# A PBR-like argument for $\psi$-ontology in terms of protective measurements

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**Abstract**

The ontological status of the wave function in quantum mechanics has been analyzed in the context of conventional projective measurements. These analyses are usually based on some nontrivial assumptions, e.g. a preparation independence assumption is needed to prove the PBR theorem. In this paper, we give a PBR-like argument for $\psi$-ontology in terms of protective measurements, by which one can directly measure the expectation values of observables on a single quantum system. The proof does not resort to nontrivial assumptions such as preparation independence assumption.

The physical meaning of the wave function has been a hot topic of debate in the foundations of quantum mechanics. A long-standing question is whether the wave function relates only to an ensemble of identically prepared systems or directly to the state of a single system (Harrigan and Spekkens 2010). Recently, Pusey, Barrett and Rudolph demonstrated that under certain assumptions including a preparation independence assumption, the wave function is a representation of the physical state of a single quantum system (Pusey, Barrett and Rudolph 2012)[1]. This poses a further interesting question, namely whether the reality of the wave function can be argued without resorting to nontrivial assumptions (cf. Colbeck and Renner 2012; Lewis et al 2012; Leifer and Maroney 2013; Patra, Pironio and Massar 2013). In this paper, we will argue that protective measurements, by which one can directly measure the expectation values of observables on a single quantum system (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993), may provide such an argument.

[1]For more discussions about the Pusey-Barrett-Rudolph theorem or PBR theorem, see Schlosshauer and Fine (2012, 2013); Wallden (2013).

The ontological status of the wave function in quantum mechanics is usually analyzed in the context of projective measurements. Although the wave function of a quantum system is in general extended over space, one can only detect the system in a random position in space by a projective measurement of position, and the probability of detecting the system in the position is given by the modulus squared of the wave function there. Thus it seems reasonable for a realist to assume that the wave function does not refer directly to the physical state of the system but only relate to the state of an ensemble of identically prepared systems. Although there are several important theorems such as the PBR theorem which reject this epistemic view of the wave function, these theorems depend on some nontrivial assumptions. By denying these nontrivial assumptions, it seems that one can still restore the epistemic view of the wave function. Moreover, it has been demonstrated that additional assumptions are always necessary to rule out the epistemic view of the wave function when considering only conventional projective measurements (Lewis et al 2012).

Thanks to the important discoveries of Yakir Aharonov and Lev Vaidman et al, it has been known that there exist other kinds of quantum measurements such as weak measurements and protective measurements (Aharonov, Albert and Vaidman 1988; Aharonov and Vaidman 1990; Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993). In particular, by a series of protective measurements on a single quantum system, one may detect the system in all regions where its wave function extends and further measure the whole wave function of the system (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993). During a protective measurement, the measured state is protected by an appropriate mechanism, e.g. via the quantum Zeno effect or via natural protection from energy conservation for adiabatic measurements of non-degenerate energy eigenstates, so that it neither changes nor becomes entangled with the state of the measuring device appreciably. In this way, such protective measurements can measure the expectation values of observables on a single quantum system, even if the system is initially not in an eigenstate of the measured observable, and the wave function of the system can also be measured as expectation values of a sufficient number of observables.

By a conventional projective measurement on a single quantum system, one obtains one of the eigenvalues of the measured observable, and the expectation value of the observable can only be obtained as the statistical average of eigenvalues for an ensemble of identically prepared systems. Thus it seems surprising that a protective measurement can yield the expectation value of the measured observable directly from a single quantum system. In fact, the appearance of expectation values as measurement results is quite natural when the measured state is not changed and the entanglement during conventional measurements does not take place as for ideal protective measurements (Aharonov, Anandan and Vaidman 1993). In this case, the evolution of the combining state is

$$|\psi(0)\rangle\,|\phi(0)\rangle \rightarrow |\psi(t)\rangle\,|\phi(t)\rangle\,, t > 0 \tag{1}$$

where $|\psi\rangle$ denotes the state of the measured system and $|\phi\rangle$ the state of the measuring device, and $|\psi(t)\rangle$ is the same as $|\psi(0)\rangle$ up to a phase factor during the measurement interval $[0, \tau]$. The interaction Hamiltonian is given as usual by $H_I = g(t)PA$, where $P$ is the conjugate momentum of the pointer variable $X$ of the device, and the time-dependent coupling strength $g(t)$ is a smooth function normalized to $\int dtg(t) = 1$ during the measurement interval $\tau$, and $g(0) = g(\tau) = 0$. Then by Ehrenfest's theorem we have

$$\frac{d}{dt}\langle\psi(t)\phi(t)|X\,|\psi(t)\phi(t)\rangle = g(t)\langle\psi(0)|A\,|\psi(0)\rangle\,, \tag{2}$$

which further leads to

$$\langle\phi(\tau)|X\,|\phi(\tau)\rangle - \langle\phi(0)|X\,|\phi(0)\rangle = \langle\psi(0)|A\,|\psi(0)\rangle\,. \tag{3}$$

This means that the shift of the center of the pointer of the device gives the expectation value of the measured observable in the measured state.

That the wave function of a single prepared system can be measured by protective measurements can also be illustrated with a specific example (Aharonov and Vaidman 1993). Consider a quantum system in a discrete nondegenerate energy eigenstate $\psi(x)$. In this case, the measured system itself supplies the protection of the state due to energy conservation and no artificial protection is needed. We take the measured observable $A_n$ to be (normalized) projection operators on small spatial regions $V_n$ having volume $v_n$:

$$A_n = \begin{cases} \frac{1}{v_n}, & \text{if } x \in V_n, \\ 0, & \text{if } x \notin V_n. \end{cases} \tag{4}$$

An adiabatic measurement of $A_n$ then yields

$$\langle A_n\rangle = \frac{1}{v_n}\int_{V_n} |\psi(x)|^2 dv, \tag{5}$$

which is the average of the density $\rho(x) = |\psi(x)|^2$ over the small region $V_n$. Similarly, we can adiabatically measure another observable $B_n = \frac{\hbar}{2mi}(A_n\nabla + \nabla A_n)$. The measurement yields

$$\langle B_n\rangle = \frac{1}{v_n}\int_{V_n} \frac{\hbar}{2mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)dv = \frac{1}{v_n}\int_{V_n} j(x)dv. \tag{6}$$

This is the average value of the flux density $j(x)$ in the region $V_n$. Then when $v_n \rightarrow 0$ and after performing measurements in sufficiently many regions $V_n$ we can measure $\rho(x)$ and $j(x)$ everywhere in space. Since the wave function $\psi(x,t)$ can be uniquely expressed by $\rho(x,t)$ and $j(x,t)$ (except for an overall phase factor), the above protective measurements can obtain the wave function of the measured system.

Since the wave function can be measured from a single quantum system by a series of protective measurements, it seems natural to assume that the wave function refers directly to the physical state of the system. Several authors, including the discoverers of protective measurements, have given similar arguments supporting this implication of protective measurements for the ontological status of the wave function (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993; Anandan 1993; Dickson 1995; Gao 2013, 2014; Hetzroni and Rohrlich 2014). However, these analyses are also subject to some objections (Unruh 1994; Dass and Qureshi 1999; Schlosshauer and Claringbold 2014)[2]. It is still debatable whether protective measurements imply the reality of the wave function. In the following, we will give a new argument for $\psi$-ontology in terms of protective measurements along the line of reasoning of PBR theorem.

The PBR theorem is based on three assumptions (Pusey, Barrett and Rudolph 2012). The first one is that if a quantum system is prepared such that quantum theory assigns a pure state, then after preparation the system has a well defined set of physical properties, usually denoted by $\lambda$. This assumption is necessary for the analysis of the ontological status of the wave function, since if such physical properties don't exist, it is meaningless to ask whether or not the wave function describes them. The second assumption is called preparation independence assumption, which states that it is possible to prepare multiple systems such that their physical properties are uncorrelated. This assumption is nontrivial, and it has been replaced by certain seemingly weaker assumption in other relevant theorems (Colbeck and Renner 2012; Patra, Pironio and Massar 2013). The third assumption is that when a measurement is performed, the behaviour of the measuring device is only determined by the complete physical state of the system, along with the physical properties of the measuring device (see also Harrigan and Spekkens 2010). Our following arguments will be based on this generally accepted assumption.

For an ideal projective measurement $M$, this assumption means that the physical state or ontic state $\lambda$ of a system determines the probability $p(k|\lambda, M)$ of different outcomes $k$ for the measurement $M$ on the system. While for an ideal protective measurement, this assumption will mean that the ontic state $\lambda$ of a system determines the definite result of the protective measurement on the system. Based on this inference, we can give a simpler PBR-like argument for $\psi$-ontology in terms of protective measurements.

For two different quantum states such as two nonorthogonal states, select an observable whose expectation values in these two states are different.

[2]See Gao (2014) for a brief review of and answers to these objections.

Although these two states need different protection procedures, the ideal protective measurements of the observable on the two (protected) states such as adiabatic measurements are the same, and the results of the measurements will be different with certainty. If there exists a probability $p > 0$ that these two (protected) quantum states correspond to the same ontic state $\lambda$, then according to the above inference, the results of protective measurements of the above observable on these two states will be the same with probability $p > 0$. This leads to a contradiction. Therefore, the two (protected) quantum states correspond to different ontic states. This result is not surprising, since two (protected) quantum states of a single system can be distinguished with certainty by ideal protective measurements.

It can be further argued that a quantum state $\psi$, whether it is protected or not, corresponds to the same distribution of $\lambda$, though the ontic state of a single system may change when the system changes from an unprotected situation to a protected situation[3]. The reason is that a protected state and the original unprotected state yield the same probability of outcomes for any projective measurement. Thus, the above result also shows that two (unprotected) quantum states correspond to two ontic states. In other words, the quantum state represents the physical state of a single quantum system.

In fact, we can give a more direct argument for $\psi$-ontology in terms of protective measurements. As stated above, for an arbitrary (ideal) protective measurement, the ontic state $\lambda$ of a quantum system determines the definite result of the protective measurement on the system, namely the expectation value of the measured observable in the measured quantum state. Since the expectation values of a sufficient number of observables in a quantum state can uniquely determine the quantum state, the ontic state $\lambda$ of a system will uniquely determine the quantum state of the system. This proves $\psi$-ontology.

One may object that we should consider realistic protective measurements in the above arguments, while a realistic protective measurement can never be performed on a single quantum system with absolute certainty. For example, for a realistic protective measurement of an observable $A$ on a nondegenerate energy eigenstate whose measurement time $T$ is finite, there is always a tiny probability proportional to $1/T^2$ to obtain a different outcome $\langle A \rangle_{\perp}$, where $\perp$ refers to a normalized state in the subspace normal to the measured state as picked out by the first order perturbation theory. In this case, according to the above third assumption, the probability of different outcomes should be also determined by the ontic state of the measuring device and the measuring time, as well as by the ontic state of the measured

[3]For the cases of the measured state being a nondegenerate energy eigenstate, no artificial protection is needed, and there is no difference between an unprotected situation and a protected situation. Thus this result is obvious.

system[4].

However, on the one hand, although a realistic protective measurement with finite measurement time $T$ can never be performed on a single quantum system with certainty, the uncertainty can be made arbitrarily small when the measurement time $T$ approaches infinity. Concretely speaking, the probability distribution of different outcomes will approach a $\delta-$function localized in the expectation value of the measured observable in the measured state in the limit. On the other hand, when the measurement time $T$ approaches infinity, the probability of different outcomes will be determined only by the ontic state of the measured system. No matter what the ontic state of the measuring device is, the probability distribution of different outcomes will always approach a $\delta-$function depending only on the measured state of the system. Based on this result, we can give a similar PBR-like argument for $\psi$-ontology as above.

For two different quantum states, select an observable whose expectation values in these two states are different. Then the overlap of the probability distributions of the results of protective measurements of the observable on these two states can be arbitrarily close to zero (e.g. when the measurement time $T$ approaches infinity for adiabatic protective measurements). If there exists a non-zero probability $p$ that these two quantum states correspond to the same ontic state $\lambda$ in reality, then since the same $\lambda$ yields the same probability distribution of measurement results (when the measurement time $T$ approaches infinity) according to the above result, the overlap of the probability distributions of the results of protective measurements of the above observable on these two states will be not smaller than $p$. Since $p > 0$ is a *determinate* number, this leads to a contradiction.

Several comments are in order before concluding this paper. First of all, the above arguments in terms of protective measurements only consider a single quantum system, and thus avoid the preparation independence assumption for multiple systems used by the PBR theorem. Next, our argument in terms of ideal protective measurements does not depend on the origin of the Born probabilities. Unlike the PBR argument in terms of projective measurements, it is not necessary for our argument to assume that the probability of different outcomes of a projective measurement on a quantum system is determined by the ontic state of the system at the time of measurement (cf. Drezet 2014). Thirdly, it is worth noting that the principle of protective measurement does not depend on the solution to the measurement problem, and it only replies on the established parts of quantum mechanics, namely the linear Schrödinger dynamics and the Born rule. In fact, the above arguments only reply on the results of protective measure-

[4]Similarly, the probability of different outcomes of a realistic projective measurement will be also determined by the ontic state of the measuring device and the measuring time, as well as by the ontic state of the measured system. It will be interesting to see whether the PBR theorem can also be proved for realistic projective measurements.

ments in the limit case such as when the measurement time $T$ approaches infinity. Thus the arguments do not depend on the precise Born rule, for example, it will be enough if very small probability amplitude corresponds to very small probability of a measurement outcome.

Finally, we note that there might also exist other components of the underlying physical state, which are not measureable by protective measurements and not described by the wave function, e.g. the positions of the particles in the de Broglie-Bohm theory or Bohmian mechanics. In this case, according to our arguments, the wave function still represents the underlying physical state, though it is not a complete representation. Certainly, the wave function also plays an epistemic role by giving the probability distribution of measurement results according to the Born rule. However, this role will be secondary and determined by the complete quantum dynamics that describes the measurement process, e.g. the collapse dynamics in dynamical collapse theories.

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