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Two Axioms for Implication Algebras

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Abstract It is well-known that the implicational fragment of the classical propositional calculus has a single axiom. By contrast, here we show that the corresponding equational class defined by the implicational reduct of Boolean algebra cannot be defined by a single axiom. However, it can be defined by two identities. By a deep theorem of Alfred Tarski, it follows that this variety has an independent basis with n identities for all $n > 1$. Furthermore, it follows that no equational theory defined by any of the six well-known orthomodular implications is one-based.

The implicational fragment of the 2-element Boolean algebra is the class of all algebras of type $\langle 2 \rangle$ having a single binary operation $\rightarrow$ with the interpretation that $x \rightarrow y = x' \vee y$. Abbott [1] first defined these *implication algebras* by the following three identities:

$$(x \rightarrow y) \rightarrow x = x \tag{1}$$

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \tag{2}$$

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \tag{3}$$

In 1948, Łukasiewicz [4] proved that the implicational fragment of 2-valued logic is one-based, that is, it can be defined by a single axiom. The shortest single axiom is thanks to Tursman [8]:

$$i(i(i(x,y),z),i(i(z,x),i(u,x))).$$

It is natural to ask whether this is also possible for the implicational reduct of Boolean algebras. In Appendix 4 of Grätzer [2], W. Taylor claims D.H. Potts proved the following result in Potts [6], however, Potts' paper makes no mention of implication algebras and is focused entirely on semilattices.

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Theorem 1 *Implication algebras are not one-based.*

Proof If the variety of implication algebras has a single axiom, then it must be of the form $f(x_1, \ldots, x_n) = x_i$ for some $1 \leq i \leq n$. Otherwise, the zero algebra $x \to y = 0$ for all x, y will be a model and $(x \to y) \to x = x$ would never be a consequence. Here are some examples of such "absorption laws" valid in all implication algebras (generated by the program Prover9 by McCune [5]):

$$
\begin{aligned}
(((x \to y) \to z) \to z) \to x &= x \\
(x \to (y \to (z \to u))) \to z &= z \\
((x \to y) \to (z \to (((x \to u) \to v) \to (w \to y)))) \to v &= v \\
(((x \to (y \to z)) \to u) \to u) \to y &= y \\
(x \to (((y \to z) \to u) \to u)) \to y &= y \\
(x \to (y \to (z \to (u \to v)))) \to u &= u
\end{aligned}
$$

Notice that in all these examples, the last variable of the left side is exactly the same as the single variable on the right side. This is no accident. In fact, $f \to g = f' \vee g \geq g$, since the basic operation $x \to y = x' \vee y$ is always $\geq y$. We have, by induction, $f(x_1, \ldots, x_n) \geq x_n$. Hence if $f(x_1, \ldots, x_n) = x_i$ is an identity valid in all implication algebras then we must have that $i = n$. Now the right projection, $x \to y = y$ for all x and y, is always a model for such an identity. However, then (2) can never follow. $\square$

Corollary 1 *Let L be an orthomodular lattice and let $\to$ be any one of the six well-known orthomodular implications (see p. 239 in Kalmbach [3]). Then the equational theory of $(L; \to)$ has no single axiom.*

Proof Such a potential single axiom must be a consequence of the strongest Boolean implication, namely, $x \to y = x' \vee y$ and hence it must be of the form $f(x_1, \ldots, x_n) = x_n$. Thus $x \to y = y$ will be a model. In particular, no identity of the form $h = g$ where h and g have different last variables will be a consequence of $f(x_1, \ldots, x_n) = x_n$. However, in all such implication algebras, the join operation $x \vee y$ is a derived term, say $j(x, y)$. For example, $(x \to y) \to y = x \vee y$ in the Boolean case. However, $x \vee y = y \vee x$ is true in every OML. This means that the identity $j(x, y) = j(y, x)$ must be a consequence of the potential single axiom $f(x_1, \ldots, x_n) = x_n$, which is clearly impossible. $\square$

So, it is not possible to define implication algebras by a single identity, but in the following theorem we show that it is possible to reduce the number of axioms to two.

Theorem 2 *The identities* (1) *and*

$$x \to (y \to ((z \to u) \to u)) = y \to (x \to ((u \to z) \to z)) \tag{4}$$

are a basis for the equational theory of implication algebras.

Proof The following is based on an automated proof generated by the program Prover9, which is given in Appendix A. Setting $u = z$ in (4) yields

$$x \to (y \to ((z \to z) \to z) = y \to (x \to ((z \to z) \to z)$$

and since $(z \to z) \to z = z$ by (1), we may write the above as

$$x \to (y \to z) = y \to (x \to z)$$

which is (3). Now we show that (2) can be derived from (1), (3), and (4). We will make use of the fact that

$$\begin{aligned} x \to (x \to y) &= ((x \to y) \to x) \to (x \to y) \\ &= x \to y. \end{aligned} \tag{5}$$

First, apply (4) to obtain the following identity

$$\begin{aligned} &(((y \to x) \to x) \to z) \to (u \to ((x \to y) \to y)) \\ &= u \to ((((y \to x) \to x) \to z) \to ((y \to x) \to x)) \\ &= u \to ((y \to x) \to x). \end{aligned} \tag{6}$$

Then, using (3),

$$\begin{aligned} &(x \to y) \to ((((y \to x) \to x) \to z) \to y) \\ &= (((y \to x) \to x) \to z) \to ((x \to y) \to y) \\ &= (((y \to x) \to x) \to z) \to ((((x \to y) \to y) \to u) \to ((x \to y) \to y)) \\ &= (((x \to y) \to y) \to u) \to ((y \to x) \to x) \qquad \text{by (6)} \\ &= (y \to x) \to ((((x \to y) \to y) \to u) \to x). \end{aligned} \tag{7}$$

Setting $u = (((y \to x) \to x) \to z)$ in (6) and applying (5),

$$\begin{aligned} &(((y \to x) \to x) \to z) \to ((x \to y) \to y) \\ &= (((y \to x) \to x) \to z) \to ((y \to x) \to x) \\ &= (y \to x) \to x. \end{aligned}$$

Using (3), rewrite the left side of the above to obtain

$$(x \to y) \to ((((y \to x) \to x) \to z) \to y) = (y \to x) \to x. \tag{8}$$

Combining the above and (7), we have

$$(y \to x) \to x = (y \to x) \to ((((x \to y) \to y) \to u) \to x).$$

Finally, by applying (8),

$$(y \to x) \to ((((x \to y) \to y) \to u) \to x) = (x \to y) \to y$$

and so we have $(x \to y) \to y = (y \to x) \to x$. Thus (2) and (3) follow from (1) and (4) and the proof is complete. $\square$

Corollary 2 *By a theorem of Alfred Tarksi [7], the complete equational spectrum for this variety is $(2, \omega)$, i.e. for all $n \geq 2$, there exists an independent n basis for implication algebras.*

Appendix A

The following proof was discovered by the automated theorem prover Prover9. The program generated 787 identities, but only those necessary to the proof are displayed below. Note that **1** and **2** are (4) and (1) respectively. The first generated identity, **4**, is (3), while the final identity, **10**, is (2). Simplified justifications are given after each step.

1	$x \to (y \to ((z \to u) \to u)) = y \to (x \to ((u \to z) \to z)).$	[input]
2	$(x \to y) \to x = x.$	[input]
3	$(c2 \to c1) \to c1! = (c1 \to c2) \to c2.$	[input]
4	$x \to (y \to z) = y \to (x \to z).$	[2,1,2]
5	$(((x \to y) \to y) \to z) \to (u \to ((y \to x) \to x)) = u \to ((x \to y) \to y).$	[2,1]
6	$x \to (x \to y) = x \to y.$	[2,2]
7	$x \to (((x \to y) \to z) \to y) = x \to y.$	[4,2]
8	$(x \to y) \to ((((y \to x) \to x) \to z) \to y) = (y \to x) \to ((((x \to y) \to y) \to u) \to x).$	[2,5,4,4]
9	$(x \to y) \to ((((y \to x) \to x) \to z) \to y) = (y \to x) \to x.$	[5,6,4,7,4]
10	$(x \to y) \to y = (y \to x) \to x.$	[8,9,9]

References

[1] Abbott, J. C., "Implicational algebras," *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, vol. 11 (59) (1967), pp. 3–23 (1968). Zbl 0172.0200. MR MR0239958 (39:1312).

[2] Grätzer, G., *Universal algebra*, second edition, Springer-Verlag, New York, 1979. MR MR538623 (80g:08001).

[3] Kalmbach, G., *Orthomodular lattices*, volume 18 of *London Mathematical Society Monographs*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1983. MR MR716496 (85f:06012).

[4] Łukasiewicz, J., "The shortest axiom of the implicational calculus of propositions," *Proc. Roy. Irish Acad. Sect. A.*, vol. 52 (1948), pp. 25–33. MR MR0026017 (10,93c).

[5] McCune, W., "Prover9," , Version June-2006C.

[6] Potts, D. H., "Axioms for semi-lattices," *Canad. Math. Bull.*, vol. 8 (1965), p. 519. Zbl 0135.03101. MR MR0180511 (31:4746).

[7] Tarski, A., "Equational logic and equational theories of algebras," pp. 275–288 in *Contributions to Math. Logic (Colloquium, Hannover, 1966)*, North-Holland, Amsterdam, 1968. Zbl 0209.01402. MR MR0237410 (38:5692).

[8] Tursman, R., "The shortest axioms of the implicational calculus," *Notre Dame J. Formal Logic*, vol. 9 (1968), pp. 351–358. Zbl 0165.01001. MR MR0242634 (39:3964).

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