# ON THE IMPOSSIBILITY OF USING ANALOGUE MACHINES TO CALCULATE NON-COMPUTABLE FUNCTIONS 

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## Introduction

A number of examples have been given of physical systems (both classical and quantum mechanical) which when provided with a (continuously variable) computable input will give a non-computable output. It has been suggested that these systems might allow one to design analogue machines which would calculate the values of some number-theoretic noncomputable function. Analysis of the examples show that the suggestion is wrong. In $\S 4$ I claim that given a reasonable definition of analogue machine it will always be wrong. The claim is to be read not so much as a dogmatic assertion, but rather as a challenge.

In §'s 1 and 2 I discuss analogue machines, and lay down some conditions which I believe they must satisfy. In $\S 3$ I discuss the particular forms which a paradigm undecidable problem (or non-computable function) may take. In §'s 5 and 6 I justify any claim for two particular examples lying within the range of classical physics, and in $\S 7$ I justify it for two (closely connected) examples from quantum mechanics, and discuss, very briefly, other possible quantum mechanical situations. $\S 8$ contains various remarks and comments. In $\S 9$ I consider the suggestion made by Penrose that a (future) theory of quantum gravity may predict non-locally-determined, and perhaps non-computable patterns of growth for microsopic structures. My conclusion is that such a theory will have to have non-computability built into it.

## 1. Analogue machines

By a continuously variable quantity ('CVQ') I mean a physical quantity which is represented mathematically by a point in a metric space - e.g., by a real number, or a point of Hilbert space. This is not put forward as an exact definition, but as an indication of how I use the term. For CVQ's very natural definitions of 'computable' have been given in Pour-El \& Richards (1989); I shall to this book as CAP. Roughly speaking ' $x$ is computable' means that $x$ is the limit of a sequence of finitely presented approximations and a modulus of convergence for the sequence can be computed.

In the theoretical treatment of a physical device the CVQ's have exact values, and no bound is place, a priori on their magnitude. But when such a device is to be used as an analogue machine to perform some calculation then there will be an upper limit $x$ on the

[^0]size of a CVQ (an electric circuit will melt if the current is too large) and a lower limit $\epsilon$ on the accuracy with which it can be controlled or measured. Numerical values for $x$ and $\epsilon$ depend, of course, on the choice of units for the particular CVQ considered, but the ratio $x / \epsilon$ does not; so we define the precision ratio (PR for short) of the CVQ to be $x / \epsilon$. In the theory of the machines there may be different variables having the same physical dimension; these are to be counted as distinct CVQ's and may have different precision rations. The 'independent' variable time is also a CVQ and has a PR; when an analogue machine is to be used some limit must be placed on its run-time.

We are concerned with matters of principle rather than of practice, so although a given analogue machine will have definition precision ratios, we do not place any bound on the PR's that may be attained by some machine.

We are primarily - sometimes only - concerned with those CVQ's which are inputs and outputs of the machine. We shall be interested in cases where these may be continuously controlled or continuously recorded functions; in such cases the relevant $x$ and $\epsilon$ will be given by some norm for the functions. Most usually the uniform norm will be appropriate, but some machines one might want to say, the $L^{2}$ norm.

Even discretely varying quantities such as natural numbers have precision ratios attached to them; perfect accuracy (say $\epsilon<1 / 2$ ) may be attainable, but still there is a bound on the size: one cannot place more than $N$ balls in a given box nor record more than $N$ events with a given geiger counter. In particular if an analogue machine incorporates a battery of digital computers then a PR (which depends both on the programme used and on the hardware) can be assigned to each of them; note that it does not depend on the placing of the decimal point.

In what follows we shall be concerned with the orders of magnitude of PR's rather than with precise values or upper bounds.

## 2. Specification of analogue machines

A specification for an analogue machine is a finite list of instructions which would, in principle, enable a technician or engineer to construct it; descriptions of the apparatus used in a (published) account of an experiment, do, although greatly abbreviated, have this form. If the correct operation of the machines requires particular precision ratios for certain quantities, then the instructions will specify tolerances for certain components $\xi^{1]}$. For example a machine might require a cam whose ideal shape ideal shape would be given by $r=f(\theta)$ where $f$ is some mathematical function. Then the instructions would indicate how the function $f$ could be computed (e.g., $f(\theta)=2+\sin ^{2} \theta \mathrm{cms}$ for $0 \leq \theta \leq 360^{\circ}$ ) and give a permitted tolerance (e.g., $\pm 10^{-3} \mathrm{cms}$ ). Tolerances can be given as precision ratios ( $3.10^{3}$ in the example). A specification will determine either explicitly or implicitly the PR's in the quantities (including outputs and inputs) occurring in the machine.

[^1]
## 3. Undecidable problems

In the examples known to me it is proposed that there might be an analogue machine which with input $j(\in \mathbb{N})$ would output 'Yes' or 'No' to questions of the form ? $j \in A$ ? where $A$ is some standard recursively enumerable non-recursive set - for example the set which represents the halting problem. I shall only consider proposed machines of this kind. I describe two ways of representing the set $A$.
3.1. There is a total computable function $a: \mathbb{N} \rightarrow \mathbb{N}$ which enumerates $A$ without repetitions. (This is the notation used throughout CAP).

The waiting-time function $\nu$ is defined by

$$
\begin{equation*}
\nu(j) \simeq \mu n . a(n)=j \tag{3.1}
\end{equation*}
$$

This is a partial recursive function whose domain is $A$ and which is not bounded by any total computable function. For any particular analogue machine there is an upper bound $J$ on the inputs it can accept. I define

$$
\begin{equation*}
\beta(J)=\operatorname{Max}\{\nu(j): j<J \& j \in A\} \tag{3.2}
\end{equation*}
$$

(with $\operatorname{Max} \emptyset=0$ ). This is a total function which is not computable; indeed it eventually majorises every computable function.
3.2. There is a polynomial $P_{A}(y, \vec{x})$ such that

$$
\begin{equation*}
j \in A \leftrightarrow(\exists \vec{m}) P_{A}(j, \vec{m})=0 \tag{3.3}
\end{equation*}
$$

where the variables of $\vec{m}\left(=m_{1}, m_{2}, \ldots, m_{k}\right)$ range over the natural numbers.
In this case we define

$$
\begin{equation*}
\nu(j) \simeq(\mu n)(\exists \vec{m}<n) P_{A}(j, \vec{m})=0 \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta(J)=\operatorname{Max}\{\nu(j): j \in A \& j<J\} . \tag{3.5}
\end{equation*}
$$

Then $\nu$ and $\beta$ have the same properties as in 3.1. Observe that, if $j \in A$, then

$$
\begin{equation*}
\forall \vec{m}<\nu(j) P_{A}(j, \vec{m}) \neq 0 \tag{3.6}
\end{equation*}
$$

Various explicit definitions of suitable polynomials have been given. For each of these, if $P_{A}(j, \vec{m})=0$ then at least one of the $m_{i}$ encodes a particular sequence which lists the first so many values of some recursive function. So, taking $i=1$, we may suppose that

$$
\begin{equation*}
P_{A}(j, \vec{m})=0 \text { and } P_{A}\left(j, m^{\prime}, m_{2}, \ldots, m_{k}\right) \neq 0 \tag{3.7}
\end{equation*}
$$

where $\left|m^{\prime}-m\right|=1$.

## 4. The Claim

Since a given machine cannot handle numbers greater than some bound we consider a given $J$ and the questions $? j \in A$ ? for $j<J$. Now I make the following

CLAIM. Let $J$ be given. Then one cannot design an analogue machine (whose behaviour is governed by standard physical laws) which will give correct answers to all the questions ? $j \in A$ ? for $j<J$ unless one knows a bound $\beta$ for $\beta(J)$.

I call this a claim rather than a conjecture because I do not think one could prove it unless one placed severe restrictions on the notion of 'analogue machine', and this I do not wish to do. ${ }^{2}$ But I believe that if someone proposes an analogue machine for settling $? j \in A$ ? for $j<J$ then it can be shown that either they have (surreptiously?) made use of a bound for $\beta(j)$, or that not all the given answers will be correct. To illustrate the significance of the wording of the claim, suppose (what is quite plausible) that someone proves that $j \notin A$ for all $j<J=10$; then he can design a machine which always outputs ' NO ' for $j<J$. But, because of his proof he does in fact know that $\beta(J)=0$.

Of course if one knows a $B$ as above then one does not need an analogue machines to settle $? j \in A$ ? One simple computes $a(n)$ (as in 3.1) on $P_{A}\left(j, m_{1}, \ldots, m_{k}\right)$ (as in 3.2) for all $n<B$ or for all $m_{1}, \ldots, m_{k}<B$.

## 5. First example (see CAP pp 51-53)

Let

$$
\phi(x)= \begin{cases}e^{-\frac{x^{2}}{1-x^{2}}} & \text { for }|x| \leq 1  \tag{5.1}\\ 0 & \text { for }|x| \geq 1\end{cases}
$$

$\phi$ is an infinitely differentiable function $\left(\in C^{\infty}\right)$ though it is not analytic. Let

$$
\begin{equation*}
\psi_{n}(x)=4^{-a(n)} \phi\left(2^{-(n+a(n)+2)}\left(x-2^{-a(n)}\right)\right), \tag{5.2}
\end{equation*}
$$

where $a$ is as in $\S 3.1$. The graph of $\psi_{n}(x)$ is a blip of height $4^{-a(n)}$ centred on $2^{-a(n)}$, and having a width of $2^{-(n+a(n)+1)}$. If $m \neq n$ then the supports of $\psi_{m}, \psi_{n}$ do not intersect. Set

$$
\begin{equation*}
f^{\prime}(x)=\sum_{n=0}^{\infty} \psi_{n}(x) \tag{5.3}
\end{equation*}
$$

$f^{\prime}$ has a continuous but unbounded derivative, and $f^{\prime}(x)=0$ for $x>5 / 4$. Since

$$
f^{\prime}\left(2^{-j}\right)= \begin{cases}4^{-j} & \text { if } j \in A  \tag{5.4}\\ 0 & \text { if } j \notin A\end{cases}
$$

[^2]$f^{\prime}$ is not a computable function.
Let
\[

$$
\begin{equation*}
\Phi_{n}(x)=\int_{0}^{x} \psi_{n}(x) d x \tag{5.5}
\end{equation*}
$$

\]

The graph of $\Phi_{n}$ is a smoothed out step function with initial value 0 (at $x=0$ ) and a final value lying between 0 and $2^{-n}$.

Now take

$$
f(x)=\sum_{n=0}^{\infty} \Phi_{n}(x)
$$

$f$ is a computable function and its derivative is indeed the $f^{\prime}$ given by (5.3). Note that $\|f\|$, the uniform norm of $f$, is less than 2 . To settle $j \in A$ the idea is to feed $f$ into an (analogue) differentiator, and then to observe whether the output $f^{\prime}(x)$ is zero or not at $x=2^{-j}$. For definiteness let us suppose that we control the current $i_{1}$ in a circuit $C_{1}$ inductively to a passive circuit $C_{2}$ and observe whether the current $i_{2}$ in $C_{2}$ is zero or not at time $2^{-j}$. The claim for this machine is justified on two counts.
5.1. Because of the narrowness of the blip $\psi_{n}$, the measurement of the time $2^{-j}$ at which $i_{2}$ is observed must have, for $j \in A$, a precision ratio of order $2^{-\nu(j)}$ if the observed value of $i_{2}$ is to be different from zero.
5.2. For $j \in A$, let

$$
f_{j}(x)=f(x)-\Phi_{\nu(j)}(x)
$$

Then $f_{j}^{\prime}\left(2^{-j}\right)=0$. So if the machine is to give the answer YES for this $j$, then $i_{1}$ must satisfy

$$
\left|i_{1}(t)-f(t)\right|<\Phi_{\nu(j)}(t) \leq 2^{-\nu(j)}
$$

So unless the precision ratio for the uniform norm of $i_{1}$ is better than $2^{\beta(J)}$ the machine will give wrong answers for some $j<J$.
5.3. Thus to design a machine which will give correct answers for all $j<J$ we need to know $\beta(J)$.

## 6. Second Example

In their (1991) Doria \& Costa showed how a function defined in Richardson (1968) could theoretically be used in the construction (based solely on classical dynamics) of a device which would settle questions of the form $? j \in A$ ?. They write
'Our example is intended to be seen as a Gedanken experiment, as we do not wish to consider at the moment the certainly formidable question of its implementation.'

I shall show that its implementation by an analogue machine requires knowledge of a bound for $\beta(J)$.
6.1. Let $k \geq 1$ be given and let $\mathscr{L}$ be the class of all real-valued functions of $k+1$ or fewer real variables which can be get by composition from the following initial functions:
(i) + and $\times$;
(ii) sin;
(iii) projection functions $\lambda \vec{x} \cdot x_{i}$;
(iv) constant functions $\lambda \vec{x}$. , where $c$ is either $\pi$ or a rational number.

Let $P_{A}$ be the polynomial of $\S 2$ (3.3). Richardson shows how one can define a function $F\left(u, x_{1}, \ldots, x_{k}\right)$ in $\mathscr{L}$ having the following properties.
(1) $F$ is an even function of each of the $x_{i}$.
(2) $F\left(u, x_{1}, \ldots, x_{k}\right) \geq 0$
(3) $F\left(j, x_{1}, \ldots, x_{k}\right)>1$ if $j \notin A$.
(4) If $F\left(j, x_{1}, \ldots, x_{k}\right) \leq 1$ then $P_{A}\left(j,\left\langle x_{1}^{2}\right\rangle, \ldots,\left\langle x_{k}^{2}\right\rangle\right)=0$ and $F\left(j,\left\langle x_{1}^{2}\right\rangle, \ldots,\left\langle x_{k}^{2}\right\rangle\right)=0$ where $\left\langle x_{i}^{2}\right\rangle$ denotes the natural number nearest to $x_{i}^{2}$. Hence in this case $j \in A$.
(5) To calculate $F\left(j, x_{1}, \ldots, x_{k}\right)$ it is necessary first to calculate $P_{A}\left(j, x_{1}^{2}, \ldots, x_{k}^{2}\right)$
6.2. Let $\rho$ either be the function $\phi$ of $\S 5$, or be given by

$$
\rho(x)=\frac{1}{2}(|x-1|-(x-1)) .
$$

In either case $\rho(x)=0$ for $x \geq 1$ and $\rho(0)=1$. If we extend $\mathscr{L}$ to $\mathscr{L}^{+}$by taking $\rho$ as a further initial function then either all the functions in $\mathscr{L}^{+}$belong to $C^{\infty}$ or they are all continuous piecewise analytic functions.

Now set

$$
\begin{equation*}
H(u, \vec{x})=\rho(F(u, \vec{x}))\left(\vec{x}=x_{1}, \ldots, x_{k}\right), \tag{6.1}
\end{equation*}
$$

and write $H_{j}(\vec{x})$ for $H(j, \vec{x})$. Then by 6.1 (3), (4), we have

$$
\begin{gather*}
H_{j}(\vec{x})=0 \text { for all } \vec{x} \text { if } j \notin A,  \tag{6.2}\\
\exists \vec{x} H_{j}(\vec{x})=1 \text { if } j \in A . \tag{6.3}
\end{gather*}
$$

But, by (3.6) and 6.1 (4) we see that, for $j \in A$,

$$
\begin{equation*}
H_{j}(\vec{x})=0 \text { if } x_{1}^{2}, \ldots, x_{k}^{2}<\nu(j)-1 \tag{6.4}
\end{equation*}
$$

Thus if an analogue machine is going to use $H_{j}$ to settle $? j \in A$ ? and if $j \in A$, then the machine will have to calculate $P\left(j, y_{1}, \ldots, y_{k}\right)$ for some values $y_{1}, \ldots, y_{k}$ one at least of which - say $y_{i}$ - is greater than $\nu(j)-1$. And by $(3.7)$ the value of one of the $y^{\prime}$ 's $-y_{i}$, say - must be accurate to within 1. Hence, for $j \in A$, the inputs $y_{1}, \ldots, y_{k}$ for the calculation of $H_{j}\left(y_{1}, \ldots, y_{k}\right)$ need to have a precision ratio of at least $\nu(j)^{3}$. This is also true if $H_{j}$ is calculated by a digital computer. Thus the claim is proved for this example.

[^3]6.3. Richardson, and following him, Da Costa and Doria make the problem look simpler by coding the $k$-plot $\vec{x}$ by a single real number $t$. Richardson defines decoding functions $(t)_{1}, \ldots,(t)_{k}$ (in $\mathscr{L}$ ) with the following property:

Given $\epsilon>0$ and $x_{1}, \ldots, x_{k}$ one can find $t$ so that

$$
\begin{equation*}
\left|x_{i}-(t)_{i}\right|<\epsilon \text { for } 1 \leq i \leq k \tag{6.5}
\end{equation*}
$$

The functions he defines also satisfy

$$
\begin{equation*}
(t)_{i} \leq t \tag{6.6}
\end{equation*}
$$

Now define a function $B_{j}$ by

$$
\begin{equation*}
B_{j}(t)=H_{j}\left((t)_{1}, \ldots,(t)_{k}\right) \tag{6.7}
\end{equation*}
$$

Then

$$
\begin{equation*}
B_{j}(t)=0 \text { for all } t, \text { if } j \notin A, \tag{6.8}
\end{equation*}
$$

while if $j \in A$ then for any $z<1$

$$
\begin{equation*}
\exists t\left(B_{j}(t)>z\right) \tag{6.9}
\end{equation*}
$$

But, by (6.4) and (6.6) above we also have

$$
\begin{equation*}
B_{j}(t)=0 \text { if } t^{2}<\nu(j)-1 \tag{6.10}
\end{equation*}
$$

Any attempt to distinguish between (6.8) and (6.9) will yield further justifications for my claim. For example, Da Costa and Doria define

$$
\begin{equation*}
K(j)=\int_{0}^{\infty} B_{j}(t) \gamma(t) d t \tag{6.11}
\end{equation*}
$$

where $\gamma(t)$ is a cut off factor inserted to ensure that the integral converges. (The exact nature of $B_{j}$ depends both on the distribution of the zeros of $P_{A}$ and on the particular decoding functions; in any case $B_{j}$ will be highly oscillatory, and, if $P_{A}$ has 'rather few' zeros I think it likely that $\int_{0}^{\infty} B_{j}(t) d t$ will be of order $\left.\nu(j)^{-1}\right)$.

To specify and analogue machines which, for $j<J$ and $j \in A$ will output a non zero approximate value for $K(j)$ one will have to specify a value $B$ say, to replace $\infty$ as the upper limit of integration. But, by (6.9) above, one will then be able to compute a bound for $\beta(J)$ from $B$. And because of the cut off factor $\gamma, 6.9$ shows that $K(j)$ will be small of order $\nu(j)^{-1}$. Da Costa and Doria propose switching from one dynamical system to another, according to whether $K(j)=0$ or $K(j)>0$. An analogue machine which will correctly effect this switching will thus require, for the CVQ corresponding to $K(j)$ a precision ratio of order $\beta(J)$. Thus, in all, there are three different factors in the specification of the proposed machine which requires a knowledge of a bound for $\beta(J)$.

## 7. Quantum Mechanical machines

7.1. Both my examples depend on specifying a self-adjoint operator $T$ on, say, Hilbert space (e.g. specifying the Hamiltonian for some quantum-mechanical system) and making observations on its spectrum to settle $? j \in A$ ?.

The first example is due to Pour-El and Richards (CAP pp. 190-191). They show that a
certain $T$ may be constructed as a computable limit of a sequence of computable operators $T_{n}$ with the following properties.
(1) Let $\lambda_{j}(j \geq 0)$ be a computable bounded sequence of real numbers. Then if $j \notin A$ the spectrum of $T$ has $\lambda_{j}$ as an eigenvalue (corresponding to a line in spectranalytic terms), while if $j \in A$ the spectrum has a continuous band of width $2.2^{-\nu(j)}$ centered on $\lambda_{j}$. The factor $2^{-\nu(j)}$ ensures that the sequence $T_{n}$ has a computable modulus of convergence. To make observation easy one could take

$$
\lambda_{j}=5-4.2^{-j}
$$

and then there will be a gap between the bands (if present) around $\lambda_{j}$ and $\lambda_{j+1}$ to separate the lines or bands around $\lambda_{j}$ and $\lambda_{j+1}$ one only needs a precision of the order $2^{j}$; but to distinguish a line at $\lambda_{j}$ and a band around $\lambda_{j}$; one needs a precision of order $2^{\gamma(j)}$. Thus as in the previous examples, to settle $? j \in A$ ? correctly for $j<J$ one needs to know a bound on $\beta(J)$ in order to ensure that the measyrements made will have the required precision. Another justification for my claim in this example is best illustrated by another example, which is a simplification of one given in Gandy (1991). Namely let the sequence $\left\{\lambda_{n}\right\}$ be defined by

$$
\lambda_{n}=2^{-a(n)},
$$

and let $S$ be a compact operator with these values of $\lambda_{n}$ as its eigenvalues. To decide $? j \in A$ ? it is only necessary to observe, with say, a precision $2^{j+1}$, whether or not there is a line at $2^{-j}$. (Of course, on physical spectroscopy what one observes is transitions from one $\lambda$ to another, but this does not affect the argument.) So the question becomes: could one design a quantum mechanical device which would have, for some observable, an approximation $S^{\prime}$ to $S$ whose eigenvalues for $j<J$ would be close to $S$ ? IT will be recalled that a design must allow one to compute approximate values for all relevant parameters and must specify allowed tolerances. I do not know, except in particular cases like atomic and molecular spectra, how one might construct a system which would approximate a given operator for a given observable. But it is obvious, for both $S$ and $T$, that one would need to know, at least approximately, the entries in the first $\beta(J)$ rows of their representing matrices (wrt some chosen orthonormal basis). But this justifies the claim ${ }^{4}$.
7.2. The wave functions for a quantum mechanical system may result from the superposition of infinitely many more easily defined wave functions and so correspond to the parallel working of infinitely many separate machines. This suggests a possible method for designing a quantum-mechanical device which would give correct answers to the questions $? j \in A$ ? However the quantum computer described by Deutsch (1985) cannot do this, although it can use superposition greatly to reduce the run time for certain decidable problems.
7.3. Refinements in experimental technique allow one to build analogue machines whose behaviour depends on a single quantum (e.g., a single photon). Experiments with such

[^4]devices confirm the often counter-intuitive predictions of standard quantum theory. Could they provide a disproof of my claim? I do not know of any example for this.

## 8. Discussion

8.1. When one shows that a given number-theoretic function is computable, or that a given number-theoretic problem is decidable, one does not place bounds on the run-time or the size of the memory - unless, of course, one is concerned with problems of complexity. That is, one is not concerned with precision ratios. So it may look as if I have placed unfair restrictions on analogue machines. But suppose one has proved that a certain programme will give correct answers to a problem $? j \in X$ ?. Then, given $J$, one can compute bounds on the time and space required to settle $? j \in X$ ? correctly for all $j<J$. But this is exactly what I claim cannot be done for analogue machines intended to settle non-decidable problems.
8.2. Cascades of events and chain reactions allow one (as in a photon multiplier) greatly to amplify the scale of an event. This is, in effect, a reduction of precision ratios. Could this be used to overcome the objections raised by my claim? The answer is 'No', because only when one knows a bound for $\beta(J)$ can one determine how much amplification is needed.
8.3. In CAP (and Pour-El \& Richards (1979)) other examples are given of differential equiations (in particular the wave equations) which will give a non-computable output for a computable input. The claim can be justified for these using the ideas of $\S 5$.
8.4. Kreisel has discussed calculation by analogue machines in a number of place; see, in particular, his (1974), (1982), and (199). Some of his comments and analysis are illuminating, and have helped me in getting my ideas stragith. But one of his points is that there are more interesting, more sensible, and more relevant questions to ask than the (logical) question with which I am concerned.
8.5. Penrose, in his (1989) and (1994), has argued that the human brain can be thought of as an analogue machine which can, in principle, settle undecidable problems. Firstly, he believes that mathematical results which can, at least in principle, be produced by human intelligence, cannot, even in principle, be produced by artificial intelligence - that is by some fixed programme $P$. Note that $P$ need not be itself directly responsible for the mathematical statements which the machine outputs. $P$ may be like an operating system, for example it may, by a process similar to natural selection, use mutations and tests of fitness to direct the (continual) evolution of subprogrammes for doing mathematics. But this possibility does not, straightforwardly, invalidate Penrose's argument justifying his belief. A concise version of Penrose's argument is given in Gandy (1994). Secondly Penrose believes that the sentences uttered or written by people are caused by physical and chemical events in their brains.

To allow for non-algorithmic actions in the brain, Penrose postulates a - not yet completely formulate - future theory which he calls CQG (for Correct Quantum Gravity). This will have consequences both for cosmology (concerning the direction of time's arrow) and for quantum theory (accounting for the collapse of real (not subjective) wave functions). He suggests ways
in which such a theory may allow for the growth of microscopic structures (such as quasicrystals, synapses and micro tubules in neurons) in ways which are not locally determined nor computable. It seems worthwhile to consider (rather naively) such patterns of grwoth from a mathematical point of view.

## 9. Patterns of growth

I consider a pattern of possible growth as being displayed on a tree. At each node $P$ there is a finite label which represents a particular structure $S_{P}$ at a particular stage of growth for example, a particular quasi-crystal. If this structure $S_{P}$ is capable of growth then there will be a finite number of nodes $P_{1}, \ldots, P_{k}$ immediately below $P$; each of the structures $S_{P_{1}}, \ldots, S_{P_{k}}$ arises from $S_{P}$ by a single step of growth (for example, by the addition of a single molecule). Two distinct structures $S_{P}$ and $S_{Q}$ may, in one step, grow into the same structure. Hence a node may have two different immediate predecessors; these trees are not the same as those standardly used in recursion theory. A node $P$ and the corresponding structure $S_{P}$ are fertile if there is an infinite path through $P$. If $P$ is not fertile then, however $S_{P}$ may grow, after a finite number of steps it will become a structure which can grown no more.

Now we suppose that the label representing any structure $S$ is (coded by) a finite sequence $u$ of 0's and 1's. We may suppose that the significant features of $S$ can be computed from $u$. An infinite path gives an infinite sequence $u_{1}, u_{2}, \ldots$, of binary sequences. We define the growth function $\gamma$ along the path by $\gamma\left(u_{n}\right)=u_{n+1}$. If the sequence is computable then so is $\gamma$; in particular there is a Turing machine $M$ which, when presented with $u_{n}$ on its tape, will eventually replace it by $u_{n+1}$. Now the action of $M$ is certainly locally determined; it will, for example, in general, inspect each of the digits in $u_{n}$. We shall say that $\gamma$ (and the infinite sequence) are potentially locally determined.
9.1. Suppose we are given a tree of structures and a growth function $\gamma$ which satisfies the following conditions:
(i) If $u$ codes a fertile structure $S$, then $\gamma(u)$ codes a fertile structure into which $S$ can grown in a single step.
(ii) The function $\gamma$ is not potentially locally determined.

Then, starting from any fertile structure $S$ and iterating $\gamma$ will produce a non-computable infinite sequence of structures.

If one could examine, say, the first $J$ structures in this sequence one could compute the first $J$ values of some non-computable function. The precision ratio of observation has to be sufficiently large to enable one to determine the codes $u$ for these $J$ structures; it might well be a computable function of $J$.
9.2. Since quasi-crystals have been observed which contain a very large number of molecules, Penrose suggests that their growth is not a matter of chance, but is governed by some - as yet unformulated - laws of non-local actions. If, further, the theory involved actions which were not even potentially locally determined, then it would allow analogue machines to produce non-recursive functions. One would not expect the theory to be totally deterministic; indeed it is plausible that there are at least two distinct infinite paths through any fertile point of
the tree, and hence continuum many such. Although each path yields a non-computable function, one cannot use it to settle a specified undecidable problem.

But for the growth of microstructures in the brain, which determine how neurons behave and how they affect each other, one would expect that certain particular paths would be selected on would be permitted.
9.3. The definition of 'potentially locally determined' can be made quite general by considering, in place of the Turing machine $M$, any mechanism which satisfies the principles of Gandy (1980) - in particular, of course, the principle of 'local causation'. And then one has a converse to 9.2 - if the growth function along an infinite path is potentially locally determined, then the sequence of structures along it is computable.
9.4. It is well-known that there are binary trees whose nodes form a recursive set, which have infinite paths but no computable infinite paths; using this fact one can for example describe a finite set of tiles which can tile the whole plane, but only in a non-computable way (see Hanf (1974)). Using the notion of trial and error predicates (see Putnam (1965)) we can see how the lattermost infinite path, $\lambda$ say, might be grown. A node is specified by a finite binary sequence $u$ which describes (with 0 for 'Left' and 1 for 'Right') the path from the vertex leading to it, and we consider $u$ also as the structure starting at $u$. The size of this is just the length of $u$. Now we define a computable sequence $u_{n}$ of nodes on the tree as follows.
(i) $u_{0}=()$ (the vertex of the tree).
(ii) If $u_{n}$ is not terminal (has nodes of the tree below it) then

$$
u_{n+1}=u_{n} 0
$$

(iii) Suppose $u_{n}$ is terminal and has the form $v 0$ or $v 011 \ldots 1$ then

$$
u_{n+1}=v 1
$$

Since no node on $\lambda$ is terminal, none of the $u_{n}$ can lie on the right of $\lambda$. Below any node $v$ which lies to the left of $\lambda$ (e.g.; 10 if $\lambda(1)=1 \lambda(2)=1$ ) there can only be finitely many nodes of the tree (since $v$ cannot be fertile). Hence for some $n$ we must have a $u_{n}$ lying to the right of $v$. Thuse for any $J$ there will be an $n_{J}$ such that $u_{n_{J}}=\lambda(1), \lambda(2), \ldots, \lambda(J-1)$.
9.5. At first sight it might look as if this process of trial and error growth could be accomodated in some reasonable physical theory. But this is an illusion; for not only is $n_{J}$ not computable from $J$, but there can be no computable bound on the lenghts of the sequences $u_{n}$ with $n<n_{J}$ which have to be explored before $u_{n_{J}}$ is arrived at. And so the process considered is analogous to a trial and error process for deciding if $j \in A$ (as in §3) - one simple looks ahead to see if, for some $n, a(n)=j$.
9.6. Penrose suggests that in a theory of quantum gravity the process of growth would be represented by a superposition of wave functions each corresponding to a particular pattern of growth, and that the effect of gravity would be to collapse the wave function, so that only constituents corresponding to patterns of growth capable of producing large structures would survive. To picture this process on the binary tree let the potential size, $\pi(v)$ of a node $v$ be the maximum length of all nodes $u$ extending (or lying below) $v$. If $v$ is fertile
we set $\pi(v)=\infty$. Then the proposed theory would ensure that any permitted vertex would grow to some node of great size, though (in the simple form in which I stated it) it would not guarantee growth along an infinite path. It would well be that for a given $J$ there would be a $k_{J}$ such that any node of size greater than $k_{J}$ would agree with $\lambda$ at the first $J$ places. But this fact will not allow us to compute values of $\lambda$ from observations on large structures which have developed, unless we know some (necessarily non-computable) bounds for $k_{J}$. If a theory of growth of the kind considered is to stand up against our claim it looks as if some kind of non-computability must be built into the theory - for example into the way in which gravity determines the collapse of wave functions.

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On the impossibility of using analogue machines to calculate non-compatable functions.
(R.O.Gandy, 9 squitchey Lane, $O$ reform $0 \times 2>0$ ) Sep 93.
Introduction A number of examples have been given of physical systems (both classical and quartern mechanical) which when provided with a (continuously variable) computable input will give a ron-computable output. It has been suggested that such systems might allow one to design analogue machines which would calculate the values of some number-theoretic non-computabl function. Analysis of the examples shew that the suggestion is wrong. In 24 I claim that given a reasonable definition of analogue machine' it will allays be wrong. The cain is to be read not so much as a dogmatic assertion, bat rather as a challenge.

In z's 1 al 2 I discuss analogue machines, ad Lay down some conditions which I believe they must. salify. In $\$ 3$ I discuss the particular forms which a paradigm undecidable problem (or nor-compulable function) may take. In $\}$ 's 5 and 6 I justify my claim for two particular examples lying within Ae range of classical physics, an in 37 justify it for two (closely connected) examples from quatutorim mechanics, and decals, very briefles, other possible quantum mechanical situations. S8 contains mani zeus remarks ad comments. In $\$ 9$ I consider the suggestion made by Pennose that a (future) theory of quantum gravity may predict tion-locally-determised, oed perhaps nom-cempuliche patterns of growth for microscopic stradares. My caduxem is hat such a theory will hone to here non-cimputability bret into it.
21 Analogue machines. By a continucresly variable quantity ( ${ }^{6} C V Q$ ') I mean a physical quantity which is represented matamatically by a point in a metric space, e.g., by real number, or a point of Hill ert space. Thin is not put Forward as an exact definition, but as an indication
of how I use the term. Far CVQ's very natural definitions of 'computable' have been given din. Pour-EL $\&$ Richards $(19899)$; I shall report to his Look as CAP. Roughly speak in y ' $x$ i computable' means that $x$ is the linnet of a sequence of finitely presented approximations and a modulus of convergence for the sequence can be computed.

In a theoretied treatment of a physical device the CVQ's have exact values, ad no bound is placed, a prion on their magnitude. But when such a device is $t$ be used as on analogue machine to perform some calculation then there will be an upper limit $x$ on the size of a CVQ (an electric circuit will melt of the current a lo large) and a lower (imition He decuracy with whin it can be controlled on measured. Numerical values fr $x$ ad $\varepsilon$ depend, Of course, on the choice of units for the porticalor $C V Q$ considered, Bat the ratio $X /\left(\frac{\varepsilon}{(P R \text { for shat } \text { does act; }}\right.$; so we define the precicion nation of the $C V Q$ to he $x / E$. Ia the theory of he machine there may he different variables having the same physical dimension; these ane to bo counted as deitinet $\subset V Q$ 's ad may haver different precision ratios. The 'midepantant variable time is oho a CVQ ad hus a PR; ishen an and loge machine is to be used some limit must be placed on its run tire.

We ane concerned with molters of principle rather then of practice so although
a given analogue machme will hove definite preciein ratios, we do nat plate anzio boéndye on the $P R$ s alithert may be attaisied by some machine.

We are primarily - sometimes only concerned with those CVQ's which ane inputs and outputs of the machine. We shall he interested in cores where these may be contincererly controlled or continuously recorded fandeins; in such cases the relaviont $X$ ad $E$ will be given by some norm for the function. Most usually the uniform norm will be appropriate, wet for sorn machines one might wont to axe, sous, the $L^{2}$ norm.
"Even discretely varying quantities such as natural numbers have precision nations attacked to them; perfect accuracy $\left(\operatorname{say} \varepsilon<\frac{1}{2}\right)$ (may be altaicividble, but still bare is a bound on size one connect plat moore then $N$ balls in a gain bore nor record move than Nevents with a. given geiger counter. In porticular if an analogue machine incorporates a battery of digital computers then a PR (wrued depends both on the programme used and on the hardware) can be assigned to each of them: note that it does no l' depend on the placing of Ac decimal point.

In what follows we shat be concerned with the orders of magnitude of $P R$ 's rather the with precise values or appear bounds.
32. Specification of analogue machines

A specification for an analogue mactisne is a finite list of instructions whig would, in principle, enable a technicion or engineer $t$ construct it ; descriptions of the apparatus cred in a (published) accant of an experiment, der, although greatly a blreviated, hare this form. If the correct operation of the machine requires particular precision rations for certain quantities. ten Ace instructions will specify tolerances for certain components ${ }^{1}$. For example a machine might

1 When A.M. Turing was building his speech encoder ('Delilah') he found tat if it was to work, some of the components had to hove a tighter hon usual tolerance on their values; these were more expentizis ton the standers components and - ot beset in Ae case of resist andes - had a gold spot do indicate tat Ry wren e decorate $\&$ wither (Ithaki) 1\%.
require a cam whose ideal shape wald be gavin by $r=f(\theta)$
who ne $f$ ix some mathematical fundeis. Then The instructions would indic ate how the function $f$ could be computed (egg-, $f(\theta)=2+\sin ^{2} \theta$ ems for $0 \leqslant \theta \leq 360^{\circ}$ ) and give a permitted to france (egg., $\pm 10^{-3} \mathrm{cms}$ ). Tolerances can he given as precision ratios $\left(3.10^{3}\right.$ in the example). A specification will determine either explicitly or implicitly the $P R$ 's of the quantities (inducing outputs and inputs) occurring in the machine.

23 Undecidable problems
In the excomples known to me it is proposed that theme might be an analogue machine which win input $J$ ( $E N$ ) would output 'Yes' or 'No' to questions of the form? j $A$ ? whore $A$ is some standing recursively exumevadi nou-recersive set -for example the set. winch represents the halting problem. I shall only considerposod machines of Rim kind. I descilite two ways of representing the set $A$
31 Theme is a cotol. computable function $a: \mathbb{N} \rightarrow \mathbb{N}$ which enumereles A without repetitions. (Thin is the notation used throughout $C A P$ ) The waiting-time function $\sim$ in defined ely
(1) $\quad)(j) \simeq \mu n$. a $(n)=j$.

This in a partial recursiere function arose domain in $A$ and which is nat bounded by any total computable function. e For any paricieler analogue machine there in an upper bound $J$ on Re inputs it car a ceept. I define.
(2) $\beta(J)=\operatorname{Mare}\{\nu(j): j<J थ j+A\}$
(with Moose $\phi=0$ ). Thin is a total fametion which is not compulable indeed it eventarilly majanises every computable function. is
32 There is a polynornial $P_{A}(y, \vec{x})$ such Dat ${ }^{\text {a }}$
(3) $j \in A \longrightarrow(\exists \vec{m}) P(j, \vec{m})=0$,
wee the vorialles of $\vec{m}\left(=m, m, \ldots, m_{k}\right)$ mange overall the nations inumbers.

In this case we define
(4) $\sim(j) \simeq(\mu n)(\exists \vec{m}<n) P_{A}(j, \vec{m})=0$,
and
(5) $\beta(J)=\operatorname{Max}\{\gamma(j): j \in A \propto j<J)$.
ashen $\omega$ ad $\beta$ home. Ae some properties as in 3.1. observe $A_{0}$, it $j \in A$, then
(6) $\forall \vec{m}<\nu(j) P_{A}(j, \vec{m}) \neq 0$.

Various explicit definitions of suitable polymonids horn been given. For each of Ruse, if Pokich lists the first so many values dp some recarsive function so, taking $i=1$, we may suppose that
(7) $P_{A}(j, \vec{m})=0$ add $P_{A}\left(j, m^{\prime}, m_{2}, \ldots, m_{k}\right) \neq 0$ where $\left|m^{\prime}-m\right|=1$.
34 The Cain Since a giom machine connect hassle numbers greater than some bound we conciob a given $J$ and the question's $2, j \in A$ ? "for $j<J$. Now I make the following section 5 and 6 be con recent $C$ on b $P$ ?

CLAIM Let $J$ be given. Then one connst design an analogue machine (whore he houviour is governed by stand and physical lows) which will give correct answers to all the questions ? $j \in A$ ? for $j<J$ unless one knows a bound $B$ for $\beta(J)$.

I call this a claim rather than a conjecture because I de nit think one could prove it unless one placed severe restrictions on the notion of 'analogue machine, and finis I dr not wish to $d \sigma$. But I believe that if someone proposes 1 Pour- EL in her ( 1974 ), gives a definition (based on differential analysers) of class of continuous functions which they and characterizes the class of continuous functions inestions of precision but can generate. She is net concerned with questions of could he app hid Ibolieve that the methods used in $\delta^{5}$ on lo ta ty e she cmsiders. to justify my dim for all mach an analogue machine for settling? jed (sureptioush?) made use he shown that either they have (hat not all the given answers will be of a bound for $\beta(J)$, or that not all the giver of A he wording of correct. To illustrate the quit plausible) that someone proves Re claim, suppose (what is quite plausen the can design a that $j \notin A$ for all $j<J=10$ ' ' $N O$ ' for $j<J$. Bat, because machine which always out puts act know tat $\beta(J)=0$ of this proof he does $B$ as above then one Of course, if one knows a $B$ as above then ere machine to settle? $j \in A^{2}$. does not need an analogue ( 3.1 ) or $P_{A}\left(j, m_{1}, \ldots, m_{k}\right)$ One simply computes $a(n)$ (cs for for $m_{1}, \ldots, m_{k}<B$. (as in 32) for a (l $n<B$ oral

$$
\text { End } 2\} 4
$$

25. First excomple (see carp $51-53$ )

$$
\text { Let } \begin{align*}
\phi(x) & =e^{-\frac{x^{2}}{1-x^{2}}} & & \text { for }|x| \leqslant 1  \tag{1}\\
& =0 & & \text { fo }|x| \geqslant 1
\end{align*}
$$

$$
=0 \quad \text { for }|x| \geqslant 1
$$

$\phi$ is an infinitely differentiable function $\left(\in C^{\infty}\right)$ Tough itionot analytic.
(2) Let $\psi_{n}(x)=4^{-a(n)} \phi\left(2^{-(n+a(n)+2)}\left(x-2^{-a(n)}\right)\right)$,
where $a$ is as in 23.1.) The graph of $\psi_{n}$ is a blip of height $4^{-a(n)}$ centred on $2^{-a(n)}$, ad having $a$ width of $2^{-(n+a(n)+1)}$. If $m \neq u$ then Te supports of $\psi_{m}, \psi_{n}$ do not intersect.
(3) Set $f^{\prime}(x)=\sum_{n=0}^{\infty} \psi_{n}(x)$;

If ${ }^{\prime}$ hus a continuous Gut unbounded derivative, and 1 in
$f^{\prime}(x)=0$ for $x>5 / 4$. Since $f^{\prime}(x)=0$ for $x>5 / 4$. Since
(4)

$$
\begin{array}{rlrl}
f^{\prime}\left(2^{-i}\right) & =4^{-j} & & \text { of } j \in A \\
& =0 & \text { it } j \notin A
\end{array}
$$

$F^{\prime}$ is not a computable function.
Let

$$
\Phi_{n}(x)=\int_{0}^{x} \psi_{n}(x) d x .
$$

The graph of In in a smoothed out step function with in it cal value $O$ (at $x=0$ ) addafinal value lying between 0 and $2^{-n}$.
Now take

$$
\begin{aligned}
& \text { take }=\sum_{n=0}^{\infty} \Phi_{n}(x) . \\
& f(x)=\text { function }
\end{aligned}
$$

$f$ is a computable function and its devirative is indeed He $f^{\prime}$ gin by (3). Norite Nat $\| f f l l$ an the uni form
 te ilea is to feed f pinto ankli(ancodlogue) differentiator, , and then to ob o bserve

Whether the output $f^{\prime}(x)$ is zero on not at $x=2^{-j}$. For definiteness let us suppose $t$ ant we control the current $i$ in a circuit $C_{1}$ to be $f(t)$ ( $t$ for time) and couple $C_{1}$ inductively is a passive circuit $C_{2}$ and observe whet ines the current $i_{2}$ in $C_{2}$ is zero or net of time $\dot{z}^{-j}$. The claim for thin machine is justified on two counts
5.1 Because of the narrowness of the bligh $\Psi_{n}$, the measurement of the time $2^{-j}$ at which $i_{2}$ is observed must have, for $j \in A$, a precision ratio of border $2^{-\nu(j)}$ if in wibseraed value of $i_{2}$ is to he different from zero.
5.2 For $j \in A$. Let

$$
f_{j}(x)=f(x)-\Phi_{D(j)}(x)
$$

Then $f_{0}^{\prime}\left(2^{-j}\right)=0$. So if the machine is to give the answer YES for this $j$, then $i$, must satisfy

$$
\begin{aligned}
& \text { must satisfy } \\
& \left|i_{1}(t)-F(t)\right|<\Phi_{\nu(j)}(t) \leqslant 2^{-\nu(j)}
\end{aligned}
$$

So unless the precision ratio $f \sigma^{2}$ the uniform norm of $i$, is better than $2^{\beta(J)}$ the machine will give wrong answers for some $j<J$.
5!3n. Thus to design a machine
which will give correct answers for all $j<J$ we need $t$ know $\beta(J)$
IG second Example
In their (1991) Donia \& Costa phoued how a function defined. (ing. Riehundson (1968) could, theoretically be used bim Aettionstunctions. duosedysolely on classical dynamics) of a device which would settle questions of the form? $j \in A$ ?.
They unites

- Our example is intended to be seen as a Gedanken experiment, as we do not wink te consider at the moment de certainly formidable quest in of its implementation?

I shall h that its implemental ain le an analigue machine requires knowledge of a bound for $(J)$.
6. 1 Let $k \geqslant 1$ be given and let $g$ be the class of all nolo rolued functions of $k+1$ or fewer real variables which can be get by composit ion from the following initial functions:-
(i) $+\operatorname{ad} x ;$
(ii) $\sin$;
(ii) projection functions $\lambda \vec{x} \cdot x_{i}$; (iv) constant functions $\lambda \vec{x}, C$, whin C is either $\pi$ or a rational number.
Let $P_{A}$ be the polynomial of 22 (3). Richardson shows how one can define a function $F\left(u, x_{1}, \ldots, x_{k}\right)$ in $f$ hewing $A$ following properties.
(1) $F$ is an even function of each of the $x_{i}$.

$$
\begin{aligned}
& \text { (2) } F\left(u, x_{1}, \ldots, x_{k}\right) \geqslant 0 \\
& \text { (3) } F\left(j, x_{1}, \ldots, x_{k}\right)>1 \text { if } j \notin A \text {. }
\end{aligned}
$$

(4) If $F\left(j, x_{1}, \ldots, x_{k}\right) \leqslant 1$
then $P_{A}\left(j,\left\langle x_{1}^{2}\right\rangle, \ldots,\left\langle x_{k}^{2}\right\rangle\right)=0$

$$
\text { and } \quad F\left(j,\left\langle x_{1}^{2}\right\rangle, \cdots,\left\langle x_{k}^{2}\right\rangle\right)=0
$$

where $\left\langle x_{i}^{2}\right\rangle$ denotes the natural number nearest $t$ $x_{i}^{2}$. Hence in this case $j \in A$.
(5) To calculate $F\left(j, x_{1}, \ldots, x_{k}\right)$ it is necessorg first to calculate $P_{A}\left(j, x_{1}^{2}, \ldots, x_{k}^{2}\right)$
6.2. Let $p$ either be the foundation $\phi$ of 25 , or be given lu s $\rho(x)=1 / 2(|x-1|-(x-1))$.
In either case $\rho(x)=0$ for $x \geqslant 1$ and $\rho(0)=1$. If we extend $\xi^{\prime} \xi^{+}$by taking $\rho$ as a fur the ${ }^{+}$ initial function then either all Ae functions in $\delta^{+}$ belong $t c^{\infty}$ or they are all continuous picceivise analytic functions.

Now set
(1) $H(u, \vec{x})=j(F(u, \vec{x})) \quad\left(\vec{x}=x_{1}, \ldots, x_{k}\right)$, and unite $H_{j}(\vec{x})$ for $H(j, \vec{x})$. Then by $6,1(3),(4)$,
(2) $H_{j}(\vec{x})=0$ for all $\vec{x}$. if $j \notin A$,
(3) $\exists \vec{x} H_{j}(\vec{x})=1$ if $j \in A$.

But, by $3.2(6)$ and $6.1(4)$ we see that, for $j c A$, (4) $H_{j}(\vec{x})=0$ if $x_{1}^{2}, \ldots, x_{k}^{2}<2(j)-1$.

Thus if an analogue machine in going to use? H $j$ to settle? $j \in A$ ? and of $j \in A A$, then the machine will hove to calculate $P\left(j, y_{1}, \ldots, y_{k}\right)$ for some ur lues $y_{1}, \ldots, y_{k}$ one at least of which - say $y_{i}$ - is greater than $2(j)-1$. And by $3.2(7)$ the value woffione of the $y^{\prime} s-y_{i}$, say mast he accurate to within 1. Hence, for $j \in A$,
the inputs $y_{1}, \ldots, y_{n}$ for the calculation of $H_{j}\left(y_{1}, \ldots, y_{k}\right)$ need ts have a precision ratio of at least $2(j)^{1}$. This is ado tune of $\mathrm{H}_{j}$ is calculated by a digital computer. Thus the claim is proved for Ais example. 6.3. Richardson, and following him, $A$ a Costa ad Donia moke the problem look simpler by coding the $k$-plat $\vec{x}$ by a single real number $t$. Riehundson defines decoding functions $(t)_{1, \ldots,}(t)_{k}($ inf $)$ with the following property:
given $\varepsilon>0$ ad $x_{1}, \ldots, x_{k}$ one can find $t$ so that

$$
\text { (1) }\left|x_{i}-(t)_{i}\right|<\varepsilon \quad \text { for } \quad 1 \leqslant i \leqslant 1 \text {. }
$$

The functions he defines also satiety
(2) $(t)_{i} \leqslant t$.

Now define a function $B_{j}$ bs

$$
\text { (b) } B_{j}(t)=H_{j}\left((t)_{1}, \ldots,(t)_{k}\right) \text {. }
$$

Than (4) $B_{j}(t)=0$ for all $t$, if $j \notin A$, while if $j \in A$ then for any $z<1$

$$
\text { (5) } \exists t\left(B_{j}(t)>z\right)
$$

But, by $6.2(4)$ and (2) above we do have

$$
\text { (6) } B_{j}(t)=0 \text { if } t^{2}<2(j)-1 \text {. }
$$

Any attempt to distinguish between (4) and ( $(5)$ will yield farther justificat ions for my claim. For example, Da Costa ad Doric
1 Even if different $P R$ 's were used for $g_{1}, \ldots$, In $_{R}$ I believe the clair would stand: for the $m$, in $3.2(7)$ codes a computation sequence, so its size will. certainly increase with $o(j)$.
define
(b) $K(j)=\int_{0}^{\infty} B_{j}(t) r(t) d t$
where $r(t)$ is a cut off factor inserted to ensure Beat $A_{c}$ integral converges. (The exacts nature " of $B_{j}$ depends lath on the distribution of Pellizencs of $P_{A}$, and on te particular decoding functions; in any core $B$, will be highly oscillatory, ad sp if i $P_{A}$ os has 'rather few' zeros I think it likely test $\int_{0}^{\infty} B_{j}(t) d t$ will he of order $\left.2(j)^{-1}\right)$.

To specify an analogue machine which, for $j<J$ and $j \in A$ will output a non zero approx inmate value for $K(j)$ one will hare to specify a value $B$ sur, to, replace $\infty$ as the upper limit of integration. Bat, by (5) abarne, wo will ten be able to compute a bound for $\beta(J)$ from $B 2$ And, because of the cut off factor $r_{g}$ : (5) show's tat $K(j)$ will he small of onder " $(j)^{-1}$ Da Costa ad Dovia propose switching from one dyumical system to another, according to whet hr $K(j)=0 \underset{\text { or }}{\text { or }} k(j)>0$. An analogue machine which will, effect $A$ switching will thais require, for the CVQ corresponding $f K(j)$ a precinia nation of order $\beta(J)$ it it hues, in all, there are three different factors in Ac specifical of the proposed mach mine which require an knowledge of abound for $\beta(J)$.

Q27 Quantum Nedonical machines
T-1 Both my examples depend on specifying a self-adjoint qurater $T$ for, sods, Hilbert space (eg. specifying te Homiltoncion for sone quantum-mechenceal systonn) aol making arserrations on its spectrum $t$ seth ? $j \in A$ ?.

The first example in due $\&$ Pour - $E L$ ad Richards (CAP PP 190-191). They show that a certain $T$ may be constructed as a computille limit of a sequence of compatible operates $T_{n}$ with $t=$ following properties.
(1) Let $\lambda_{j}(\dot{j} \geqslant 0)$ be a computable bended sequence of real numbers. Then if $j \notin A$ the spectrum of $\tau$ hes $\lambda_{j}$ as an eigen value (cornsponding $t$ a cone in spectioncoinic terms), while if $j \in A$ the spectrum has a contimerous hand of width $2.2^{-\nu(j)}$ combed on $\lambda_{j}$. The factor $2^{-\nu(j)}$ ensusac. mat the sequeria $T_{n}$ has a compatible modules of convergence. To make observation easy one mould take

$$
\lambda_{j}=5-4.2^{-j}
$$

ad then thine will he a gap between the bonds (if present) around $\lambda_{j}$ ad $\lambda_{j+1}$
To separate the lines or bands around $\lambda_{j}$ ad $\lambda_{j+1}$ one only needs a precincin 7 he order $2^{j}$; but to distinguish between a line at $\lambda_{j}$ a a a bend around $\lambda_{j}$ one needs a preizien of order $2^{r(j)}$. Thess
as on A previous examples. $t$ sett le ? $j \in A$ ? correctly fo $j<J$ one needs $t$ know a bound on $\geqslant(J)$ in coder to ensure tat Au measurements made will here the required precision. Another justificath for my claim in Ais example is best ill astrutul by another example, which is a sirmplifieats of one gives in Candy (1991). Namely Let the sequence $\left\{\lambda_{n}\right\}$ be defoid by

$$
\lambda_{n}=2^{-a(n)}
$$

one let $S$ be compact operator with the se values of $\lambda_{n}$ as its eigenvalues. To decile ? j $\in A$ ? it is only necessong to obserne, with say, a precision $2^{j+1}$, whether or net thence in a line at $2^{-j}$. (of course, a physical spectroscopy wat one breves in transitions from one $\lambda$ to anolor, but An doers at effect the argument.) So the question he comes: could mi design guantun mechanical device arid would have, for some observable, an approxionstion S' to $S$ whose eigenvalues for $j<J$ would he close te those of S? It will be recalled Ant a design must allow one to compute appnoxinoate values for all nelevent parameters air muspecify allowed tolerances. I do net know, except in particular cases lithe atomecic ad molecular spectra, how one might constnect a system which would approxionat a given poerater for a given observable. But it is
obvious, for both $S$ and $T$, that one woald need te know, at least ap pure innately, the entries in the first $p(J)$ rows of their representing matrices (cent some chosen or Phonormal basis). But Pis jasti/ies the claire?.
1 Both $S$ and $T$ ane 'effective by determiñed' operators. The interest of Pis concept lies not in examples like those gain above. but in the fact that the authors can (with considerable labour) give a general characterization, in terms of computabith. for the spectra of search operates.

The wave function
7.2 for a quantum mechanical system may result prom the superposition of infinitely many move adsily defined wave functions ad so correspond to the parallel working of infincetaly many sepereste machines. Thin suggests a possible method for designing a quantum - mechanical device whir would give correct answers to the questions? $j \in A$ ? However the quantum computer descrichol by Deatsch (1985) canned do Phi, although it can use superposition greatly to reduce te run time for certain decidable problems.
6.7.3 Refinements ion experimental technique allow one to build analogue machines whose behaviour depends on a single quantum (egg,, a single photon). Experiments
with such devices conform the of len counter-intuitine predictions of standard quantum Ronny. Could they provide a disprait of my claim? I co not know of any er ample for this.

3 Discussion
8.1 When one shows Ant a given number. beoretie function is computable, or Rat es giver number. Now lie pwillim is decidable ane does not place bounds on the un leone or the size of He memory unless, of course, one is concerned with problems of con laxity. That in, one is not concerned with precision ratios. So it may bock s I have placed anfoer restrictions on analogue machines Bat sappose one has proved that a certion programme will give correct answers to a problem? $j \in X$ ?. Then, given $J$, one can compute bounds on the tine and spar required to sect le? se $^{\prime} X$ ? carnet th for all $j<J$. But thin is exactly when t I claim cannot be done for analogue machines intended "to settle a non-decidable problems.

8.2 Cascades of events ad chain reactions allow one (as on a photon-multiplier) greatly to amplify the scale of an event: This is, in effect, a reduction of precision ratios. Could thin he used to overcome the objections rained by mg claion? The answer in 'No', because only when one knows a brand for $\beta(I)$ con one determine how much amplification is needed. 8,3 InICAR AXP OTher examples are gavin of differential equations (in particular A wave equation) which will give a nox-computible output for a computable input. The clair can he justified for these using the ideas of $\& 5$. 8.4. Kreisel has discussed calculation by analogue machines in a number of places; see. on porciculor $\operatorname{lin}(1974),(1982)$ and (199). Some of his comments and and loges ane illuminating, ad have helped me in getting my ileas straight. Bet one of hin points is that there are more interesting, more sessile at move relevert questions to ask then the (logical) question with which 2 arm hone concerned. 8.5. Pen rose, $h$ has angered (1994), brain can bo thought of as an analogue machine Which can, ir principle, settle undecidable. problems. Firstly, he belienes that mathematical reacts which can, at least in principle, be produced by human intelligence, cannot, even in principle, be produced by artificial in te Uigence - Nat in by some fired
programme. $P$. Note that $P$ inced noses bel
itself directly responsible for the motematical statements which the machine outputs. $P$ may be like an operating system; for example it may, by a process similar to natural selection, use mutations and tests of fires to direct the (contioncial) evolution of sulprognammes for doing mathematics. But this possibility does not, straightforwardly, invalidate Penvare's argument justifying his belief. A concise version of Penrose's argument is given on Goudy (1994). secondly Pennose believes that the sentences uttered or written by pegole are caused by physical ad chemical events in their brains.

To allow for non-alogorithmic actions in the Grain, penvose postulates a

- not yet completely formulated - future Bheory which he calls CQG (for Correct Quontom Gravitg). This will have consequences bol for cosmology (concerning the direction of time's arrow) and for quantum theory (accounting for the collapse of real (not subjective) wave functions.) He suggests ways on which such a thong may allow for Be growth of mieroseopie structures (such as quasicrystals, synapses and müro. (ubules in neuroses) on ways which ane not wally determined nor computable. It seems worth while to consider (rather naively) such patterns of growth from a mathematical point of view.

29. Patterns of growth I consider ai pattern of possible grouse on as being di played on a tree. At each node $p$ there is a finite label which represents a particular structure $S_{p}$ at a particular stage Of growth - for example, a particular quasi-crystal. If thin structure $S_{p}$ is capable of
growth then theme will be a finite numalor of nodes $P_{1}, \ldots, P_{k}$ immediately below $P$; each of the structures $S_{p}, \ldots, S_{P_{R}}$ arioes from $S_{p}$ by a single step of grown (for example, by the addition of a songle molecule). Two distend stmetures $S_{P}$ at $S_{Q}$ may, on one step, quod int te sane struet ire Hence, a noddle may hove two different immediate predecessors; these trees one net the save thou e standardly used in recursion theory. A node pad the corresponding structure $S_{p}$ are fertile if there is an inflict saith through P. If $P$ is net fertile then, howeour $S_{p}$ may oprow, after a fri it nombr of steps it will become a structure which cen grow no mane.

Now me suppose that the label representing any structure $S$ is (coded $b y$ ) a finite seginsice $u$ ग. O's al I's. We may suppose that the significant features of $s$ can be computed from $u$. An infinite path gules an infinite sequence $u_{1}, u_{2}, \cdots$, of binorg sequences. We define the growth function $Y$ along the path by $\gamma\left(u_{n}\right)=u_{n+1}$. If the sequence is computable then so is $r$; in porticulor three is a Tuning machine M which, when presented with $u_{n}$ on its tope, will, eventually, replace it by $u_{n+1}$. Now the action of MAs certainly locally determined; it will, for excemple, in general, inspect each of the digits in $u_{n}$. We shall say that $r$ (ad the infinite sequerce) are potentially locally determined.
9.1. Suppore we are given a thee of structunes and a growth function $r$ which satiffries It following conditions: -
(i) If $u$ codes a fertile structurne/ithm $r(a)$ codes a fertile structure int which S can gnow in a single step.
(ii) The furection $r$ is not pstentially toraltn determined.

Then, starting from ano fertite struilone $S$ ad iterating $r$ will prodice a nou-cimpulalle poid infinite sequence of stnuctures If one could excamine. saly, the furst $J$ structares in this sequence one could compet the first J values of some non-computalle fundion. The precerion rotio of ofseroutien has to be suffictingty lange to enalle one 6 delerminn $t_{e}$ codes u for these $J$ structerres; it mighl well 1.c a compulable fandion of $J$.
9.2 Since quasi-cngstals have been bhrenged Whide contain a bery lange number of midecules, Pearose sagjests that their gnouste a net a mattr of chunce, fut in goverined by sorse - as get anformulalel - lavss of non-local altiono. If, further, the theery invobexed deltions which were not even pitentially locally determisned, then it would allow anabogue machisnes $t \frac{p}{}$ podece non-recursine functions. One would nd expect The theory to be totally deterninictic; कndeed it a is plausivle Aot thene are at least two distinct
infinite paths through any fertile point of the tree, and hence continuum many such. AL (hough each path yields a nou-computable function, one carnot use it to settle c a specified condecidalle problem.

But for the growth of microstructares in the Grain, which determine how neurons behave ad how key affect each other one maraud expect that certain particular path's would he selected on would be permitted..
9.3 A.) The definition of "potentially locally determined' can be made quite general by considering, in place of the Turing machicie $M$, an mechasum which setinfies A principles of $G$ andy $(1980)$ - in particaler, I carse, the principle of 'bal causation'. And Aten one has a converse 69,2 - if the grow function along an infinite path is potentially locally determined. Hen the sequence of structures along it is computable.
49.4 It is well known fats there are $\underset{\underset{\sim}{*} \text { binary tres whose nodes form a recursive }}{\underset{\sim}{*}}$ set, which here e inforit paths but no computable infinite paths; using Rinitfuct one, can for example describe a sunset of tiles which can tile $A_{e}$ whole plane, bet t only in a non-computable, way, Using A notion of 'trial ad evror predicates (see Putnam (1965)) we can see how the leftermost infiririte) spa'ch it say, night be ugrourn. A node is specified by a 'finite binary sequence u which descriches (with O for 'Left' and 1. for 'Right') the pall from He vertex leading
ts it, and we consider uetste as the structure standing at $U$. The size of this is just the length of $U$. Now we define a 23
computable sequence $u_{n}$ of nodes on the
tree as follows.
(i) $u_{0}=$ ( ) (The vertere of the tree)
(i) If $u_{n}$ is not terminal (has nodes of Pe tree below it) then

$$
u_{n+1}=u_{n} 0
$$

(iii) Suppose $u_{n}$ is terminal and has the form $v O$ or vol...
then

$$
u_{n+1}=v 1
$$

Since no noddle on $\lambda$ is terminal, none of.
 mode $v$ which lies to the left of $\lambda(\operatorname{eg} ; 10$ if $\lambda(1)=1$ $\lambda(2)=1)$ thence can only be finitely merry nodes of Pe tree (since $V$ cannot be fertile). Hence for some $n$ we most hares $u_{n}$ lying to the right of $v$. Thus for $m$ g $J$ there will be an $n$ such nat $u_{n_{J}}=\lambda(1), \lambda(2), \ldots \lambda(J-1)$.
9.5 Att first sight it might lark as if inn $^{j}$ prows of trial an error growth could be accomodited in some reasonable physical theory. Bat A en is an illusion; for not only is $n \mathrm{~J}$ nit computable from $J$, but then can bu e $\rightarrow$ no computable bound' on the lengths of De sequencesilu with $n<n_{j}$ whin have to he explored before $u_{n} J$ is arrived at. And so the processin conseitered is analogous te a trial ad error process for deciding if if $\in A(\operatorname{ssi} 33)$ - one simply woks ahead to see of, for some $n, a(n)=j$.
9.6 Penrose suggests that in a theory If quantum gravity file process of growth mould he represented by a superposition of wave functions each corruponding t a particular pattern of grow the ad pert the effect of gravity wrulel be to collapse the wave function, so font only constituents cornerponding $t$ patterns of growth capable of proilucing lange structures would survive. To picture. An process i on The binary tree let the potential size, $\Pi(v)$ I a node $v$ be te mace imus beng th of all snodess $u$ eerectending (or lye cig below) $v^{\prime}$. If $v$ is fertile we set $\pi(v)=\infty$. Then the proposed theory would ensure that avar permitted vertex would gaia to some node of great size., though (in Ae simple form in which I "h ore stated it) it word et not guarantee grow $h$ along an infinite pith. It could well be that fir a give. J those would be a ky such Hast ony node of size grate ton ky world agree musth $\lambda$ at the first $J$ places. But then fact will not allow us le compute vi lues If $\lambda$ from sbsevvaluens on large structures which have deevlepod, unless we know some (necessarily nom-conpatalle) bounds for $k J$. If a theory of gnewith If the bind considered is to stand up against our claim it looks as if some kind of non-computability mast be bieilt int to tHeory - for excomple into th way in which gravity determines Ae collapse of wave functions.

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[^0]:    ${ }^{\dagger}$ Typeset by Aran Nayebi on August 27, 2013. A.N. is grateful to S. Barry Cooper and Philip Welch for providing a photocopy of Gandy's original handwritten manuscript (attached at the end of this document), as well as Solomon Feferman for suggesting to typeset it and his support.

[^1]:    ${ }^{1}$ When A.M. Turing was building his speech encoder ('Delilah') he found that if it was to work, some of the components had to have a tighter than usual tolerance on their values; these were more expensive than the standard components and - at least in the case of resistances - had a gold spot to indicate that they were accurate to within (I think) $1 \%$.

[^2]:    ${ }^{2}$ Pour-El in her (1974) gives a definition (based on differential analysers) of 'General Purpose Analogue Computers' and characterizes the class of continuous functions which they can generate. She is not concerned with questions of precision, but I believe that the methods used in $\S 5$ and $\S 6$ can be applied to justify my claim for all machines of the type she considers.

[^3]:    ${ }^{3}$ Even if different PR's were used for $y_{1}, \ldots, y_{k}$ I believe the claim would stand: for the $m_{1}$ in (3.7) codes a computation sequence, so its size will certainly increase with $\nu(j)$.

[^4]:    ${ }^{4}$ Both $S$ and $T$ are 'effectively determined' operators. The interest of this concept lies not in examples like those given above but in the fact that the authors can (with considerable labour) give a general characterization, in terms of computability, for the spectra of such operators.

