple looking principle turned out to have tremendous methodological superiority, as this was effectively synonymous to take *two* Bohr orbits into account instead of *one*, as was so far tried with.

As a consequence of this decision, it is not difficult to appreciate that, if some quantities are supposed to be associated with two orbits (stationary states of motion) instead of one, a convenient way to write them down is in terms of a tabular form like this

$$
\begin{matrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{matrix}
$$

an array of numbers known as matrix whose elements can be clearly *interpreted* as expressions involving *two* states of motion instead of *one*.

In fact, glimpses of something which can be termed as an operational attitude were evidently underlying these *early* technical moves right from the beginning among the Göttingen group. Heisenberg formulated much later, the essence of their early attempts as "what we observe is not nature in itself, but nature exposed to our method of questioning" [20, p. 57].

It is difficult to overlook the *Kantian undertone* behind this strategy. Interestingly enough Heisenberg tried to construct the formalism of *new* mechanics corresponding as closer as possible to that of *old* classical mechanics. The classical equation of motion $\frac{d^2}{dt^2} x = f(x)$ was replaced by their quantum analogues. Classical position q was replaced by its quantum analogue $\hat{Q}$ and classical momentum p by $\hat{P}$, where $\hat{P}$ and $\hat{Q}$ are matrix equivalents determined completely by intensity and frequency of the emitted or absorbed radiation of atom pertaining to two orbits. The exact quantum condition satisfied by these matrices is famously

$$
\left[\hat{Q}, \hat{P}\right] = \hat{Q}\hat{P} - \hat{P}\hat{Q} = \iota\hbar\hat{I} \tag{4}
$$

which can be described as an epistemic extension of $[p, q]_{\text{classical}} = 0$. This is the well known noncommutative quantum analogue of classical commutation where $\hbar$ appears explicitly.

This reflects the fact that, the most significant way matrix formalism differs, in principle, from its classical counterpart is in terms of the way the role of our "method of enquiry" is *formally internalized in theory*. This internalization is also precisely how Heisenberg's matrix endorsed the primacy of process (in quantum mechanical context) over instantaneous state. Indeed the notion of stationary state does not appear in matrix mechanics. As Muller put it aptly

> The absence of states in matrix mechanics was not a mathematical oversight of the founding fathers. On the contrary, Heisenberg counted the abolition of such unobservable relics from the old quantum theory, wherein (stationary) states were identified with electron orbits, as a personal victory. [21]

3.2 Awful interlude: outcome of response was a handful of equations without realistic interpretation

Having the exotic equations in both camps— Schrödinger's deterministic wave equation as well as Heisenberg's matrix mechanical toolkit, these can be described to have marked the end of first stage of manufacturing process or development of some kind of *uninterpreted object language* of the theory. Schrödinger had no idea of what his wave function Ψ designating states, stands for. Schrödinger is known to have changed his mind regarding the interpretation of quantum mechanics at least three times during his struggle with the theory. Schrödinger's earlier conclusion about the Ψ-function as intermediate-level concept was eventually modified with Born's probabilistic interpretation of Ψ-function. He is known to have decided to teach quantum mechanics from 1928 according to the mainstream version with which the sticker Copenhagen attached later.

So the situation during late 1920s was quite baffling— wave mechanical language developed by Schrödinger alone, though looked comfortable due to its link with familiar functional analytical set up, principle of superposition implicit within the language due to linearity of Schrödinger's equation turned out to be inadequate to make any epistemic sense of its own without any reference to measuring device—the infamous *Measurement problem*. On the other hand, Heisenberg, Born and Jordan, though successful to develop a noncommutative language of matrix algebra, which was supposed to talk directly about the observable level right from the beginning, was inadequate to make any ontological sense of state.

This way, the two versions of quantum mechanics, which came almost simultaneously (in 1924–1925) into being, the version by de Broglie–Schrödinger and the version by Heisenberg, Born and Jordan turned out to be strikingly dissimilar in question of *completeness* as well

as *self consistency*—Schrödinger's ontological version without viable epistemology of its own, and Heisenberg's epistemic version without any ontological commitments of its own. However, in spite of this striking dissimilarity, we can characterize both approaches as different looking attempts to extend classical mathematical language in quantum mechanical terms.

In view of this situation, Born's subsequent 1926 postulate [22] can be unambiguously identified as a *metarule*, which can be described as the formal semantic norms that imposes restrictions on the language of "external observer" not pertaining to the level of what is being observed (level for which the superposition principle holds). And in that way, we can say that what follows Born rule is part of the *metalanguage* of the same theory—that served as the bridge between Schrödinger's approach and others in Göttingen and Cambridge. According to Born, $|\Psi|^2$, rather than Ψ alone, can be attributed an *epistemic sense* for our level of experience. Ψ alone as probability measure is epistemically void and can only be made sense of in terms of an *anonymous referent* in context of an ensemble. Born described the situation later

> Schrödinger thought that his wave theory made it possible to return to deterministic classical physics. He proposed (and he has recently emphasized his proposal anew), to dispense with the particle representation entirely, and instead of speaking of electrons as particles, to consider them as continuous density distribution $|\Psi|^2$ (or electric density $e|\Psi|^2$). To us in Göttingen this interpretation seemed unacceptable in face of well established experimental facts. [19]

But Born's rule [22], though it turned out to be good enough to calculate the probability of outcome realized for our level of experience, *epitomized* almost the crux of a new interpretational problem of its own—probability measure Ψ rendered itself difficult to be interpreted as something characterizing preexisting property, that is to say, in terms of classical ignorance [23].

For our present purpose, we need not get into any further details of these implications of Born's interpretation.

4 What about semantic inadequacy of these uninterpreted languages in view of Nature's challenge?

We can take stock at this stage, whether these responses (developed before Born's interpretation in 1926 [22]) were really adequate to capture Nature's challenge we mentioned before.

So far the wave mechanical version of the responses is concerned, the *duality*, as we mentioned, is known to have earned its conceptual justification within the framework of Hamilton's dynamics, presupposing unambiguous (particle) identity. That is precisely why, (paradoxically) as a consequence, the strange statistical features of photon denying any *mark of identification* were not possible to be prioritized within the same framework at the same time.

This rendered wave mechanical formalism effectively as a formal counterpart of our classical intuitive belief committed to the very possibility to create *marks of distinctions* (discernibility) or labeling among exactly identical or even indistinguishable copies.

If $|\Psi_1\rangle$ describes the state of the first system and $|\Psi_2\rangle$ describes the state of the second system, the joint system in wave mechanical formalism is

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_1\rangle|\Psi_2\rangle \pm |\Psi_2\rangle|\Psi_1\rangle \right) \qquad (5)$$

where the plus sign holds for *bosons* and the minus sign holds for *fermions*. But one must note that the *indices* 1 and 2 effectively stands here for making distinction or discernibility between "this" system from "that" system, though this distinguishability was not allowed, as we discussed, in Nature's original ontological scheme.

This way, the strategy of labeling seems to call for an inevitable but unwarranted compromise in terms of *as if* sense. The standard practice is to think that, though it is not really possible to talk about the particles as distinguishable or discernible, indistinguishable particles are believed to support label/tag in the sense of recreating *as if* "this one" and "that one" are true at the level of ontology to render themselves at least provisionally distinguishable.

So the notion of indistinguishability is epistemically captured, within the standard framework of wave mechanics, only to the extent it can be made sense by symmetry under permutation or exchange of these labels. Of course, it is easy to check that, this exchange symmetry is compatible with more joint probability of photons to sit together.

But quite naturally, this symmetrization strategy left itself vulnerable to a suspicion about the real ontological significance of these labels. This seems to be mock identity, but still they seem to mimic Nature's scheme to the practical extent.

So, only permutation symmetry does not seem to be adequate to capture the whole semantic "essence of failure" to confirm numerical identity. In fact skipping the technical details apart, it can be stated that the reduced states of each of the electrons in a two-electron system is identical. This is a serious threat to the individuality of the electrons.

Schrödinger later went even further to dispense with the notion of *sameness* also. Schrödinger "begged us" to believe during his Dublin Lectures in 1950 [24, p. 17], that quantum particles being not actually supportive to any mark of distinction or labels do not even instantiate the concept of sameness. And in that way, all of them can be replaced to *leave a collection still same*—a situation far away from what standard set theory is assigned to capture in extensional sense.

On the other hand, as we mentioned, Heisenberg, Bohr and Jordan were, at the outset, more away from the challenge to capture the essence of failure of ordinary particle ontology, as their problem was rather differently focused from the very beginning.

So the standard formalism, in terms of two different recipes, can be described to have captured epistemically different sense of *failure of spatio-temporal mode of individuation*—Schrödinger's version in terms of mock identity, and Heisenberg's version in terms of matrix endorsed by absence of the notion of state. It was soon understood that both formalisms are alternative isomorphic versions of the same underlying mathematical structure, and that is why they ensure empirical equivalence. We need not get into that details here. What do all these have to do with inconsistency as a whole?

4.1 What does it mean to accept a contradiction or inconsistency in physical theory?

So the "peculiar mixture" Jaynes referred to—so far the standard formal manufacturing is concerned, can be described as about mixing a token of failure of standard logic with a fragment of functional analysis relying on standard logic itself. This was followed by coupling with Born metarule or a formal recipe of the role of observer.

So the overall development of the standard version as we narrated here can be described as endorsed by an implicit mark of *inconsistency* in an extended sense. This has an undoubtedly oxymoronic flavor. Of course , the standard sense of inconsistency is a bit restricted—a theory T (a set of rules closed under deduction) is described as inconsistent if it contains a theorem L, whose negation $\neg L$ is also a theorem. Otherwise T is consistent.

The issue is basically about making sense of a question like—what does it mean to accept a contradiction or inconsistency in Physical theory? Does quantum theory really host something like L and $\neg L$ together?

The question analyzed further begs to resolve that, whether contradiction has an epistemological or ontological character.

Admitting contradiction at the level of ontology or object language effectively looks for a semantic possibility

to ensure the notion of *unambiguous being* corresponding to or compatible with A and not-A, where A is any proposition or state of affairs. But the indispensable principle of non-contradiction in classical logic $\neg(A \wedge \neg A)$ states, that, it cannot be the case that a proposition and its negation are both true. In other words, the concept of conjunction is meaningless in this context. The situation can also be stated using second order language $\forall x \neg [P(x) \wedge \neg P(x)]$, where P is one place predicate, and x is a variable.

This formula seems to capture the *empirical fact* that an *object cannot possess and not possess a property*. So it can be easily appreciated that, within the scope of semantics of standard classical logic, those propositions or conjunction can only be interpreted to denote membership of an empty set. However, though seems to be meaningless or semantically void within the framework of standard logic, the manufacturing process of the uninterpreted language of standard quantum mechanics, as we discussed, seems to have instantiated, throughout its development, a wider sense of juxtaposition of A and not-A. Apart from this wider sense of juxtaposition, wave particle duality is often cited in recent literature as a more specific instance of clubbing A and $\neg A$ together. Wave particle duality is often claimed in some recent literature as an example of how nature hosts contradictory aspects (see as a representative article, for example, [25]). But this seems to be a forced misinterpretation in view of the existence of *wave packet*.

This is how *inconsistencies of different orders* continued to come into play in the course of *Theory building* of quantum mechanics mostly as "epistemic howler".

Einstein was famously more welcome to his General Theory of Relativity (1915) [26, 27] than quantum theory, and one of the reasons that can be made sense in retrospective is that he could not accept the possibility that both can be simultaneously true–while General Theory of Relativity confirms an unambiguous principle of individuation (consistency?) in the sense of committing implicitly to the notion that "everything just *is*", quantum theory, on the other hand, we have seen to have done a violence to the notion of spatio-temporal *is* or separability. So Einstein's universe cannot operate on the basis of a coupled mechanism supportive to hostage of both. And he was ready to disown quantum theory (at least the compelling interpretation pressed upon by Bohr) in favor of General Theory of Relativity. Einstein's difficulties with *photon ontology*, in contrast to "possible substantiation" of his favored ontological attitude in General Theory of Relativity, is an interesting episode in modern History of Physics, which recent science historians [28] claim to have led Einstein to think about the famous Einstein–Podolsky–Rosen paper [29]. These all are because he could not allow Nature to be inconsistent to host both. Strangely enough,

Bohr—though definitely known to have not shared Einstein's ontological attitude, was equally skeptic to any such possibility of admitting contradiction.

It is curious to note that Bohr, though seems to be more with the spirit of new mechanics, was notoriously skeptic, more like a partially disguised Aristotelian, to make any ontological sense of *Togetherness as union of contradictory aspects*. "Even the mathematical scheme does not help", he lamented seeing contradiction to lurk behind, as reported by Werner Heisenberg later, "I first want to understand how nature actually avoids contradictions" [30].

4.2 Birth of quantum logic in different versions

4.2.1 Two different variants of logic connected by the common thread of inconsistency

Bohr's skepticism against admitting contradiction is clear again in *Atomic Theory and the Description of Nature* (1934), where his response to the apparent contradiction was complementarity [31]. Later, Léon Rosenfeld wrote

> Complementarity denotes the logical relation, of quite a new type, between concepts which are mutually exclusive, and which therefore cannot be considered at the same time because that would lead to logical mistakes, but which nevertheless must both be used in order to give a complete description of the situation. [32, p. 385]

In fact, Bohr was the first to have developed an informal sketch of what we might call *quantum logic* even before John von Neumann's celebrated 1932 book [33] which is usually credited to have had the first seed of quantum logic [34]. Traditional quantum logical approach within the logico-algebraic framework, initiated by von Neumann, starts typically with the questions "What a quantum mechanical proposition would look like referring to the basic constraint imposed by uncertainty relation or what constitutes an *Event* in quantum mechanical context?" and, "How to translate a quantum mechanical event in terms of Hilbert Space Language?" That is to say, how to *internalize* the measuring apparatus or *observer* formally within the Theory? John von Neumann famously suggested a formal characterization based essentially on the algebraic structure associated with the collection of all the projection operators in Hilbert space. This is equivalent to talk in terms of a partially ordered set or lattice. We will not get here into this logico-algebraic approach.

But Bohr never articulated clearly that, what formal sense can possibly be made of complementarity as "logical relation, of quite a new type". But, *what constitutes*

the newness can be understood to be a clear departure from the scope of classical logic, where a true proposition cannot rule out another true proposition. Classical reality is flatly inclusive in the sense that there is no problem to take the descriptive features of a classical system *together* at a time.

Many of Bohr's followers tried to develop a logical counterpart of Bohr's idea of Complementarity.

Unlike von Neumann, Bohr was famously *not supportive* to the idea to *internalize* the *measuring apparatus* formally within the Theory. Bohr's response to the Measurement problem as well as to the meta-theoretical issues had its starting point with the apparatus/observer taken as *external* and ontologically different from the quantum system in question, which led him (effectively) to the notion of *complementarity*. Though this left many questions almost undecidedly open, Bohr himself was hardly sympathetic to formalization as an answer to every conceptual difficulty. However there were series of attempts by others to formalize Bohr's emphatically advocated idea of complementarity.

Strauss (1936) used a propositional calculus based on partial Boolean algebra of projection operators acting on Hilbert space to formally capture complementarity [35]. His intention was to develop a bivalent propositional logic, say with A and B as complementary propositions which can both be True/False, but not their conjunction $A \wedge B$. This amounts to say that, the quantum logical connectives are not *Truth-functional*, that is to say, the Truth value of a compound proposition is not determined by the Truth value of its constituents.

However Février (1937) [36] and Reichenbach (1944) [37] introduced a third indeterminate truth value, while von Weizsäcker advocated many-valued complementarity logic [38, 39]. Février's logic resembles Łukasiewicz logic as she also proposed an impossible third value to the conjunction of complementary propositions. In fact this is another way to speak about the failure of distributive Law as she recognized the impossibility of having a conjunct of complementary propositions.

4.2.2 Backdrop of Schrödinger logic

It is important to note that the failure of *discernibility* noted during the early days of development of quantum mechanics is synonymous to a failure of Leibniz's Principle of Identity of Indiscernibles, which reads using higher order predicate logic

$$\forall x \, \forall y \, [\forall P \, [P(x) \leftrightarrow P(y)] \rightarrow x \equiv y] \qquad (6)$$

Leibniz famously pointed out that if certain objects are not identical, there must be some quality (a given property) that distinguishes them. In section 9 of his *Discourse*

on Metaphysics, he notes that, "it is not true that two substances can resemble each other completely and differ only numerically, *solo numero*" [40, p. 14]. Stated otherwise, no two objects can share all the intrinsic qualitative properties. If there is no property whatsoever that would allow distinction, Leibniz's Principle implies that we are talking about just one thing instead of two. But standard quantum mechanics seems to advocate an ontology admitting truly indiscernible, but distinct entities. But this conclusion is clearly far from admissible, for example, while talking about two electrons, as the composite system would surely has, twice the charge of a single electron, twice the mass. We cannot flatly accept the implication of Leibniz's Principle.

How about recreating or saving discernibility somehow? Attempts to save Leibniz's Principle constitute a huge literature mostly by the philosophical logicians. In fact Bohmian mechanics can be seen, in a certain sense as an interesting variant attempting to recreate *discernibility* by introducing hidden parameters in the original discourse [41]. As such, there is no logical injunction to think that (presupposing of course in some sense the validity of *being where it is*), given all intrinsic properties are same, there seems to be space for other (may be hidden/non-empirical or counterfactual) properties and relations not considered in original (universe of) discourse, in a tricky sense, for example, described by Quine

> Ontology is indeed doubly relative. Specifying the universe of a theory makes sense only relative to some background theory, and only relative to some choice of a manual of translation of the one theory into the other. [...] Identity is thus of a piece with ontology. Accordingly it is involved in the same relativity, as may be readily illustrated. Imagine a fragment of economic theory. Suppose its universe comprises persons, but its predicates are incapable of distinguishing between persons whose incomes are equal. The interpersonal relation of equality of income enjoys, within the theory, the substitutivity property of the identity relation itself; the two relations are indistinguishable. It is only relative to a background theory, in which more can be said of personal identity than equality of income, that we are able even to appreciate the above account of the fragment of economic theory, hinging as the account does on a contrast between persons and incomes. [42, pp. 54-55]

This is precisely what happens when we treat indistinguishability/indiscernibility within a classical framework such as Zermelo–Fraenkel set theory which encompasses classical logic. Technically speaking certain mathematical structures (built in set theory) can be considered as non-rigid so that once we work within these structures, we can regard some objects as indiscernible relative to all the predicates and relations defined in the structure (Quine's *group of people*, indiscernible with respect to the *income predicate*). And it is subsequently a question of possibility to extending or modifying this structure (rigid extension) to accommodate new properties and relations not considered in original discourse.

In a series of recent papers Simon Saunders, Fred Muller, Michael Seevinck have collectively argued against the standard folklore of *failure* of Leibniz's Principle, that some non-trivial version of Leibniz's principle is upheld in quantum mechanics [43–45]. They argued that all particles—fermions, paraparticles, anyons and even bosons, can be weakly discerned by some physical relation. However, Adam Caulton argued that their arguments make some illegitimate appeal to non-symmetric, that is, *permutation non-invariant* quantities, and therefore cannot be accepted to go through [46, 47]. Sometimes Hilbert–Bernays axiom (1934), which mentions every primitive predicate quantifying in each argument place, is also used as an explicit first order logical substitute to Leibniz's Principle [47].

4.2.3 Schrödinger from other way around: No need to recreate discernibility or mark of identification

Schrödinger's proposal was from an altogether other way around for a new interpretation that can accommodate the failure rather than to recreate or save discernibility (as ontologically prior) instantiated by quantum particles. In a series of public lectures in Dublin (February 1950), Schrödinger appealed

> we have yet been compelled to dismiss the idea that such a particle is an individual entity which in principle retains its 'sameness' for ever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no 'sameness' at all. [24, p. 17]

This is effectively a warning against standard logic as well as traditional quantum logic with *identity* as well as the mathematics based on it. But Schrödinger did not formally elaborate his proposal which was initiated later during the early 1990s by a group of Brazilian logicians under the leadership of Newton da Costa. A substantial bulk of the recent literature of Philosophy of Science during the last 30 years or so is devoted to make formal sense of the *possibility of metaphysics of quantum non-individual within the framework of quasi-set theory*. Use

of the term *quasi-set* follows a suggestion in Brazilian logician Newton da Costa [2].

During the last 50 years or so there is slow but steady developments in the line inspired by Schrödinger by different logicians as well as mathematicians, for example, like Heinz Post, Yuri Manin and others acknowledging the semantic inadequacy of standard Set theory. What is being tried is basically to develop different versions of metaphysics of *non-individual* which is not subscriptive to particle ontology or discernibility right from the beginning; particle concept itself along with the possibility of creating tag or label for it was considered to be ontologically surplus.

However, we see many of the founding fathers of quantum mechanics like Max Born and Paul Langevin to talk about the sort of approach to the metaphysics of non-individuals. But they hardly talked in terms of further formal requirements. Heinz Post in 1963 proposed to consider the non-individuality of quantum objects right from the beginning [48].

Manin is reasonably clearer about what is needed

> I would like to point out that it is rather an extrapolation of common sense physics, where we can distinguish things, count them, put them in order, etc. New quantum physics has shown us models of entities with quite different behavior. Even "sets" of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the "set" of grains of sand. [. . .] The twentieth century return to Middle Age scholastics taught us a lot about formalisms. Probably it is time to look outside again. [49]

Schrödinger's original spirit of dispensing altogether with the notions of *identity* and *sameness* is captured by hosting non-individuals in the ontology of the theory at the cost of assuming *background theory* (metalanguage in which we can speak about our object language and describe the Semantic concepts) as a *quasi-set theory* instead of *ordinary set theory* underlying Hilbert space formulation.

In a quasi-set theory $\aleph$, the property "being identical with *a*", for a certain term *a*, cannot be considered among the properties of the object *a*. For the elements of $\aleph$, non-individuality is taken into account by making room for entities for which it does not make sense to assert that they are identical to themselves or different from each other in a class. So classical theory of Identity is not allowed to be applicable for them. Classical theory of Identity characterizes the objects as individual in a sense that they can always be distinguished from each other either for having a certain property or by existence of

a set to which it belongs to, but not in others. In other words, their membership function is clearly bi-valent.

As Krause put it, "non-individuals, taken as indistinguishable in the object theory, cannot be distinguished even in the background theory, for they lack the concept of identity" [50]. However, an adequate metaphysics of non-individual is still in its infancy.

4.3 How to connect these apparently different approaches?

Finally compared to the challenges posed by Nature, which one is a better logical expression? Though it is difficult to compare, but we have argued here that the logical sense of "scrambling" or "unscrambling" has definitely to do with the role played by *inconsistency* or *contradiction* in different sense. In fact one of the fundamental extensions of the concept of *complementarity* was famously the introduction of *freedom of choice* of experimenter. As a response to the charge of *incompleteness* raised by Einstein [29], Bohr famously introduced the concept of *freedom of choice* of experimenter—freedom of choice to measure a specific property of a complementary or noncommutative conjugate couple [51]. Bohr in this connection also discarded the "disturbance" (epistemic) account of uncertainty principle by Heisenberg in favor of indeterminacy at the level of fundamental ontology. This had a far reaching chain of consequences.

This standpoint is effectively equivalent to say that *future* is not ontologically settled in some fundamental sense. We should stop talking in terms of *pre-existing* properties which have nothing to do with experimenter's *free choice* and this cannot be captured in terms of *Boolean property structure*. So though Bohr himself was not very enthusiastic about formal expression as answer to a conceptual issue, Bohr logic—so far it is an authentic expression of *complementarity*, is essentially a matter of *non-Boolean property structure*. But the underlying framework is still a *Hilbert Space* and *ordinary Set theory* which is faithful to the notion of *particle identity*.

Schrödinger logic on the other hand is ontologically more challenging in the sense of proposing to dispense even with the notion of *identity* and *individuation* right from the beginning. Schrödinger logic this way promises a more radical *departure* from classical Semantics based on *Hilbert Space* formulation.

But one must keep in mind that a departure in a certain sense was envisaged by John von Neumann also as early as 1935 [52]. He was very much aware of the basic *inconsistency* involved in generalizing classical mathematical toolkits to handle quantum mechanical situation. He was led to a critical attitude towards the standard Hilbert Space formulation of quantum mechanics. However his grounds

of motivation were quite different from Schrödinger. Basically von Neumann was keen to interpret the algebraic structure of quantum mechanics as algebra of *random events* in the sense of non-commutative probability theory. But this cannot be achieved in a generalized Euclidean space where probabilities are viewed as *relative frequencies*. In a letter to Garrett Birkhoff (November 13, 1935), von Neumann wrote

> I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert-space any more. After all Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of "conserving the validity of all formal rules". This is very clear, if you consider the axiomatic-geometric definition of Hilbert-space, where one simply takes Weyl's axioms for a unitary-Euclidean-space, drops the condition on the existence of a finite linear basis, and replaces it by a minimum of topological assumptions (completeness + separability). Thus Hilbert-space is the straightforward generalization of Euclidean space, if one considers the *vectors* as the essential notions. [52, p. 61]

In fact von Neumann's logic is a response to the question "what to be replaced for Hilbert space?". We need not get here into further details of von Neumann's response, but the role played by Hilbert space is central to note in this context.

Needless to say that, unlike the *phenomenological* response of von Neumann, both Bohr and Schrödinger talked about some kind of *blur* at the level of fundamental ontology itself though their logics were intended for different treatments of *identity* and *individuation* concept. But Schrödinger's logical approach is more challenging as it brings us closer to a different kind of question which cuts directly through differently overlapping *metalogical* issues of practical worth. This is a question of developing adequate semantics of background theory—how far the base theory of standard quantum mechanics can be *extended* or modified to develop a *consistent* background theory with adequate semantics?

Stated in tangible terms, this is a question of semantic scope of creating *artificial* mark of identity or discernibility.

Standard Hilbert Space formulation presents some *metalogical* anomalies implying *semantic inadequacy*, which is precisely what Gleason theorem (1957) [53] and Kochen–Specker theorem (1967) [54] are about.

Summing up the logical insights, it seems that it is not feasible to talk about any unique non-standard logic of quantum mechanics, but instead *logics* for depending on *interpretation*. This point has been repeatedly emphasized by John Stachel [55]. This amounts to say that we can connect these logics through the different possible interpretations of *inconsistency* itself.

So in view of the whole development of *mixed* or *epistemically extended mathematics* in its different versions, the manifest inconsistencies of varying degree can be tried to be given a better *conceptual justification* rather than scaring them as epistemic hell. This way quantum mechanics seems to need an overarching *logical framework* that can allow internalizing the notion of inconsistency at the *object level of the language*. Several proposals are there. Logic of formal inconsistency claims to afford this purpose. This latter logic is claimed to afford very expressive logical system whose fundamental feature is the ability to recover *all consistent reasoning* while still allowing to reason under contradiction—a strategy beyond standard practice of either–or. Also from a more wider point of view of requirements of making room for inconsistency in science, *Partial structure approach* had been proposed by da Costa and Steven French [56]. This approach had been further developed by Otávio Bueno and Newton da Costa [57, 58]. This approach is fundamentally motivated, as Bueno describes, "for supplying a formal framework in which the openness and incompleteness of information dealt with in scientific practice can be accommodated" [59]. Adan Cabello's non-standard roundup and classification of various interpretation of quantum mechanics stressing somewhat this point is also helpful in this connection [60]. But let us hold further details for another occasion.

5 Postscript

Throughout this brief exposition, it is tried to emphasize that it is not really a question to single out precisely one theory or another in the history of Physics as *non-trivially inconsistent*. Different orders of mark of inconsistencies can be figured out throughout the whole developments of Physics. In fact inconsistency can be described as even a characterizing *trait of progress* of science as a whole. As we stressed, that this is inevitable, as, after all, Physics is a human creation. Our standard classical logico-mathematical artifacts as well as the functional analytical tools are primarily outcome of our psycholinguistic commitments deep-rooted in our everyday experience of a slowly varying universe faithful to the notion of *individual* of permanent type. But ironically, this very empirically motivated artifacts had almost always served as the primary guiding frameworks to talk about the domain beyond our ordinary empirical access not faithful

to the notion of *individual*. And this is how a mix-up and consequent inconsistency of different orders used to come into play. Within the human scale of empirical scope of classical reality, many difficulties can be overlooked for all our practical purposes. But the problem attains an unavoidable new dimension while *formulating the language of quantum world*. Subjective parts are rendered misleadingly difficult to be differentiated from the objective part of Theory. These all give rise basically to two kinds of "inconsistencies" at the ontological levels of the language—pertaining to particle identity itself, and that pertaining to possessed properties.

This necessitates a fresh assessment of the ontological baggage as well as the methodological manual of the language from within different non-standard frameworks, whether it is Schrödinger logic, Bohr logic or something else to serve our purpose. However, detailed comparative account of the relative merits of all the proposals outlined here is something to be developed as a research program and we leave this for future works.

Acknowledgments

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