

A Completeness Theorem for a 3-Valued Semantics for a First-order Language

Christopher Gauker

July 20, 2015

This document presents a Gentzen-style deductive calculus and proves that it is complete with respect to a 3-valued semantics for a language with quantifiers. The semantics resembles the strong Kleene semantics with respect to conjunction, disjunction and negation. The completeness proof for the sentential fragment fills in the details of a proof sketched in Arnon Avron (2003) “Classical Gentzen-type Methods in Propositional Many-valued Logics” in *Beyond Two: Theory and Application of Multiple-Valued Logics*, M. Fitting and E. Orłowska, eds., pp. 117-155. Physica Verlag. The extension to quantifiers is original but uses standard techniques.

Sentential Logic

Let SL be a sentential language with connectives \neg , \rightarrow , \wedge , \vee and standard syntax. Let $GS3$ be a Gentzen-style deductive calculus with the following rules:

$$\text{(Basis)} \quad A \Rightarrow A \text{ i.e. } \frac{\emptyset}{A \Rightarrow A}$$

$$\text{(Weakening)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}$$

$$\text{(Cut)} \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\text{(\(\perp\)\(\Rightarrow\))} \quad \neg A, A \Rightarrow \text{ i.e. } \frac{\emptyset}{\neg A, A \Rightarrow \emptyset}$$

$$\text{(\(\neg\neg\)\(\Rightarrow\))} \quad \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}$$

$$\begin{array}{l}
(\Rightarrow \neg\neg) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \\
(\rightarrow\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \\
(\Rightarrow\rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \\
(\neg\rightarrow\Rightarrow) \quad \frac{A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \rightarrow B), \Gamma \Rightarrow \Delta} \\
(\Rightarrow\neg\rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \rightarrow B)} \\
(\wedge\Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \\
(\Rightarrow\wedge) \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \\
(\neg\wedge\Rightarrow) \quad \frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \\
(\Rightarrow\neg\wedge) \quad \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \\
(\vee\Rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \\
(\Rightarrow\vee) \quad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \\
(\neg\vee\Rightarrow) \quad \frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \\
(\Rightarrow\neg\vee) \quad \frac{\Gamma \Rightarrow \Delta, \neg A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}
\end{array}$$

NB: Arbitrary rearrangements of elements before “ \Rightarrow ” and arbitrary rearrangements of elements after “ \Rightarrow ” are allowed.

Definition. A *valuation* v is an assignment to atomic sentences of SL of members of $\{Y, I, N\}$ (“yes”, “indeterminate” and “no”).

Definition. V *extends* a valuation v to every sentence of SL iff for all sentences P of SL :

1. if P is atomic: $V(P) = v(P)$

2. if $P = \neg Q$, then: (a) $V(P) = Y$ if $V(Q) = N$,
(b) $V(P) = N$ if $V(Q) = Y$,
(c) $V(P) = I$ otherwise;
3. if $P = (Q \rightarrow R)$, then: (a) $V(P) = Y$ if $V(Q) \in \{I, N\}$ or $V(R) = Y$,
(b) $V(P) = N$ if $V(Q) = Y$ and $V(R) = N$,
(c) $V(P) = I$ if $V(Q) = Y$ and $V(R) = I$;
4. if $P = (Q \wedge R)$, then: (a) $V(P) = Y$ if $V(Q) = V(R) = Y$,
(b) $V(P) = N$ if $V(Q) = N$ or $V(R) = N$,
(c) $V(P) = I$ otherwise;
5. if $P = (Q \vee R)$, then: (a) $V(P) = Y$ if $V(Q) = Y$ or $V(R) = Y$,
(b) $V(P) = N$ if $V(Q) = V(R) = N$,
(c) $V(P) = I$ otherwise;

In tables:

\neg		\rightarrow	Y	I	N	\wedge	Y	I	N	\vee	Y	I	N
Y	N	Y	Y	I	N	Y	Y	I	N	Y	Y	Y	Y
I	I	I	Y	Y	Y	I	I	I	N	I	Y	I	I
N	Y	N	Y	Y	Y	N	N	N	N	N	Y	I	N

NB: This set of connectives is not functionally complete. That is, not all truth functions on $\{Y, I, N\}$ can be defined by means of them (Avron 2003, p. 219).

Definition. A *model* for a sequence $\Gamma \Rightarrow \Delta$ is a valuation v s.t. if V extends v , then for some $P \in \Gamma, V(P) \in \{I, N\}$ or for some $P \in \Delta, V(P) = Y$.

Definition. $\frac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *valid₃* iff for every valuation v , if v is a model of each of $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n$, then it is a model of $\Gamma \Rightarrow \Delta$.

Definition. $\Gamma \models_3 \Delta$ iff $\frac{\emptyset}{\Gamma \Rightarrow \Delta}$ is valid.

Definition. Where $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n\}$, an *S-cut* is an application of (Cut) in which $A \in \left(\bigcup_{i=1}^n \Gamma_i\right) \cup \left(\bigcup_{i=1}^n \Delta_i\right)$.

Definition. An *S-proof* of $\Gamma \Rightarrow \Delta$ from a set of sequences S is a proof in which every application of (Cut) is an *S-cut*.

Definition. Where $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n\}$, $\Gamma^* \Rightarrow \Delta^*$ is *S-saturated* iff:

1. there is no *S*-proof of $\Gamma^* \Rightarrow \Delta^*$;
2. if $A \in \left(\bigcup_{i=1}^n \Gamma_i\right) \cup \left(\bigcup_{i=1}^n \Delta_i\right)$ then $A \in \Gamma^* \cup \Delta^*$;
3. (a) if $\neg\neg A \in \Gamma^*$, then $A \in \Gamma^*$,
(b) if $\neg\neg A \in \Delta^*$, then $A \in \Delta^*$;
4. (a) if $A \rightarrow B \in \Gamma^*$, then $A \in \Delta^*$ or $B \in \Gamma^*$,
(b) if $A \rightarrow B \in \Delta^*$, then $A \in \Gamma^*$ and $B \in \Delta^*$,
(c) if $\neg(A \rightarrow B) \in \Gamma^*$, then $A \in \Gamma^*$ and $\neg B \in \Gamma^*$,
(d) if $\neg(A \rightarrow B) \in \Delta^*$, then $A \in \Delta^*$ or $\neg B \in \Delta^*$;
5. (a) if $A \wedge B \in \Gamma^*$, then $A \in \Gamma^*$ and $B \in \Gamma^*$,
(b) if $A \wedge B \in \Delta^*$, then $A \in \Delta^*$ or $B \in \Delta^*$,
(c) if $\neg(A \wedge B) \in \Gamma^*$, then $\neg A \in \Gamma^*$ or $\neg B \in \Gamma^*$,
(d) if $\neg(A \wedge B) \in \Delta^*$, then $\neg A \in \Delta^*$ and $\neg B \in \Delta^*$;
6. (a) if $A \vee B \in \Gamma^*$, then $A \in \Gamma^*$ or $B \in \Gamma^*$,
(b) if $A \vee B \in \Delta^*$, then $A \in \Delta^*$ and $B \in \Delta^*$,
(c) if $\neg(A \vee B) \in \Gamma^*$, then $\neg A \in \Gamma^*$ and $\neg B \in \Gamma^*$,
(d) if $\neg(A \vee B) \in \Delta^*$, then $\neg A \in \Delta^*$ or $\neg B \in \Delta^*$.

NB: If $\Gamma^* \Rightarrow \Delta^*$ is S -saturated, then membership in Γ^* behaves like Y and membership in Δ^* behaves like I or N .

Let $s_1, s_2, \dots, s_m, \dots$ be a list of all formulas that are either subformulas or negations of subformulas in $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right) \right)$.

Construct $\Gamma^* \Rightarrow \Delta^*$ thus:

Let $\Gamma^0 = \Gamma, \Delta^0 = \Delta$.

For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}, \Delta^{i+1} = \Delta^i$, if there is no S -proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.

For all $i \geq 0$, let $\Gamma^{i+1} = \Gamma^i, \Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, if there is an S -proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$.

Let $\Gamma^* = \bigcup_{i=1}^{\infty} \Gamma^i$ and $\Delta^* = \bigcup_{i=1}^{\infty} \Delta^i$.

Observation 1: $A \in \Gamma^* \cup \Delta^*$ iff A is a subformula or a negation of a subformula of a formula in $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right) \right)$.

Lemma 1. *Suppose there is no S -proof of $\Gamma \Rightarrow \Delta$. Then:*

(i) *For each $i \geq 0$, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ has no S -proof.*

(ii) *There is no S -proof of $\Gamma^* \Rightarrow \Delta^*$.*

(iii) *Maximality: Let Γ_0 and Δ_0 be sets consisting of formulas that are either subformulas or negations of subformulas of formulas in $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right) \right)$. If $\Gamma_0 \not\subseteq \Gamma^*$ or $\Delta_0 \not\subseteq \Delta^*$, then $\Gamma_0, \Gamma^* \Rightarrow \Delta^*, \Delta_0$ has an S -proof.*

Proof. (i) By induction:

Basis: By assumption, $\Gamma^0 \Rightarrow \Delta^0$ has no S -proof.

Induction hypothesis: Suppose $\Gamma^i \Rightarrow \Delta^i$ has no S -proof.

Induction step: By the construction, either $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$ or $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$. If $\Gamma^{i+1} = \Gamma^i \cup \{s_{i+1}\}$, then, by the construction, $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$ has no S -proof. If $\Delta^{i+1} = \Delta^i \cup \{s_{i+1}\}$, then there is an S -proof of $\Gamma^i, s_{i+1} \Rightarrow \Delta^i$. Suppose $\Gamma^{i+1} \Rightarrow \Delta^{i+1}$, i.e. $\Gamma^i \Rightarrow \Delta^i, s_{i+1}$, has an S -proof. Then by (Cut), $\Gamma^i \Rightarrow \Delta^i$ has an S -proof, contrary to assumption.

(ii) Suppose there is an S -proof of $\Gamma^* \Rightarrow \Delta^*$. Since proofs are finite, there is an i such that $\Gamma_i \Rightarrow \Delta_i$ (in our construction) has an S -proof, contrary to (i).

(iii) Let Γ_0 and Δ_0 be as described. *Case (a):* $\Gamma_0 \not\subseteq \Gamma^*$. There is $\Gamma'_0 \subseteq \Gamma_0$ such that $\Gamma'_0 \neq \emptyset$ and $\Gamma'_0 \cap \Gamma^* = \emptyset$. By the construction, $\Gamma'_0 \subseteq \Delta^*$. So by (Weakening), there is an S -proof of $\Gamma^0, \Gamma^* \Rightarrow \Delta^*, \Delta^0$. *Case (b):* $\Delta_0 \not\subseteq \Delta^*$. Similarly. \square

Lemma 2. *If $\Gamma \Rightarrow \Delta$ has no S -proof, then $\Gamma^* \Rightarrow \Delta^*$ (as in the construction) is S -saturated.*

Proof. Suppose $\Gamma \Rightarrow \Delta$ has no S -proof.

Condition (1) in the definition of S -saturated: By Lemma 1(ii).

Condition (2) in the definition of S -saturated: Suppose, for a reductio, that $A \in \left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right)$ but $A \notin \Gamma^* \cup \Delta^*$. By Lemma 1, maximality, $A, \Gamma^* \Rightarrow \Delta^*$ and $\Gamma^* \Rightarrow \Delta^*, A$ have S -proofs. But $\frac{\Gamma^* \Rightarrow \Delta^*, A \quad A, \Gamma^* \Rightarrow \Delta^*}{\Gamma^* \Rightarrow \Delta^*}$ is an application of (Cut) to a member of $\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right)$. So $\Gamma^* \Rightarrow \Delta^*$ has an S -proof, contrary to Lemma 1(ii).

Condition (3): (a) Suppose $\neg\neg A \in \Gamma^*$. $\frac{A, \Gamma^* \Rightarrow \Delta^*}{\neg\neg A, \Gamma^* \Rightarrow \Delta^*}$ is an application of $(\neg\neg \Rightarrow)$. But $\neg\neg A, \Gamma^* = \Gamma^*$. So $\Gamma^* \Rightarrow \Delta^*$ has an S -proof if $A, \Gamma^* \Rightarrow \Delta^*$ has one. So $A, \Gamma^* \Rightarrow \Delta^*$ has no S -proof. So $A \notin \Delta^*$. So by the construction (and Observation 1), $A \in \Gamma^*$. (b) Suppose $\neg\neg A \in \Delta^*$. By $(\Rightarrow \neg\neg)$, $\Gamma^* \Rightarrow \Delta^*$ has an S -proof if $\Gamma^* \Rightarrow \Delta^*, A$ has one. So $\Gamma^* \Rightarrow \Delta^*, A$ has none. So $A \notin \Gamma^*$. So $A \in \Delta^*$.

Condition (4): (a) Suppose $A \rightarrow B \in \Gamma^*$. $\frac{\Gamma^* \Rightarrow \Delta^*, A \quad B, \Gamma^* \Rightarrow \Delta^*}{A \rightarrow B, \Gamma^* \Rightarrow \Delta^*}$ is an application of $(\rightarrow \Rightarrow)$. But $A \rightarrow B, \Gamma^* = \Gamma^*$. Since $\Gamma^* \Rightarrow \Delta^*$ lacks an S -proof, either (i) $\Gamma^* \Rightarrow \Delta^*, A$ has no S -proof, or (ii) $B, \Gamma^* \Rightarrow \Delta^*$ has no S -proof. Suppose (i). $A \notin \Gamma^*$. So by the construction, $A \in \Delta^*$. Suppose (ii). $B \notin \Delta^*$. So $B \in \Gamma^*$. (b) Suppose $A \rightarrow B \in \Delta^*$. $\frac{A, \Gamma^* \Rightarrow \Delta^*, B}{\Gamma^* \Rightarrow \Delta^*, A \rightarrow B}$ is an application of $(\Rightarrow \rightarrow)$. But $\Delta^*, A \rightarrow B = \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S -proof $\Gamma^*, A \Rightarrow \Delta^*, B$ has no S -proof. So $A \notin \Delta^*, B \notin \Gamma^*$, which means $A \in \Gamma^*$ and $B \in \Delta^*$. (c) Suppose $\neg(A \rightarrow B) \in \Gamma^*$. $\frac{A, \neg B, \Gamma^* \Rightarrow \Delta^*}{\neg(A \rightarrow B), \Gamma^* \Rightarrow \Delta^*}$ is an application of $(\neg \rightarrow \Rightarrow)$. But $\neg(A \rightarrow B), \Gamma^* = \Gamma^*$. So since $\Gamma^* \Rightarrow \Delta^*$ has no S -proof, $A, \neg B, \Gamma^* \Rightarrow \Delta^*$ has no S -proof. So $A \notin \Delta^*, \neg B \notin \Delta^*$, so $A \in \Gamma^*, \neg B \in \Gamma^*$. (d) Suppose $\neg(A \rightarrow B) \in \Delta^*$. $\frac{\Gamma^* \Rightarrow \Delta^*, A \quad \Gamma^* \Rightarrow \Delta^*, \neg B}{\Gamma^* \Rightarrow \Delta^*, \neg(A \rightarrow B)}$ is an application of $(\Rightarrow \neg \rightarrow)$. But $\Delta^*, \neg(A \rightarrow B) = \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S -proof, either (i) $\Gamma^* \Rightarrow \Delta^*, A$ has no S -proof, or (ii) $\Gamma^* \Rightarrow \Delta^*, \neg B$ has no S -proof. So either $A \notin \Gamma^*$ or $\neg B \notin \Gamma^*$. So either $A \in \Delta^*$ or $\neg B \in \Delta^*$.

Conditions (5) and (6): Similarly. \square

Lemma 3. *If $\Gamma^* \Rightarrow \Delta^*$, constructed from $\Gamma \Rightarrow \Delta$, is S -saturated, then there is a valuation that is a model of every sequence in S , but not a model of $\Gamma \Rightarrow \Delta$.*

Proof. Suppose $\Gamma^* \Rightarrow \Delta^*$ is S -saturated. Define valuation v as follows: For all

$$\text{atomic formulas } P \text{ of } SL, v(P) = \begin{cases} Y \text{ if } P \in \Gamma^* \\ I \text{ if } P \notin \Gamma^* \text{ and } \neg P \notin \Gamma^* \\ N \text{ if } \neg P \in \Gamma^* \end{cases} .$$

First step: v is well-defined: If $P \in \Gamma$ and $\neg P \in \Gamma^*$, then by (Weakening) $\frac{P, \neg P}{\Gamma^* \Rightarrow \Delta^*}$ will be an S -proof. Since $\Gamma^* \Rightarrow \Delta^*$ (by S -saturation) does not have an S -proof, either $P \notin \Gamma^*$ or $\neg P \notin \Gamma^*$. So the definition of v does not yield both $v(P) = Y$ and $v(P) = N$.

Second step: Suppose V extends v . Prove for all formulas P of SL , if $P \in \Gamma^*$ then $V(P) = Y$ and if $P \in \Delta^*$ then $V(P) \in \{I, N\}$.

By induction:

Basis: The thesis holds for all literals (atomic sentences and negations of atomic sentences): First, consider atomic P . (i) Suppose $P \in \Gamma^*$. $v(P) = V(P) = Y$. (ii) Suppose $P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S -proof, $P \notin \Gamma^*$. If $\neg P \notin \Gamma^*$, then $v(P) = V(P) = I \in \{I, N\}$. If $\neg P \in \Gamma^*$, then $v(P) = V(P) = N \in \{I, N\}$. Next, consider $\neg P$, where P is atomic. (i) Suppose $\neg P \in \Gamma^*$. $v(P) = V(P) = N$. $V(\neg P) = Y$. (ii) Suppose $\neg P \in \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ has no S -proof, $\neg P \notin \Gamma^*$. If $P \notin \Gamma^*$, then $v(P) = V(P) = I$ and $V(\neg P) = I \in \{I, N\}$. If $P \in \Gamma^*$, then $v(P) = V(P) = Y$ and $V(\neg P) = N \in \{I, N\}$.

Induction hypothesis: The thesis holds for $A, B, \neg A$ and $\neg B$.

Induction step: Show that it holds for $\neg\neg A, (A \rightarrow B), \neg(A \rightarrow B), (A \wedge B), \neg(A \wedge B), (A \vee B)$ and $\neg(A \vee B)$.

$(\neg\neg)$: Suppose $\neg\neg A \in \Gamma^*$. By the definition of S -saturation, $A \in \Gamma^*$. By IH, $V(A) = Y$. $V(\neg\neg A) = Y$. Suppose $\neg\neg A \in \Delta^*$. By the definition of S -saturation, $A \in \Delta^*$. By IH, $V(A) \in \{I, N\}$. $V(\neg\neg A) \in \{I, N\}$.

(\rightarrow) : Suppose $(A \rightarrow B) \in \Gamma^*$. By the definition of S -saturation, $A \in \Delta^*$ or $B \in \Gamma^*$. By IH, $V(A) \in \{I, N\}$, or $V(B) = Y$. $V(A \rightarrow B) = Y$. Suppose $(A \rightarrow B) \in \Delta^*$. By the definition of S -saturation, $A \in \Gamma^*$ and $B \in \Delta^*$. By IH, $V(A) = Y, V(B) \in \{I, N\}$. $V(A \rightarrow B) \in \{I, N\}$.

$(\neg \rightarrow)$ Suppose $\neg(A \rightarrow B) \in \Gamma^*$. By the definition of S -saturation, $A \in \Gamma^*$ and $\neg B \in \Gamma^*$. By IH, $V(A) = Y$ and $V(\neg B) = Y$, $V(\neg(A \rightarrow B)) = Y$. Suppose $\neg(A \rightarrow B) \in \Delta^*$. By the definition of S -saturation, $A \in \Delta^*$ or $\neg B \in \Delta^*$. By IH, $V(A) \in \{I, N\}$ or $V(\neg B) \in \{I, N\}$. $V(A) \in \{I, N\}$ or $V(B) \in \{Y, I\}$. $V(A \rightarrow B) \in \{Y, I\}$. $V(\neg(A \rightarrow B)) \in \{I, N\}$.

Cases $(\wedge), (\neg\wedge), (\vee), (\neg\vee)$ similarly.

Consequently, v is not a model of $\Gamma^* \Rightarrow \Delta^*$. So since (by Observation 1)

$\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$, v is not a model of $\Gamma \Rightarrow \Delta$.

Third step: Show that v is a model of every sequence in S . Let $\Gamma_i \Rightarrow \Delta_i$ be an arbitrary member of S . Show that v is a model of $\Gamma_i \Rightarrow \Delta_i$. Suppose, for reductio, that $\Gamma_i \subseteq \Gamma^*$ and $\Delta_i \subseteq \Delta^*$. In that case, by (Weakening), $\Gamma^* \Rightarrow \Delta^*$ has an S -proof, $\frac{\Gamma_i \Rightarrow \Delta_i}{\Gamma^* \Rightarrow \Delta^*}$, contrary to Lemma 1(ii). So, either $\Gamma_i \not\subseteq \Gamma^*$ or $\Delta_i \not\subseteq \Delta^*$. But by condition (2) in the definition of S -saturation (or by Observation 1), $\Gamma_i \cup \Delta_i \subseteq \Gamma^* \cup \Delta^*$. So either (i) there is $A \in \Gamma_i$ such that $A \in \Delta^*$, or (ii) there is $A \in \Delta_i$ such that $A \in \Gamma^*$. In case (i) $V(A) \in \{I, N\}$. So v is a model of $\Gamma_i \Rightarrow \Delta_i$. In case (ii) $V(A) = Y$. So v is a model of $\Gamma_i \Rightarrow \Delta_i$. \square

Completeness Theorem for GS3: *If $\frac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is valid₃, then, where $S = \{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n\}$, there is an S -proof of $\Gamma \Rightarrow \Delta$ in GM3.*

Proof. Suppose there is no S -proof of $\Gamma \Rightarrow \Delta$. Then, by Lemma 2, $\Gamma \Rightarrow \Delta$ can be extended to S -saturated $\Gamma^* \Rightarrow \Delta^*$. By Lemma 3, $\frac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not valid₃. \square

Corollary 1. *If $\Gamma \models_3 \Delta$ then there is a proof in GS3 of $\Gamma \Rightarrow \Delta$.*

Extension of these results to QL

Suppose that SL is now a language like SL , defined above, except that the atomic formulas are composed, by the usual syntax, from countably many predicates of each adicity and denumerably many variables and denumerably many individual constants. Let QL be a language containing, for each (individual) variable v , a quantifier $\forall v$. QL has the standard syntax, allowing vacuous quantification, and $\exists v$ abbreviates $\neg \forall v \neg$. In any sequence, $\Gamma \Rightarrow \Delta$, the members of $\Gamma \cup \Delta$ are sentences, not open formulas. Pn/v denotes the result of substituting n for v wherever v occurs free in P . If v is not in P , then $Pn/v = P$.

Let $GQ3$ be a Gentzen-style deductive calculus containing all of the rules of $GS3$ plus the following:

$$\begin{array}{l}
(\forall \Rightarrow) \quad \frac{Pn/v, \Gamma \Rightarrow \Delta}{\forall v P, \Gamma \Rightarrow \Delta} \\
(\Rightarrow \forall) \quad \frac{\Gamma \Rightarrow \Delta, Pn/v}{\Gamma \Rightarrow \Delta, \forall v P} \text{ where } n \text{ is not in } P \text{ and not in any member of } \Gamma \cup \Delta, \\
\text{i.e. } n \text{ is new, or } v \text{ is not in } P,
\end{array}$$

$$\begin{aligned}
(\neg\forall \Rightarrow) \quad & \frac{\neg Pn/v, \Gamma \Rightarrow \Delta}{\neg\forall v P, \Gamma \Rightarrow \Delta} \text{ where } n \text{ is not in } P \text{ and not in any member of } \Gamma \cup \Delta, \\
& \text{i.e. } n \text{ is new, or } v \text{ is not in } P, \\
(\Rightarrow \neg\forall) \quad & \frac{\Gamma \Rightarrow \Delta, \neg Pn/v}{\Gamma \Rightarrow \Delta, \neg\forall v P}
\end{aligned}$$

Define a *3-valued structure* \mathfrak{M} as a triple $\langle U, \Sigma^+, \Sigma^- \rangle$ where U , the universe, is a set of objects, and for each individual constant n , $\Sigma^+(n) = \Sigma^-(n) = \Sigma(n) \in U$. For each m -ary predicate R , $\Sigma^+(R) \subseteq U^m$, $\Sigma^-(R) \subseteq U^m$, and $\Sigma^+(R) \cap \Sigma^-(R) = \emptyset$.

Let a *structure and variable assignment* \mathfrak{M}_g be a quadruple $\langle U, \Sigma^+, \Sigma^-, g \rangle$ with U, Σ^+, Σ^- as before and g a partial function over some of the variables of QL such that for each variable v in the range of g : $g(v) \in U$.

$g[v/o]$ is a *variable assignment like* g except that $g[v/o]$ assigns o to v instead of whatever g assigned to v , if v is in the range of g , and otherwise assigns o to v , if v is in the range of g . g_\emptyset is the *empty variable assignment* with an empty range.

Associate with \mathfrak{M}_g the function h such that for each singular term t of QL that is either an individual constant of QL or a variable of QL in the range of

$$g, h(t) = \begin{cases} \Sigma(t) & \text{if } t \text{ is an individual constant,} \\ g(t) & \text{if } t \text{ is a variable.} \end{cases}$$

A *structure* $\mathfrak{M} = \mathfrak{M}_{g_\emptyset}$.

Associate with each structure and variable assignment \mathfrak{M}_g a function of the same name from formulas of QL into $\{Y, I, N\}$, as follows:

$$\mathfrak{M}_g(Rt_1t_2 \dots t_m) = Y \text{ iff } \langle h(t_1), h(t_2), \dots, h(t_m) \rangle \in \Sigma^+(R),$$

$$\mathfrak{M}_g(Rt_1t_2 \dots t_m) = N \text{ iff } \langle h(t_1), h(t_2), \dots, h(t_m) \rangle \in \Sigma^-(R),$$

$$\mathfrak{M}_g(Rt_1t_2 \dots t_m) = I \text{ otherwise,}$$

$$\mathfrak{M}_g(\neg P) = Y \text{ iff } \mathfrak{M}_g(P) = N,$$

$$\mathfrak{M}_g(\neg P) = N \text{ iff } \mathfrak{M}_g(P) = Y,$$

$$\mathfrak{M}_g(\neg P) = I \text{ otherwise,}$$

$$\mathfrak{M}_g(P \rightarrow Q) = Y \text{ iff } \mathfrak{M}_g(P) \in \{I, N\} \text{ or } \mathfrak{M}_g(Q) = Y,$$

$$\mathfrak{M}_g(P \rightarrow Q) = N \text{ iff } \mathfrak{M}_g(P) = Y \text{ and } \mathfrak{M}_g(Q) = N,$$

$$\mathfrak{M}_g(P \rightarrow Q) = I \text{ otherwise,}$$

$$\mathfrak{M}_g(\forall v Q) = Y \text{ iff for all } o \in U, \mathfrak{M}_{g[v/o]}(Q) = Y,$$

$$\mathfrak{M}_g(\forall v Q) = N \text{ iff for some } o \in U, \mathfrak{M}_{g[v/o]}(Q) = N,$$

$$\mathfrak{M}_g(\forall v Q) = I \text{ otherwise.}$$

Observation 2: If U is identical to the set of all individual constants of QL and for all singular constants n of QL , $\Sigma(n) = n$, then $\mathfrak{M}(\forall vQ) = Y$ iff for all n of QL , $\mathfrak{M}(Qn/v) = Y$, and $\mathfrak{M}(\forall vQ) = N$ iff for some n of QL , $\mathfrak{M}(Qn/v) = N$.

Observation 3: If v is not in Q , $\mathfrak{M}_g(\forall vQ) = Y$ iff $\mathfrak{M}_g(Q) = Y$ and $\mathfrak{M}_g(\forall vQ) = N$ iff $\mathfrak{M}_g(Q) = N$.

Definition. A structure \mathfrak{M} is a *model* for $\Gamma \Rightarrow \Delta$ iff either there is $P \in \Gamma$ such that $\mathfrak{M}(P) \in \{I, N\}$ or there is $P \in \Delta$ such that $\mathfrak{M}(P) = Y$.

Definition. $\frac{\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *valid*_{Q3} iff every structure that is a model for each of $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2, \dots, \Gamma_n \Rightarrow \Delta_n$ is a model for $\Gamma \Rightarrow \Delta$.

Let C be a denumerable set of individual constants not in QL . QL^+ is QL supplemented by the individual constants in C . By standard techniques, we associate with each formula P of QL^+ having exactly one free variable, two members of C , c_P^+ and c_P^- , called the *witnesses* for P , having the *same birth date*, such that for no formula Q of QL^+ whose witnesses have that same birth date or an earlier birth date does Q contain c_P^+ or c_P^- . c_P^+ is the *positive witness* for P and c_P^- is the *negative witness* for P .

Definition. The *Henkin set* \mathcal{H} for QL^+ is the set of sentences Q of QL^+ such that $Q \in \mathcal{H}$ iff

1. n is an individual constant of QL^+ and $Q = (\forall vP \rightarrow Pn/v)$ or $Q = (\neg Pn/v \rightarrow \neg \forall vP)$, or
2. v is not in P , and $Q = (P \rightarrow \forall vP)$ or $Q = (\neg \forall vP \rightarrow \neg P)$, or
3. c_P^+ is the positive witness for P and $Q = (Pc_P^+/v \rightarrow \forall vP)$, or
4. c_P^- is the negative witness for P and $Q = (\neg \forall vP \rightarrow \neg Pc_P^-/v)$.

Lemma 4. (a) $\frac{(A \rightarrow B), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$ is *provable*.

(b) $\frac{(A \rightarrow B), \Gamma \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}$ is *provable*.

Proof. (a)

$$\frac{\frac{A \Rightarrow A}{\Gamma, A \Rightarrow \Delta, B, A} \text{ (Weakening)}}{\Gamma \Rightarrow \Delta, A, (A \rightarrow B) (\Rightarrow \rightarrow)} \quad (A \rightarrow B), \Gamma \Rightarrow \Delta$$

$$\frac{}{\Gamma \Rightarrow \Delta} \text{ (Cut)}$$

(b) Similarly. □

Lemma 5. $\frac{\Gamma, (\forall vP \rightarrow Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ is provable.

$\frac{\Gamma, (Pn/v \rightarrow \forall vP) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ where n is new, or v is not in P , is provable.

$\frac{\Gamma, (\neg \forall vP \rightarrow \neg Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ where n is new, or v is not in P , is provable.

$\frac{\Gamma, (\neg Pn/v \rightarrow \neg \forall vP) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$ is provable.

Proof. By Lemma 4 and $(\forall \Rightarrow)$, $(\Rightarrow \forall)$, $(\neg \forall \Rightarrow)$ and $(\Rightarrow \neg \forall)$ respectively. For example:

$$\frac{\frac{\Gamma, (\neg \forall vP \rightarrow \neg Pn/v) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \forall vP \text{ (Lemma 4)}} \quad \frac{\frac{\Gamma, (\neg \forall vP \rightarrow \neg Pn/v) \Rightarrow \Delta}{\neg Pn/v, \Gamma \Rightarrow \Delta \text{ (Lemma 4)}}}{\neg \forall vP, \Gamma \Rightarrow \Delta \text{ (\neg \forall \Rightarrow)}}}{\Gamma \Rightarrow \Delta \text{ (Cut)}}$$

□

The Elimination Theorem Suppose every sentence in

$(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right) \right)$ is in QL . Suppose also that

$\frac{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}{\Gamma, \mathcal{H} \Rightarrow \Delta}$ is provable in $GQ3$ for QL^+ . Then

$\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is provable in $GQ3$ for QL .

Proof. Suppose the hypothesis. Since proofs are finite, there is a finite subset $\mathcal{J} \subseteq \mathcal{H}$ such that $\frac{\Gamma_1, \mathcal{J} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{J} \Rightarrow \Delta_n}{\Gamma, \mathcal{J} \Rightarrow \Delta}$ is provable. Show:

$\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is provable (in $GQ3$ for QL). By induction on the size of \mathcal{J} :

Basis: $\mathcal{J} = \emptyset$. Trivial.

Induction Hypothesis: Suppose the thesis holds when \mathcal{J} has m members ($m \geq 0$). Show that the thesis holds when \mathcal{J} has $m + 1$ members.

Case 1: At least one member Q of \mathcal{J} is of the form $(\forall vP \rightarrow Pn/v)$ or $(\neg Pn/v \rightarrow \neg \forall vP)$ or $(P \rightarrow \forall vP)$ or $(\neg \forall vP \rightarrow P)$. There is a set \mathcal{J}' such that $\mathcal{J} = \mathcal{J}' \cup \{Q\}$, $Q \notin \mathcal{J}'$. By the IH it suffices to show that if $\frac{\Gamma_1, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_n}{\Gamma, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta}$ is provable then

$\frac{\Gamma_1, \mathcal{J}' \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{J}' \Rightarrow \Delta_n}{\Gamma, \mathcal{J}' \Rightarrow \Delta}$ is provable, thus:

$$\frac{\frac{\Gamma_1, \mathcal{J}' \Rightarrow \Delta_1}{\Gamma_1, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_1 \text{ (Weakening)}} \cdots \frac{\Gamma_n, \mathcal{J}' \Rightarrow \Delta_n}{\Gamma_n, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta_n \text{ (Weakening)}}}{\Gamma, \mathcal{J}' \cup \{Q\} \Rightarrow \Delta \text{ (Supposition)}} \frac{}{\Gamma, \mathcal{J}' \Rightarrow \Delta \text{ (by Lemma 5)}}$$

Case 2: All members of \mathcal{J} are sentences of the form $(Pc_P^+/v \rightarrow \forall vP)$ or $(\neg \forall vP \rightarrow \neg Pc_P^-/v)$. Of all the witnesses in the sentences in \mathcal{J} (positive or negative), let c_P^* be one that has latest birth date. (There might be two, one positive, one negative.) c_P^* does not occur in any member of $(\Gamma \cup \Delta) \cup \left(\left(\bigcup_{i=1}^n \Gamma_i \right) \cup \left(\bigcup_{i=1}^n \Delta_i \right) \right)$ or in any other member of \mathcal{J} . Let Q be a sentence in \mathcal{J} containing c_P^* . So $Q = (Pc_P^*/v \rightarrow \forall vP)$ or $Q = (\neg \forall vP \rightarrow \neg Pc_P^*/v)$. As in Case 1, we can drop Q from the proof. \square

Definition. A *valuation* val for QL^+ is an assignment of the members of $\{Y, I, N\}$ to sentences of QL^* that are either quantified or atomic.

Definition. Val extends val to every sentence of QL^+ in accordance with tables given above for SL .

The Henkin Construction Theorem: *Suppose val is a valuation for QL^+ and Val extends val such that for all $Q \in \mathcal{H}$, $Val(Q) = Y$. Then we can construct a structure \mathfrak{M}_{Val} for QL^+ such that for all sentences P of QL^+ , $Val(P) = Y$ iff $\mathfrak{M}_{Val}(P) = Y$, and $Val(P) = N$ iff $\mathfrak{M}_{Val}(P) = N$ (By implication: $Val(P) = I$ iff $\mathfrak{M}_{Val}(P) = I$).*

Proof. Define \mathfrak{M}_{Val} thus:

U is identical to the set of individual constants of QL^+ .

For each individual constant n of QL^+ : $\Sigma(n) = n$.

For each m -place predicate R of $QL^+(QL)$:

$$\Sigma^+(R) = \{\langle n_1, n_2, \dots, n_m \rangle \mid val(Rn_1n_2 \dots n_m) = Y\},$$

$$\Sigma^-(R) = \{\langle n_1, n_2, \dots, n_m \rangle \mid val(Rn_1n_2 \dots n_m) = N\}.$$

By induction on the length of sentences:

Basis: Suppose P is atomic, i.e. $P = Rn_1n_2 \dots n_m$.

Left-to-right: Suppose $Val(Rn_1n_2 \dots n_m) = Y$. By the construction of \mathfrak{M}_{Val} , $\langle n_1, n_2, \dots, n_m \rangle \in \Sigma^+(R)$. By the construction of \mathfrak{M}_{Val} , $\langle \Sigma(n_1), \Sigma(n_2), \dots, \Sigma(n_m) \rangle \in \Sigma^+(R)$. So by the definition of \mathfrak{M}_g (as a function), $\mathfrak{M}_{Val}(Rn_1n_2 \dots n_m) = Y$. Suppose $Val(Rn_1n_2 \dots n_m) = N$. By the construction, $\langle n_1, n_2, \dots, n_m \rangle \in$

$\Sigma^-(R)$. By the construction, $\langle \Sigma(n_1), \Sigma(n_2), \dots, \Sigma(n_m) \rangle \in \Sigma^+(R)$. So $\mathfrak{M}_{Val}(Rn_1n_2\dots n_m) = N$.

Right-to-left: Suppose $\mathfrak{M}_{Val}(Rn_1n_2\dots n_m) = Y$. By the construction of \mathfrak{M}_{Val} , $\langle \Sigma(n_1), \Sigma(n_2), \dots, \Sigma(n_m) \rangle \in \Sigma^+(R)$. By the construction of \mathfrak{M}_{Val} , $\langle n_1, n_2, \dots, n_m \rangle \in \Sigma^+(R)$. $Val(Rn_1n_2\dots n_m) = Y$. Suppose $\mathfrak{M}_{Val}(Rn_1n_2\dots n_m) = N$. Similarly.

Induction hypotheses: Suppose the thesis holds for all sentences having complexity k . Show that it holds for all sentences having complexity $k+1$.

Induction step:

(\neg) : Exercise.

(\rightarrow) : Suppose $P = (Q \rightarrow R)$.

Left-to-right: Suppose $Val(Q \rightarrow R) = Y$. By the definition of Val , either $Val(Q) \in \{I, N\}$ or $Val(R) = Y$. By IH, either $\mathfrak{M}_{Val}(Q) \in \{I, N\}$ or $\mathfrak{M}_{Val}(R) = Y$. By the definition of \mathfrak{M}_g , $\mathfrak{M}_{Val}(Q \rightarrow R) = Y$. Suppose $Val(Q \rightarrow R) = N$. By the definition on Val , $Val(Q) = Y$ and $Val(R) = N$. By the definition of \mathfrak{M}_g , $\mathfrak{M}_{Val}(Q \rightarrow R) = N$.

Right-to-left: Exercise.

$(\wedge), (\vee)$: Exercise.

(\forall) : *Left-to-right:* Suppose $Val(\forall vQ) = Y$. By the definition of \mathcal{H} , for all individual constants n in QL^+ , $Val(\forall vQ \rightarrow Qn/v) = Y$. So by the definition of Val , for all individual constants n in QL^+ , $Val(Qn/v) = Y$. By the IH, for all individual constants n in QL^+ , $\mathfrak{M}_{Val}(Qn/v) = Y$. By Observation 2, $\mathfrak{M}_{Val}(\forall vQ) = Y$. Suppose $Val(\forall vQ) = N$. $Val(\neg\forall vQ) = Y$. *Case 1:* v is not in Q . Then by the construction of \mathcal{H} , $Val(\neg\forall vQ \rightarrow \neg Q) = Y$. By the definition of Val , $Val(\neg Q) = Y$. By IH, $\mathfrak{M}_{Val}(\neg Q) = Y$. By the definition of \mathfrak{M}_g , $\mathfrak{M}_{Val}(Q) = N$. By Observation 3, $\mathfrak{M}_{Val}(\forall vQ) = N$. *Case 2:* v is in Q . Then by the construction of \mathcal{H} , $Val(\neg\forall vQ \rightarrow \neg Qc_Q^-/v) = Y$. By the definition of Val , $Val(\neg Qc_Q^-/v) = Y$. By the IH, $\mathfrak{M}_{Val}(\neg Qc_Q^-/v) = Y$. By the definition of \mathfrak{M}_g , $\mathfrak{M}_{Val}(Qc_Q^-/v) = N$. By the definition of \mathfrak{M}_g , $\mathfrak{M}_{Val}(\forall vQ) = N$.

Right-to-left: Suppose $\mathfrak{M}_{Val}(\forall vQ) = Y$. *Case 1:* v is not in Q . By Observation 3, $\mathfrak{M}_{Val}(Q) = Y$. By IH, $Val(Q) = Y$. By the construction of \mathcal{H} , $Val(Q \rightarrow \forall vQ) = Y$. $Val(\forall vQ) = Y$. *Case 2:* v is in Q . By Observation 2, for all individual constants n in QL^+ , $\mathfrak{M}_{Val}(Qn/v) = Y$. In particular, $\mathfrak{M}_{Val}(Qc_Q^+/n) = Y$. By IH, $Val(Qc_Q^+/v) = Y$. By the construction of \mathcal{H} , $Val(Qc_Q^+/n \rightarrow \forall vQ) = Y$. $Val(\forall vQ) = Y$. Suppose $\mathfrak{M}_{Val}(\forall vQ) = N$. By Observation 2, there is an individual constant n of QL^+ such that $\mathfrak{M}_{Val}(Qn/v) = N$. By IH, $Val(Qn/v) = N$. $Val(\neg Qn/v) = Y$. By the construction of \mathcal{H} ,

$Val(\neg Qn/v \rightarrow \neg \forall vP) = Y$. $Val(\neg \forall vP) = Y$. $Val(\forall vP) = N$. □

Completeness Theorem for GQ3: If $\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is $valid_{Q3}$, then $\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is *provable in GQ3*.

Proof. Suppose that $\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not provable in GQ3. By the Elimination Theorem, $\frac{\Gamma \Rightarrow \Delta}{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}$ is also not provable in GQ3. So it is also not provable in GS3. By the Completeness Theorem for GS3, $\frac{\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n}{\Gamma, \mathcal{H} \Rightarrow \Delta}$ is not $valid_3$. So there is a valuation val such that val is a model for each of $\Gamma_1, \mathcal{H} \Rightarrow \Delta_1, \dots, \Gamma_n, \mathcal{H} \Rightarrow \Delta_n$, but not a model for $\Gamma, \mathcal{H} \Rightarrow \Delta$. Since val is not a model for $\Gamma, \mathcal{H} \Rightarrow \Delta$, for all $Q \in \mathcal{H}$, $Val(Q) = Y$. By the Henkin Construction Theorem, there is a structure \mathfrak{M}_{val} , such that for all $Q \in \mathcal{H}$, $\mathfrak{M}_{val}(Q) = Y$, but \mathfrak{M}_{val} is a model for each of $\Gamma_1, \Rightarrow \Delta_1, \dots, \Gamma_n, \Rightarrow \Delta_n$, but not for $\Gamma \Rightarrow \Delta$. So $\frac{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}$ is not $valid_{Q3}$. □