

SENSE, INCOMPLETE UNDERSTANDING, AND THE PROBLEM OF NORMATIVE GUIDANCE

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Summary

Frege seems committed to the thesis that the senses of the fundamental notions of arithmetic remain stable and are stably grasped by thinkers throughout history. Fully competent practitioners grasp those senses clearly and distinctly, while uncertain practitioners see them, the very same senses, “as if through a mist”. There is thus a common object of the understanding apprehended to a greater or lesser degree by thinkers of diverging conceptual competence. Frege takes the thesis to be a condition for the possibility of the rational intelligibility of mathematical practice. I argue however that the idea that senses could be grasped as a matter of degree is in tension with the constitutive theses that Frege held with regard to sense. Given those theses, there can in fact be no such thing as misty grasp of sense, since any uncertainty as to the logical features of a given sense will entail that one is getting hold of a different sense or of no sense at all. I consider various ways of resolving the tension and conclude that Frege’s thesis cannot be defended if we take it to be a thesis about our competence with concepts. This leaves unresolved what I call the problem of normative guidance, that is, the problem of explaining how the fundamental notions of logic and arithmetic can provide inferential guidance to thinkers.

KEYWORDS: FREGE, SENSE, INCOMPLETE UNDERSTANDING, NORMATIVITY, LOGICISM

Introduction

In the introduction to *Grundlagen*, Frege considers two ways of thinking of conceptual history. We could think of conceptual history, he

tells us, either as the history of our knowledge of concepts (of our attempts to fully grasp their content) or as the history of our knowledge of the meaning of concept-words instead.¹ A third reading is ruled out, the reading that would commit us to the doctrine that it is the concepts themselves, and not the degree of clarity with which we grasp their content, that change throughout history. Frege denies that possibility on normative grounds. If what we aim at in our intellectual efforts were in a state of “continual flux”, the very possibility of knowledge would be “plunged into confusion”—no norm-governed truth-oriented discourse could come into being. Significantly, later in *Grundlagen* Frege uses the same turn of phrase (*in Verwirrung stürzen*) to describe the consequences of any attempt to reject the laws of logic (§14).

According to Frege, then, abandoning the idea that concepts remain stable targets of investigation throughout history would represent as much of a threat to rational thought as the attempt to suspend the normative grip of the laws of logic on us. Or rather: it would be just as *unthinkable*. In his view what can change is only the clarity with which we grasp those stable concepts. Until we succeed in extracting them from the “extraneous integuments” that obscure their purity, concepts are apprehended “as if through a mist”, Frege concludes.²

Call the thesis that concepts and the meaning of concept-words remain stable throughout history STABILITY.³ I think it is not mere happenstance that while Frege repeatedly warned us against the related perils of abandoning STABILITY and the laws of logic, he had no positive account to offer either for the epistemology of those laws or for the phenomenology of “misty” understanding—his remarks on both issues are equally tentative and inconclusive. In my view, Frege’s caution was engendered by his realisation that the only defence of STABILITY and of the privileged epistemic standing of the basic laws that would not violate his anti-psychologism would have to rely on the notions of sense and sense-grasp, notions for which he could not however provide a fully satisfying characterisation.

In this paper, I limit myself to considering the question of whether

¹Frege (1884, vii). See also (1891b, 133).

²I modify the Austin translation by borrowing from Russell (1914/1993, 53).

³For the purposes of this paper, I take the thesis to apply only to logico-mathematical concepts.

Frege *could* have given a coherent account of misty understanding.⁴ My main claim is that he could not (his hesitation was indeed well-placed). STABILITY, as the mature Frege thinks of it, is a thesis about our grasp of the *sense* of the concept-words referring to the concepts that are the targets of our investigations. The obstacle that Frege faces is that, given the constitutive theses that he holds with regard to sense, there can be no such thing as misty grasp of a stable sense, for senses are individuated *solely* in virtue of their logical features. Any uncertainty with respect to those features means that, *contra* STABILITY, one is getting hold of a *different* sense (or of no sense altogether). Misty grasp of sense, then, is an incoherent notion because it is incompatible with the applicability of proper criteria for sense-individuation, or so I shall argue.

I am aware that the main line of my paper is rather controversial, especially given the wide currency enjoyed by Burge's sustained elaboration of the notion of incomplete understanding.⁵ Nevertheless, it seems to me that when we appeal to that notion to defend some version of STABILITY we are in fact suffering from an illusion of understanding—the notion of incomplete (or misty) understanding is itself incompletely understood, that is. More modestly, then, in this paper I aim to raise at least some *doubts* as to the tenability of a fairly entrenched interpretation of Frege's doctrine on this issue.

Perhaps less modestly, there is also a further question that I want to consider, namely, what I call the problem of normative guidance, the problem of appropriately locating what Peacocke (1992, 126) called the *constitutive normative liaisons* of a concept, the triples of circumstances, attitudes and thoughts that competent thinkers must be sensitive to in order to count as competent with a given concept.⁶ In particular, competence with a logical concept entails the ability to move between thoughts (or to have certain attitudes towards specific moves between thoughts) in ways that are constrained by the content of that

⁴I examine Frege's epistemology of basic logical knowledge in my (2017). Two recent discussions of incomplete understanding of concepts that do not focus specifically on Frege are Buehler (2014) and Smith (2015).

⁵Burge (1984, §VI, 1990, §VII). However, Burge (2012, 586) is now countenancing the possibility that mathematical concepts *may* change in the transition from partial to complete understanding.

⁶Besides Peacocke's work, I'm here thinking of the Frege-indebted literature on inferentialism and meaning-constituting principles, e.g. Horwich (1998), Eklund (2007), Boghossian (2000, 2001, 2003). I discuss some of the difficulties faced by this family of views in my (2012).

concept (say, CONJUNCTION). The problem of normative guidance, then, is the problem of explaining both the *source* of our justification with respect to correct inference and the specific way in which that source can also *guide* us in inference.⁷ In short, it is the problem of specifying the *what* and the *how* of our acting for a reason in the logical domain.

Now, Frege's most promising attempt to give an epistemology of the basic laws of logic was in terms of our grasp of the senses of logical concept-words (the *what* is the senses, the *how* is our grasp thereof).⁸ Patently, STABILITY had to apply to those senses if it had to apply anywhere, since those senses are the only plausible repositories of the normativity of logic, the only entities where truth-preserving guidance articulated in the form of *reasons* could be encased. That is to say, if challenged regarding the validity of a given inferential step, the thinker could (and *should*) appeal to the content of the senses involved to justify her reasoning move. It is senses and senses alone, then, that can give us genuinely citable reasons to infer in accordance with the laws of logic.⁹ And this is so because *i*) Fregean thoughts (and senses more generally) are individuated in virtue of their perspectival reference-determining properties and because *ii*) it is by accessing those properties that we are able to move between thoughts in truth-preserving ways.¹⁰ But given that by Frege's own lights the senses of the fundamental notions underpinning the mathematical reasoning system are in fact deeply opaque to our intellection, then if there is no coherent account of incomplete understanding of sense available to Frege, the prospects for his epistemology of logic and for

⁷There is no slip here: within a Fregean framework the rules that guide us in inference are exactly those that justify the conclusions we reach, *modulo* the Carroll regress worries that Frege (1879, §13) acknowledges.

⁸Frege (1923, 405). Note that for Frege the problem of how basic logical laws are given to us *is* the problem of how we are justified in the inferences we carry out. See Frege (1893/2013, II 265) for the move from mode of being given to justification-maker and (1879, 5; 1893/2013, vi) for the ultimate justificatory role of basic laws with respect to a deductive system.

⁹As will be clear, I share McDowell's (2005, 178) view that the point of sense is "to provide for a conception of content that fits smoothly with a conception of what rationality requires of a thinker". In other words, the content of the senses of the laws of logic states (or unfolds, as Frege would say) the requirements of rationality under which thinkers ought to operate.

¹⁰By 'perspectival reference-determining properties' I mean properties that determine reference from a particular state of (logical) information. It is essential to bear in mind that throughout this paper I confine myself to a discussion of the senses of logico-mathematical expressions, not sublunary ones.

a Fregean solution to the problem of normative guidance look utterly forlorn. What we *would* need is an account of mathematical practice that explains *how* thinkers can be viewed as responding to *the very same reasons*, appropriately located within the relevant senses, throughout successive developments of that practice. The contention of this paper is that Frege has, and can have, no such account to offer.¹¹ And if so, it is unclear what can guide and justify us in inference, for, properly speaking, the incomplete understander seems to be inferring *without any reasons at all*, or at least she is inferring by accessing different reasons than those to which the complete master of the concept will finally be sensitive.

The plan for the paper is as follows. In §1 I distinguish three versions of STABILITY. In §2 I raise the Tension between the strongest version of that thesis and what I take to be the defining marks of Frege's conception of sense. In §3 I discuss some likely responses to the tension and dismiss them all as non-Fregean in character. In §4 I examine a version of the thesis that I call SUBSUMPTION-STABILITY, a version that I think better captures Frege's realism about numbers. I close by granting that SUBSUMPTION-STABILITY still fails to locate normative guidance satisfactorily.

1 The Stability Thesis

As mooted in *Grundlagen*, STABILITY is a thesis about concepts. In later work, and with the sense/reference distinction at hand, Frege states it as a thesis about sense instead. In this section I give three versions of STABILITY and assess the textual evidence to establish which version of the thesis we might legitimately ascribe to Frege. My discussion of the thesis will be restricted to its applicability to the fundamental notions of arithmetic, notions such as *number*, *function*, *extension/value-range*, and so on.

We start with the weakest form of the thesis:¹²

¹¹I think the issues I discuss here raise a more general problem regarding concept-stability under *any* conception of what concepts are. For example, Socrates' maieutic method relied on the assumption that the conceptual material he was painfully extracting from his victims was *already* part of their conceptual repertoire but only incompletely understood and many have followed him in making that assumption. I think that assumption still lacks *substantive* defence, however.

¹²I generalize the thesis as one about mathematical, rather than purely arithmetical concepts.

CONCEPT STABILITY (C-STABILITY) Mathematical concepts remain constant throughout history

In this form, the thesis is about the reference of concept-words.¹³ This is therefore a thesis wholly about the realm of reference, making no pronouncements regarding the realm of sense. There is no question that Frege held this thesis.¹⁴ A stronger form would be the following:

SENSE STABILITY (S-STABILITY) Mathematical signs express a constant sense throughout history¹⁵

This thesis is a claim about language, it concerns the senses that get attached to concept-words and it asserts that those senses and not just their referents remain stable throughout history. Given Frege's familiar requirement that sense determine reference, S-STABILITY obviously entails C-STABILITY. The thesis is however silent as to whether thinkers at different times (or thinkers that diverge in conceptual competence) grasp *those* stably expressed senses or other senses instead.

Finally, the strongest version of the thesis, and one that addresses precisely this last question:

GRASP STABILITY (G-STABILITY) Mathematical signs express a constant sense throughout history and thinkers grasp that very same sense throughout history

Clearly, this version of the thesis is partly about our capacities; it doesn't just claim that what signs express remains stable: it also claims that what thinkers *actually* grasp—what guides their mathematical reasoning, that is—remains stable too.

At first blush, G-STABILITY seems the best way of unpacking Frege's "as if through a mist" metaphor.¹⁶ Or rather, given Frege's frequent

¹³For Frege, concepts are the referents of concept-words. See Frege (1884/1987, §51; 1980, 63; 1892/1895, 135).

¹⁴See fn. 1.

¹⁵Obviously, the thesis is to be indexed to expressions in a particular language (including meta-languages). For instance, it applies to the German word *Anzahl*. As we shall see, Frege takes it to apply to *any* rational use of a particular language. Burge (2005, 56) seems in no doubt that Frege held (at least) this version of the view: "[Frege] regarded his logicist elaboration of *ordinary arithmetical discourse* as revealing [...] the senses that arithmetical expressions *had been expressing all along*" (my emphases). Similarly, on the preceding page Burge states that with respect to the term 'limit' Frege "would have regarded it as having a constant sense and denotation". See also Burge (1990, 260): "the words that Weierstrass uses *already* express the relevant senses" (my emphasis).

¹⁶Frege (1884/1987, 21; 1914a, 211, 217; 1924/25, 270).

use of that metaphor one is strongly tempted to abduct to the conclusion that he was committed to G-STABILITY, since with that thesis in place one would be *forced* to think of sense-grasp as something that admits of degrees. And indeed this is precisely what Tyler Burge, without question the commentator who deserves the most credit for drawing attention to the significance of the mist metaphor in Frege's conception of sense, has done. For example, here's how Burge glosses Frege's (1914a, 221–222) discussion of Weierstrass:¹⁷

[T]he best mathematicians may fail to have a clear grasp of the senses or thoughts they think with. Nevertheless, [...] the expressions such mathematicians use express definite senses, definite thoughts. [...] The mathematician is thinking with the sense, and expressing it in his writing, but not grasping it clearly.

G-STABILITY and misty, or incomplete, understanding go hand in hand, then. Is Burge correct in attributing G-STABILITY to Frege? Well, there is one quote that seems to fully support Burge's reading. When Frege (1914a, 211) discusses how we can establish whether a complex sense purporting to give the logical analysis (i.e. the definition) of an established simple sense is the same as the *analysandum*, he has this to say:

[I]f we really do have a clear grasp of the sense of the simple sign, then it cannot be doubtful whether it agrees with the sense of the complex expression. If this is open to question [...], then the reason must lie in the fact that we do not have a clear grasp of the sense of the simple sign, but that its outlines are confused as if we saw it through a mist.

On the strength of this passage, Burge's attribution seems perfectly faithful to Frege's thinking, then. Another equally supportive passage is the following:

[W]hat is called the clarity of a thought in our sense of this word is really a matter of how thoroughly *it* has been assimilated or grasped.¹⁸

¹⁷Burge (1990, 260). See also (1990, 254; 1998, 353; 2005, 55). As I noted already, Burge (2012, 582–84) seems to tell a different story however, in that the *generalised* commitment to STABILITY even in its weakest form is abandoned.

¹⁸Frege (1897, 130), my emphasis. This passage, along with all other relevant ones, is discussed at length in Burge (1990, §§iv–vi).

This again suggests that the target of our intellectual efforts remains stable, untouched by any uncertainty in our cognitive efforts. There is thus a stable object of thought that we grasp, but, as Burge (1990, 259) puts it, grasping *it* can be *a matter of degree*.

There are however other passages where the situation is less clear-cut. For example, Frege (1914b, 234) gives a different slant to the mist metaphor. In trying to get hold of the correct senses, confused thinkers are so badly placed, he tells us, that “*wenn sie zufassen, sie daneben greifen*”, when they try to grasp *them*, they *fail* to get hold of them.¹⁹ The picture here is one where at best it is only S-STABILITY that is assumed, since the confused speakers are described as either grasping no sense at all or grasping a different sense altogether.

Moreover, there is also a passage in Frege (1898/1899, 165) where, in criticising the way Peano had set things up in his system, Frege speaks of *contemporary* mathematicians having “a different opinion” (*Meinung*) regarding the sense of most theorems because they all hold diverging views regarding the sense attached to the identity sign.

Again, you could take this to support only S-STABILITY.²⁰ We could read Frege, that is, as saying that while the identity sign may well convey a stable sense, practitioners are instead all grasping (or attaching to the same expressions) different senses.

The key point for our purposes however is surely revealed by what Frege says next:

What use are 100,000 theorems, if we ourselves don't know what we want to communicate by their means, if the one who uses a theorem attaches a different thought to it than the one who first

¹⁹Frege's choice of words here rules out G-STABILITY but doesn't settle the question of *what*, if anything, the uncertain reasoner grasps: *daneben greifen* can be used e.g. to describe someone who hits a wrong note on the piano or to describe a goalkeeper that misses the ball altogether. One could of course take this passage to support the claim that Frege held the view that sense-grasp *does* come in degrees. In some cases (when the mist is really thick) we don't grasp a sense at all. In other cases we grasp it but we are uncertain as to what we have grasped. See also Frege (1891b, 133): “we seek to grasp [the concept], and in the end we hope to have grasped it, though we may mistakenly have been looking for something where there was nothing”.

²⁰Frege (1891/1892, 72) is another passage that, taken on its own, may seem to support at most S-STABILITY. There Frege tells us that in ordinary *uses* of the word ‘number’ its sense is not brought forward with the clarity required for scientific use. The rest of that paper makes however clear that Frege's target is the confusion between *Vorstellung* and *Sinn*. It's not that ordinary use attaches the wrong sense: rather, *it mistakes an idea for the sense*. This, it seems to me, is still consistent with the claim that reasoners are being guided by the right sense although that sense is (provisionally) filtered through an idea.

proved it?²¹

And in the parallel passage in *Grundgesetze* Frege also adds:

If mathematicians' opinions about equality diverge, then this means nothing less than that mathematicians disagree with respect to the content of their science[,] that there is no one united mathematical science, that mathematicians do not, in fact, understand each other.²²

The upshot then is that for Frege unless we *assume* that all practitioners attach the same sense, mathematics collapses into incoherence. But given that mathematics doesn't so collapse (for all their *apparent* ignorance of senses, the confused mathematicians were still able to prove theorems after all), we *must* assume that G-STABILITY does hold. As Frege (1893/2013, II.iii §58) insists, this is in fact a demand of logic, no less (without that assumption, its laws would have no applicability across the various temporal stages of the practice).

Frege's picture, then, is clearly that in order to explain communicability and mathematical progress we must take all mathematicians as grasping the same sense, even though they *appear* to attach different, idiosyncratic ones to the expressions in question, even though they seem, on occasion, to hit on the wrong sense altogether or on no sense at all.

²¹I think passages such as these raise a problem for e.g. Weiner's (2010, 36–41) claims that *i*) Frege's definitions (at best) preserve "whatever conceptual content is already associated" with the fundamental notions of arithmetic as ordinarily used and that *ii*) the terms expressing those notions actually lacked a reference prior to his work. Weiner sees Frege as rejecting S-STABILITY (p. 45). On her view, Frege instead thought that pre-logicist arithmetical statements did not express truths but could rather be *regarded* as expressing truths. I'm unclear about two things however. First, if ordinary statements *already* have *genuine* conceptual content (and in Weiner's use, conceptual content is equivalent to the later notion of sense, given that she ties it in to the notion of definition), then how is it that they *lack* a reference? Secondly, if the conceptual content of ordinary statements is masked by extra-logical content to the point of failing to secure a reference (i.e. truth value), how can one *rationaly* regard them as being true? For more on these stubborn issues, see Dummett (1991b, 176–178) and Blanchette (2012, ch. 4).

²²Frege (1893/2013, II.iii 71, fn. 1). See also Frege (1914a, 215, 241–242; 1919, 256). As suggested by an anonymous referee, it would be instructive to compare Frege's reasoning here with the much-discussed Aristotle footnote in *Über Sinn und Bedeutung*. There, Frege moves from stability of a practice to stability of a family of related senses. Here the move is instead from instability of senses to instability of the practice. I take Frege's different treatment of the two cases to provide grist to my mill. In the case of mathematical discourse, no reliance on guesswork is to be allowed. All reasoning steps must be open to inspection, and it must be determinate at all times which sense one is attaching to one's expressions *or else the properly formal status of the language would be under threat*.

Accordingly, although the textual evidence is not *entirely* decisive I nonetheless think that we should see Frege as holding that G-STABILITY is a condition for the *rational intelligibility* of our mathematical practice. Without it in place, the practice (*and* its rational reconstruction) would collapse into outright incoherence.²³ Indeed, time and time again Frege tells us that *logic demands diachronic sense-homogeneity* as the condition for the possibility of the coming into being of any genuine definition (i.e. of any genuinely *reasons-giving* definitions, definitions that would be able to guide thinkers in inference).²⁴

Furthermore, there is also another equally important reason why G-STABILITY must be in place. If we combine Frege's (1914a, 247) striking anticipation of Quine's (1970, 81) meaning variance argument with the remarks I quote below from the same work, we can see that G-STABILITY is also required to ensure that logicism *doesn't change the subject* as a result of its process of analysis of the fundamental notions of arithmetic:

How little value is commonly placed on sense and definitions can be seen from the sharply conflicting accounts that mathematicians give of what *number* is. [...] Each of these [i.e., Weierstrass, Heine, and Thomae] attaches a different sense to the word 'number'. So the arithmetic of these three mathematicians must be quite different.[...] How can one possibly imagine that [three] quite different sciences should really have the same content? Is it only because they are [all] called arithmetic and [they all] treat of numbers, although what is called a number in one is quite different from what is called a number in the other? Or is it not the explanation rather that we have really to do with the same science; that this man *does* attach substantially the same sense to the word number as that man, only he doesn't quite succeed in grasping it correctly?²⁵

²³It's not just a question of giving a coherent *account* of the practice: it is rather that we need to view practitioners as rational, as being *on the lookout for reasons*, and for the *right* reasons at that. In response to a worry by an anonymous referee, let me point out that for Frege the crucial issue remains, at all times, that of the *rational justification* of a practice. If mathematicians establish results by methods of proof that *openly contradict* the definitions from which those proofs start, then the results, however stellar they might be, will *always* lack justification, and hence the mathematicians' beliefs will remain ungrounded. It is in this sense that the practice is incoherent, since it contradicts the explicitly given basic concepts on which it is *taken* to rest.

²⁴Much of part III of book II of *Grundgesetze* is taken up by a constant attack against piecemeal definitions precisely on the grounds discussed in the text.

²⁵Frege (1914a, 215–217), slightly modified translation. The meaning-variance passage clearly

Here it seems clear that Frege is holding that sameness of *understood* senses, before and after the logicist analysis that he is striving to provide, is essential for the diachronic and synchronic shareability of a given practice. Nothing but (complete) understanding of senses can mandate acceptance of the basic laws of arithmetic. Nothing but a common object of the understanding can ensure that the various time-slices of a practice count as slices of the same practice. Sameness of normatively guided practice, that is, entails sameness of sense for its fundamental notions, no matter how approximate the actual grasp by practitioners might be.

G-STABILITY, then, is, quite simply, a non-negotiable feature of any proper Fregean conception of these matters, or so it seems. And G-STABILITY brings with it the claim that sense-grasp may be a matter of degree, that insufficiently alert thinkers do grasp the correct senses but only with incomplete understanding, as if through an epistemic mist.

2 The Tension

The burden of this section is to argue for a claim that seems to scupper the conception of normative guidance outlined in the introduction. I want to argue that although Frege was indeed implicitly committed to G-STABILITY, he did not and *could not* provide an account of misty understanding that would be compatible with his conception of sense. My main claim will be that by Frege's own lights *there can be no such thing as misty grasp of a sense*. Sense-grasp, that is, *cannot come in degrees*, for there is a tension between the very idea that senses could be tentatively grasped and the constitutive theses that Frege held with regard to sense—what is grasped in incomplete understanding *cannot* be a Fregean sense.²⁶ And if so, the incomplete understander lacks reasons, *logically-grounded* reasons, that is, for inferring the way she does. Much as Frege needed to have G-

locates normative guidance in sense-understanding: “so long as I understand the words ‘straight line’, ‘parallel’ and ‘intersect’ as I do, I cannot but accept the parallel axiom”. See also Frege (1980, 183, fn. 7).

²⁶Burge (1990, 247) raises and dismisses a related tension that exploits the move from incomplete understanding to grasp of a vague concept which in turn would not be a concept at all by Frege's lights. The tension I raise here concerns instead the *possibility* of the *transition* from incomplete to complete understanding of stable senses that Burge (and Frege) assume.

STABILITY in place in order to realise his philosophical goals, then, it was his own conception of sense that made that requirement impossible to satisfy.

I'll now try to sustain this claim. First, we should note that G-STABILITY entails, as we observed at the end of the last section, that *there is a common object of the understanding* shared by both partial and full understanding, a stable sense, a stable provider of reasons that one tracks as one moves from tentative to fully secure grasp of its content.

Secondly, we should also note that Frege seems to give two contrasting models of misty understanding.

In the *Grundlagen* introduction, misty understanding is *confused* understanding. We somehow hook onto the logical core of the concepts, but that core is obscured by foreign and dispensable features. On this model, we take in too much content. Alternatively, we could say that we fail to recognise the logical core as logical and the foreign additions as what they really are, elements of the content that have no logical import. Either way, confused understanding is *unfocused* understanding, allowing extraneous stuff into the target content. The question then is whether by Frege's own lights unfocused understanding can properly be said to be understanding of the same sense that one grasps in properly focused understanding.

In later work, Frege's model is different: misty understanding is *incomplete* or *partial* understanding. It's not that we take in too much; this time we take in too little instead, we leave out essential parts of the content. Partial understanding, on this model, is understanding of partial content. There is support for this attribution in the following passage, again from his discussion of Weierstrass's epistemic situation:

One can draw correct inferences even though part of the sense always remains obscure to us.²⁷

If we set aside the concerns that one could (or perhaps should) have regarding Frege's often *extremely* cautious way of characterising the misty understanding predicament, it seems clear that what Frege is saying here is that *partial* mastery of content is precisely that: one has only mastered *part* of the concept, one is *missing out* some elements

²⁷Frege (1914b, 240).

of the sense's content.²⁸

In itself the move, as Frege outlines it, is harmless: it is indeed possible to infer correctly even if part of the sense is beyond reach because it is conceivable that partial masters of a logical concept could be drawing valid inferences at all times while not drawing *all* the valid inferences that they could draw, were the content of the sense available in its entirety.

What is *not* harmless, however, is Frege's apparent claim that incomplete masters of a sense can *grasp* (and deploy) *the very same sense* grasped by complete masters even though part of its content is in the penumbra.²⁹ If we miss out part of the sense, then *contra* Burge (1990, 260) we cannot be "thinking with *the* sense"; we can only be thinking with *another* sense, or perhaps even *with no sense at all*.

In other words, Frege's conception of sense leaves *no room whatsoever* for the contrast between mistaken application of a correct procedure and correct application of a mistaken procedure.³⁰ Or to phrase

²⁸The notion of understanding is at least as fuzzy as that of sense and sense-grasp. In the philosophy of language, it is often glossed in terms of knowledge. Pettit (2002, 521) labelled this approach the *epistemic view* of understanding, whereby one equates understanding an expression *e* with knowing what *e* means. If we adopted this view, partial understanding of sense would be partial knowledge of sense. But sense, at least with respect to mathematical notions, is something that contains purely propositional content. And I struggle to make sense of any notion of partial knowledge of content that is not knowledge of partial content. Any alternative notion would have to reify senses (in effect treating knowledge of senses as a form of *de re* knowledge), thus triggering the regress discussed in fn. 40, p. 19 below. A further alternative strategy also discussed below would be to include psychological and non-conceptual capacities to account for misty grasp (see e.g. Burge 2012, 577), thus blocking the move from misty sense-grasp to grasp of misty senses. This strategy would fairly obviously fall foul of Frege's anti-psychologism though. This is perhaps the place to deal with another possible complaint. In raising the tension, I move from considerations to do with sense-*grasp* to conclusions about *what* is being grasped (I move, just as Burge (1990, 247) does, from uncertain grasp to grasp of uncertainly defined content). I think the move is perfectly legitimate. If senses are *ways of thinking*, then sense-grasp is thinking with *a* sense, that is, making *the particular way* of thinking of the referent prescribed by the sense one's own way of thinking of that referent. If so, misty sense-grasp can only be a misty way of thinking of the referent, precisely because to grasp a sense is *nothing other* than to deploy *it*. Note that this remark also applies when we think of sense-grasp as an ability (see p. 20 below). If sense-grasp is an ability, sense-content too will have ability-like constituents. However, as I argue below, at this point we have psychologised sense beyond comfort. See Kremer (2010, 281ff.).

²⁹To my knowledge, Textor (2011, 147–148) is the only commentator that has questioned the coherence of Frege's talk of partial sense-grasp.

³⁰Dummett's (1973, 93) procedural view of sense is criticised in Burge (1990, 244). Let me dispose of a possible objection here. One might complain that there's a way of saving the contrast discussed in the text. We could say, with McGee (1985, 469), that on occasion we may be able to

it less controversially: it obliterates the distinction between confusedly thinking with the correct sense and clearly thinking with the wrong sense, between a way of thinking that is unfaithful to the content of the correct sense and a way of thinking that is instead faithful to a wrong, neighbouring sense. Given that conception, Frege simply lacks the resources to tell the two cases apart.

Now, Frege's two models of incomplete understanding are clearly distinct and for my main claim that misty grasp of sense is incompatible with G-STABILITY to go through I need to tackle them separately. In particular, while I think that there is a *clear* and indeed stark problem for the Fregean model of partial understanding, I am less confident that the earlier model of unfocused understanding gives rise to the tension in the same direct manner.

At this point, one might in any case complain that since Frege did not provide firm criteria of sense-*identity*, it is unclear how one *could* even begin to argue for my claim that he cannot hold on to G-STABILITY. In response, I would however say that Frege has said enough about sense to provide us with criteria for sense-*individuation* that suffice to sustain the conclusion that failure to grasp the *entire* content of a given sense (or failure to grasp that content as the uncontaminated content it is) would amount to failure to grasp *that* sense as the sense it is.

Let me now try to substantiate this response while also preparing the ground for stating the tension more precisely.

I think we can get a good enough grip on Frege's general conception of sense from what he says about one specific kind of sense, namely, the sense of a declarative sentence, the thought that the sentence, properly disambiguated, expresses.³¹ A thought, Frege tells us, is that for which the question of truth and falsity can arise, it is something not to be identified with the content of anyone's consciousness, and

judge that e.g. an instance of a rule of inference is valid without being in a position to recognise or understand the full content of the rule, that is, without being able to tell *which rule* it is (*mutatis mutandis* for logico-mathematical concepts more generally). I think the move is *very* interesting. It would however provide no solace to the Fregean, since STABILITY is a thesis about rules, not instances. I should perhaps also make clear that I consider the inability to articulate that contrast to be a *merit* of Frege's conception, for I think that the contrast is ultimately illusory. And its illusory nature masks a *general* problem about concept-grasp that Frege's conception usefully brings into view.

³¹Frege (1914a, 206) makes it clear that by 'sentence' (*Satz*) he means utterance-tokens too, not just written tokens.

it stands in no need of interpretation; a thought remains unaffected by mood and colouring, and finally it is that which gets preserved in translation across languages or in surface linguistic transformations.³²

In short, a thought contains nothing but what is of interest to logic, nothing but what contributes to the determination of its truth condition, nothing but the reference-determining properties belonging to its constituent parts. It leaves no room for psychological or language-specific idiosyncrasies.³³ We can reach the same conclusion from another direction as well. We can, that is, start from Frege's (1897, 142) dictum that grammar is "a mixture of the logical and the psychological". We then recall that for Frege the task of logic is to isolate and insulate the purely logical kernel in grammar—all and only that which matters to inference.³⁴ Thought, then, is what we get from grammar when we subtract any locally-inflected, that is, language- or thinker-relative factors, be they linguistic or psychological.³⁵

If this is what a thought, i.e. sentential sense, is, it then seems as if it can be so only if sub-sentential sense is also devoid of any extralogical features or else those features would presumably be transmitted upwards by the sense-compositional operations. It thus seems to follow that sense in general contains no features other than logical

³²Frege (1918-19b, 353; 1914a, 209, 1918-19a, 376; 1906, 315; 1897, 140-141, 1906a, 193, 1906b, 198, 1918-19b, 357, 1891a, 357, 1892b, 161, 1980, 67, 1906a, 193; 1897, 141-142, 1914a, 206, 216) respectively.

³³It is unfortunate that Frege's most famous essay, *Über Sinn und Bedeutung*, should have encouraged the idea that sense *could* include extra-logical features. Burge (1979, 239) may well be right that "Frege's focus on logic blurred his vision of epistemology" in the case of the sense of indexical expressions. A more appropriate reading would however be that the sublunary examples were mere heuristics that Frege employed to bolster his explication of a *novel* notion, that of sense, that was required to account for the *mathematical* fact that the same entity can be recognised in different ways—Frege (1884, §62, 1897, 142). In the context we are discussing in this paper, namely, that of the fundamental notions of arithmetic, there is no room for any psychological-cognitive infiltration since here we are dealing with *nonindexical* thoughts only. Sense, as far as those notions are concerned, can only contain what is of value to *logical* knowledge. And logic has no truck with properties other than reference-determining ones. Indeed, Frege (1918-19b, 352) makes it *very* clear that *any* psychological infiltrations would nullify one's *justification* for a given claim.

³⁴Frege (1879, 6, 12; 1879/1891, 5; 1880/1881a, 33, fn.; 1897, 142; 1980, 79). One quote for all: "One should make only those distinctions [between thoughts] with which the laws of logic are concerned" (1980, 67). Note that within Frege's framework, nothing but reference-determining properties are of relevance to inference (it is reference-determining properties that prevent us from moving from names of the True to names of the False).

³⁵See the entire passage at Frege (1897, 141-144) as well as (1893/2013, xix), (1914a, 216) and (1892b, 162, fn. 7).

ones—no features that could be shed (or *misread*) with impunity as far as logic is concerned, that is.³⁶

2.1 Confronting the Tension

I am now in a position to state the tension. Let me start by raising it with respect to Frege's second model of misty understanding. On that model, misty understanding is understanding of partial content. But if senses only contain logical material and if, as seems clear, senses are individuated purely in virtue of those features, then leaving out essential logical features of a given sense straightforwardly entails that any thinker who only grasps part of that sense is in fact getting hold of another sense, or rather: of no sense at all (since crucial reference-determining material is being left out). In missing out part of the sense, that is, we are in fact missing out *essential* marks (to use Frege's term) that make that sense the sense it is. That is to say, any variation in the apprehended profile of a sense entails a difference in apprehended sense.³⁷ Accordingly, the austere conception of sense that I have attributed to Frege has left no room for dispensable features of sense, features that one might fail to grasp without affecting the indi-

³⁶And so Frege's (1892b, 158) famous metaphor that sense *contains* the mode of being given of the referent *must* be unpacked as: the mode of being given *for the purposes of logic*. Similarly, one could give a quick little argument for the same conclusion reached in the text. Frege (1980, 191) tells us that the laws of number must be developed by purely logical means (see also *ibid.*: 141). Sense is the essential means to that development (since it grounds the normative status of the basic laws—Frege 1923, 405). Hence, sense must be a purely logical notion or else numbers would be given to us by means that are not purely logical.

³⁷For arguments to that effect see e.g. Diamond (1984, 124) and Dummett (1990b, 277; 1989, 306, 311). As Sullivan (2004, 687) usefully puts it: "there is no such thing as a faulty expression of a content which even so expresses that content". *Mutatis mutandis*, this applies to *grasp* of sense-content too. There is a possibility that I do not discuss in the text, namely, the possibility that the parts the thinker leaves out are in fact logical in character—the incompletely understood sense *overdetermines* reference. The issues raised by this possibility are rather complex. Let me just say here that this option would be no threat to my account. First, Frege *never* claims that in misty understanding one is grasping part of the sense clearly and the whole sense confusedly. Rather, what Frege claims is that it is the sense *as a whole* that is grasped confusedly, if at all. Secondly, if the sense-part we grasp is already a genuine Fregean sense, as it must, then it already has reference-determining properties. But by the familiar cognitive criterion of sense difference, the thinker could always *rationaly* doubt it is co-referential with the complete sense. If so, Frege's (1895, 125) crucial requirement that the step from sense to reference be determinate "beyond doubt" would be violated. Furthermore, to think that a sense could have redundant logical features would be to confuse a sense with the words used to express it, a mistake against which Frege (1899-1906, 167) had warned us in no uncertain terms. Finally, as Frege (1880/1881b, 43) makes clear, logically superfluous material "reduces" the content of a thought (and hence it determines a *different* thought).

viduation conditions for that sense.³⁸

With the first model the tension is less straightforward to raise. In fact, in this case the risk is that I may be establishing *too much*—if things are as bad as I claim then there is no prospect of ever being able to make any progress in our reflective enquiries. Let's see why.

Recall that on the first model, misty understanding is unfocused understanding of sense. What happens here is that the logical kernel of the sense is partly hidden from sight because of “foreign” accretions. The model divides into two cases depending on whether the undesirable accretions are logical or non-logical.

In the first case, the problem is uncertainty as to the logical features of sense (in the way in which for instance Frege might have been confused about the content of Basic Law V, assuming that somewhere in the Third Realm there is an unimpeachable version of that law that Frege was trying to reach as if through a mist). I think this case is easy to deal with. The reasoning I deployed to deal with partial understanding, if sound, will apply here too: the additional but troublesome logical material, that is, will individuate a different sense anyway, leading again to a failure of STABILITY.³⁹ This is particularly clear in the case of Basic Law V where the mistily apprehended sense provides wholly unreliable normative guidance, since it licences, in conjunction with the rest of Frege's system, the derivation of any proposition whatsoever. The properly grasped sense had better be *other* than the mistily grasped one, then.

In the second case, the extraneous additions are non-logical in character and so they will presumably play no role in guiding inference. For the same reason, they will play no part in individuating the sense as distinct from the one that will eventually be grasped after

³⁸In other words, given that conception, there is no room for the possibility of the sort of “projection beyond actual understanding and use” with regard to sense-content that Burge (1990: 261) argues for. And there is a very good reason for that impossibility. As Kripke (2001: 255) correctly notes, we specify a Fregean sense by specifying the reference it determines. But with concepts, a misty specification of reference is a specification of a different concept, for individuation conditions for Fregean *logical* concepts do not admit of degree.

³⁹To use one of Frege's (1892b, 158) own examples: the sense of ‘point of intersection of *a*, *b* and *c*’ is clearly distinct from the sense of ‘point of intersection of *a* and *b*’ and it seems implausible to claim that someone grasping the latter is confusedly grasping the former. Moreover, in the case of concept-words (our *only* concern in this paper), the concept-definition grasped by the full understander is, patently, not one that contains redundancies. It is rather one that makes the concept, at last, *precise* (e.g. totally defined over all cases), and like all *elegant* theoretical definitions, it does so without redundancies.

the mist has been dispelled. If so, the tension seems not to arise at all. But the problem with this sub-case for the first model is that it seems to commit Frege to the claim that we could grasp, as *part* of grasping a sense, non-logical features too. So, what we end up grasping is neither the same sense as the one we would grasp if we could get rid of the impurities, nor a different sense (for those impurities make no contribution to determining the sense as the sense it is). Rather, what we grasp is not a sense at all, properly speaking, precisely because we are grasping the impurities *as constituents* of that putative sense, rather than as *external* add-ons. We are making, in effect, a category mistake as to the individuating features of that sense. And so, once again, we have a failure of STABILITY.

The reasoning that leads to this conclusion seems sound, given the austere conception of sense. But as I said, it also seems to prove too much. If the process of logicist analysis consists in clarifying the received senses of the fundamental notion of arithmetic, and if those notions are enveloped in impurities to be removed by analysis, how can one even *claim* that the analysis targets *those* notions in the first place? Here we seem to face a deeper paradox of analysis, a paradox not just about informativeness but about expressibility too. If a sense can only come fully into view as the sense it is only *after* we have provided the appropriate *analysans* how can we possibly *say* what *is* the target *analysandum* before the process of analysis is completed? Analysis, if the reasoning above is sound, seems just like a shot in the dark.

Unpalatable as this conclusion might be, however, at this stage I can't see a way for Frege to avoid reaching it. Given Frege's constitutive commitment to an austere conception of sense, that is, G-STABILITY and misty understanding can only be incompatible phenomena, since we have examined all the models of misty understanding that Frege seems to have considered and we have found them all wanting.

This seems to leave us with a rather nasty dilemma. Either we give up STABILITY, thereby abandoning Frege's preferred characterisation of the philosophical significance of logicism, or we give up some of the constitutive theses of sense, thus allowing unwanted psychological elements back into the grammar of inference.

In the next section, I examine some ways of thinking of incomplete

understanding that Frege did not consider, as well as some other suggestions one could make to escape the tension.

3 Responding to the Tension

I think there are two broad strategies one might want to pursue in response to the tension. One strategy would involve modifying both the notion of sense and that of understanding. The other strategy would instead embrace the loss of G-STABILITY, the consequences be damned.⁴⁰

Under the first line of response we attempt to escape the tension by suitably enriching the notion of sense, just enough to defuse the tension, and no more. After all, Frege (1892b, 158) told us that senses *contain* modes of reference-determination, not that those modes *exhaust* the content of sense. There might thus be other material, perhaps non-propositional, that “surrounds” and supports the sense’s kernel, the part that does the reference-determining job. The question now is whether the resultant, less austere notion is compatible with the key tenets of the Fregean conception of sense. It is useful here to introduce a new term for sense, namely, *conception*. Senses, on this proposal, are conceptions of the target concept (recall that we are restricting our discussion to the senses of concept-words).⁴¹ Conceptions may be richer than senses because they are not just given to us by stating bodies of propositions.⁴² They are, rather, explained in terms of (and constituted by) bodies of *information*, information that may well include sub-propositional states as well.⁴³ Clearly, this

⁴⁰The suggestion that we take senses as objects and thus recover more familiar models of object-tracking (as opposed to the highly problematic notion of concept-tracking) can be quickly dispensed with. Senses are ways referents are given to us, but they are not (and cannot) themselves (be) given to us. Any temptation to think otherwise would trigger a vicious regress, for if senses too came with modes of presentation, so would also those higher-order ways of being given, and so on without end. See Bell (1987), Dummett (1986, 1989, 1990a,b), Beaney (1996, 219–220), Kremer (2010, 288–289), and Burge (2012, 574–575).

⁴¹The conception/concept distinction was first applied to Frege’s framework in Wiggins (1980/2001, 10; 1984, 128, fn. 6; 1994, 207). Burge’s (1989, 282) early term for conceptions had been ‘conceptual explications’. The distinction, as used in a Fregean context, is by and large orthogonal to the way it has been wielded by Hart (1961/2012, 246) and Rawls (1971/1999, 5).

⁴²As *Grundlagen* §62 tells us, to give a sense is to give the sense of a sentence in which the expression whose sense it is occurs. And so in the case of the fundamental notions of arithmetic, their senses are given as *propositional* constituents of definitions.

⁴³E.g. Burge (2012, 577) includes psychological states that can be “conceptual or non-conceptual,

revised and less austere notion of sense opens up the way to a less austere notion of understanding as well.

For illustration, let's look at two sense-enriching suggestions that fully respect G-STABILITY by construing sense-grasp (and sense-deployment) as the *ability* to think (more or less efficiently) with a concept (*through* the sense).⁴⁴

The first suggestion diagnoses incomplete understanding as defective understanding at the explicit level only. Conceptions, that is, split up into two levels, implicit and explicit. To possess a concept is simply to be able to tap into one's implicit conception even though one may be unable explicitly to articulate its content. The explicitly given conception is defective, the implicit conception (the ground-level ability, as it were) is correct and it underpins appropriate use of the concept. On this view, incomplete understanding is implicit grasp of the right content, despite one's shortcomings at the level of explicit articulation of that content.⁴⁵

A variant suggestion (Burge 2012, 579, fn. 59) is that the incomplete understander has grasped the right conceptual material, but not quite in the right order. One possesses all the constituent components, but one lacks the proper structuring for those components, that is.⁴⁶ The partial master of the concept, then, is not leaving out parts of the content, but rather of *the way* the content is structured. Or rather: the understander already possesses the conceptual material that *could* fully articulate the content one thinks when deploying the concept, but has not gathered together that material *as what one really thinks* when one thinks of *that* concept with the (faulty) conception.

Misty grasp so construed is, as Burge (2012, 580) notes, a failure of *performance* rather than of competence, it is uncertain deployment cognitive or not⁷.

⁴⁴I extract them from the taxonomy given in Burge (2012, 573-589) (see also Burge 2006, 165). Burge often speaks of understanding as an ability, and one that mobilises non-conceptual resources too (e.g. 2003a, 394). For Dummett's view of understanding as a "capacity to engage in a common practice" see e.g. his (1973, 293; 1981, 245).

⁴⁵The view is articulated and defended in Peacocke (1998a,b). For criticism see Schiffer (1998), Rey (1998) and Burge (2003b; 2003c, 517). Note that Peacocke doesn't generally embrace G-STABILITY (1998b, 63). For him, implicit conceptions can evolve as our reflective efforts increase their precision and in the case of *set* he thinks that C-STABILITY fails too.

⁴⁶This tallies well with Frege's (1884/1987, §88; 1880/1881a, 34) insistence that definitions be more than a list, that they organise the characteristic marks of a concept in the appropriate, "organic" way.

of understanding-*qua*-ability—an implicitly assimilated stable sense is got hold of by insufficiently polished skills, as it were.

Would these two options be acceptable to the Fregean? I don't think they would, and, very briskly, for the following reasons. First, it is not clear on what basis one could uniquely identify abilities of this sort with the underlying implicit conceptions, nor is it at all clear what explanatory value would accrue from that purported identification (i.e. we'd be losing a grip on the old fitting/guiding contrast that Quine (1972) rightly insisted upon, we'd be unable to tell whether the conceptions merely fit patterns of usage as opposed to *determining* them).⁴⁷ Secondly, and relatedly, if we think of conceptions as implicitly-grounded abilities we are in effect thinking of them as *dispositions* and thus as unsuitable, in the eyes of many (and *certainly* of Frege), to play a genuinely normative role.

Furthermore, with these suggestions at hand we are now thinking of senses as entities that “type-individuate psychological competencies” and that are themselves “aspects of certain psychological kinds” (Burge 2012, 576), including, as already noted, non-propositional ones. But unless we buy Burge's (2012, 577) claim that, properly seen, Frege's philosophy not only de-psychologizes logic but psychology as well, so that *all* world-involving states, including psychological ones, are to be explained in terms of truth and reference, we are highly unlikely to accept that this enrichment of the austere conception of sense would accommodate Frege's hostility towards any anchoring of logic to notions that bear local, contingent relations to the specific make-up of beings such as we are.

To put it pithily, it seems to me any attempt to preserve G-STABILITY by making sense-grasp stable only at the implicit level faces the problem that we won't be able to characterise our acting (in this case: our reasoning about mathematics, our doing proofs and so on) as acting done *for* a reason, indeed as acting done *in the light of* a reason. To the extent that implicit conceptions are no more than tacitly followed dispositions, they do not and can not provide reasons, certainly not reasons of the kind that Frege (1898/1899, 157) wanted to locate in senses and sense-grasp in order to answer the how-question concern-

⁴⁷I'm very brisk on the first point here because a fuller discussion would merely duplicate Rey (1998, §§1.1-1.2). On the second point, see the analogous discussion about tacit knowledge of meaning-theorems in Evans (1981) and Wright (1986, 227).

ing mathematics that he so cared about:

If you ask what constitutes the value of mathematical knowledge, the answer must be: not so much what is known as how it is known.⁴⁸

Dispositions won't do anything to answer the how-question in terms of *effective* normative guidance, in terms of the normative liaisons between our attitudes towards a particular inference step and *what we take to be the content of the relevant concepts being appealed to* in the circumstances.

In particular, whenever Frege discusses these matters (see e.g. 1914a, 221), the crucial missing justificatory element in non-logicist mathematical practice is said to be sufficient clarity and rigour in the *definition* (i.e. the sense) attached to an expression *and in the extent to which use is actually being guided by the definition*. What so exercises Frege, that is, is the glaring hiatus between openly given definitions and actual practice in pre- and non-logicist mathematics. More precisely, it is the *conflict* between overtly espoused definitions and actual proof-conduct that makes it impossible for Frege to treat the mathematician as rationally justified in what she does. E.g. Weierstrass gave himself rules *only to flout them*, in a wilful display of inverted *akrasia*, as it were (he did the right thing by disregarding the wrong rules he'd explicitly adopted and *defended*).⁴⁹

As far as Frege is concerned, then, settled first-order patterns in a defective practice, no matter how correct they may be, are not enough to provide a vindication of our knowledge claims precisely because neither the explicitly given rules nor the implicitly adhered-to dispositions would count as reasons in the appropriate sense (remember: what matters is the *how* not the *what*). Only properly formulated definitions can provide rational guidance for our practice, and what lies on either side of the divide between complete and incomplete mastery of our concepts must be stuff of the same kind (whereas *ex hypothesi* what we have here is a fairly brute transition *from dispositions to*

⁴⁸Note that to provide a satisfactory how-answer of this kind would in effect be to solve the problem of normative guidance.

⁴⁹Frege's diagnosis of Weierstrass' situation is a clear instance of the problem of normative guidance made manifest. The senses that Weierstrass consciously grasped, as reflected in his definitions, were defective, and hence incapable of providing full guidance. Something else guided him to true, but unjustified, conclusions, and our problem is precisely to specify *what* did so.

definitions).

I now want to consider instead what seems the most obvious way of weakening G-STABILITY. Under this suggestion, one would bite the bullet concerning the epistemic costs of losing the normative guidance given by stable senses and just say that C-STABILITY is the best a Fregean can hope for (something like a reflective equilibrium strategy will be required to assess competing senses).⁵⁰ We concede that it is the reference only (the concept) and not the sense (the conception) that remains stable. After all, as Frege (1891b, 133) peremptorily said, “for a logical concept there is no development, no history”, and this contrasts with our understanding (i.e. our conception) that, as he also noted (1893/2013, II, 256) while reeling from the raw shock of Russell’s paradox, often requires “correction”.

On this proposal, then, the claim is that we do succeed in *deploying* the stable concept in our judgements even if the conception we adopted to do so is defective.⁵¹ One perfectly natural way of unfolding this suggestion would be to say that misty understanding is grasp of a sense that is *sufficiently similar* to the one we will grasp after the appropriate conceptual analysis has been carried out.⁵² All that normative guidance demands, one might add, is the requirement that there be a sufficiently large fragment intersecting the two senses even though they are not strictly speaking the same.⁵³

Would C-STABILITY so construed be compatible with Frege’s other views? I’m afraid I don’t think that it would.

First, if the two senses across the complete/incomplete divide do

⁵⁰I am again excluding the radical option to say that neither conceptions nor concepts remain stable. As I made clear at the opening, this would run counter to Frege’s insistence that a move of this kind would altogether incapacitate our cognitive activities. As a variant of this radical option, Burge (2012, 582–588) draws yet another distinction between fairly generic *ur-concepts* that get sharpened or broadened by successive elaborations. Burge then argues that while *in some cases* concept-use that precedes full explication of the concept’s content may be guided by those generic concepts, in other cases, notably the mathematical ones of concern to us, even *ur-concepts* may undergo “several tracks of elaboration” (p. 587) so that in fact a new concept is developed (p. 586).

⁵¹This is e.g. the diagnostics offered for conceptual history more generally by Stephen Schiffer and reported in Higginbotham (1998, 155).

⁵²Burge (1979, 223) rules out this option: “Frege did not think that the sense of a nonindexical expression [...] varies with the speaker”.

⁵³In different terminology, the proposal is in effect inviting us to distinguish between concept-possession and concept-mastery. Incomplete understanding is compatible with concept-possession (i.e. minimal competence with the concept). Concept-mastery requires instead full understanding (i.e. thorough understanding of the internal structure of concepts).

differ, then although of course it is perfectly possible (and indeed highly *likely*) that they *might* co-refer we would nevertheless lack a *logical* guarantee that they do *in fact* co-refer, that they pick the same concept, the same fundamental notion. In short, that they are demonstrably *subject-matter-preserving* (the danger, that is, is that distinct senses, even if similar, *may* define significantly distinct concepts).⁵⁴

Secondly, and taking up this last point, recall that the senses at stake in our investigation are in fact *definitions* of the fundamental concepts of arithmetic, such as e.g. the concept of number.⁵⁵ These definitions are intended to make manifest the logical content of those notions. Accordingly, if we tried to explain misty understanding in terms of a family of sufficiently similar senses, we would run afoul of Frege's (1893/2013, II.iii §60, 73) fulminations against giving *double explanations* of the *same* sign. Such explanations would give rise to "doubt whether the definitions contradict each other".⁵⁶ We would require, Frege adds, *at least* a proof that no contradiction could be engendered by the diverging definitions—we would need a fresh warrant for each definition and for each further modification we give as our understanding gets progressively more secure.

Crucially, Frege's suggested repair against double explanations is to treat the signs thus defined as *different* signs.⁵⁷ So, in effect, positing sufficiently similar senses amounts to saying that they attach to different signs altogether, notwithstanding the surface identity of those

⁵⁴The passage in Frege (1914b, 240) is puzzling. Frege first states that senses are "not to be identified with the content of anyone's consciousness", largely because of their synchronic unfathomability (e.g. we can never apprehend all at once the sense attached to the expression 'integral'). He then adds that despite this one can confidently deploy incompletely grasped senses in correctly drawn inferences. But if we can only access a *segment* of a sense at a given point in time because of our cognitive limitations, what *logical* guarantee do we have that the segment we accessed is a segment of *that very concept* that we can *never* grasp in its entirety?

⁵⁵See e.g. part IV of Frege (1884). A passage in the very late Frege (1924/25, 271) is, again, admittedly rather puzzling, since there Frege tells us that *function* is a logically simple notion that resists analysis and can only be elucidated (see also 1893/2013, 4 and 1906, 300). At least for the case of *number*, however, he seems to have continued to think that it *is* analysable by logical means, or else it would really be hard to make *any* sense of his entire project. Dummett (1981, ch. 14; 1991b, ch. 14) are useful. See also Weiner (2001) for a different assessment of Frege's position.

⁵⁶Frege (1893/2013, II.iii 141) expresses similar worries. See also Frege (1980, 113-15).

⁵⁷This is paralleled in Frege's (1914a, 213) remarks against polysemy: a unique sense must be attached to an expression fit for scientific purposes. It does not suffice that it be determinate *which* separate senses attach to a single expression: in scientific discourse, senses and signs must be in a 1-1 correspondence. See also Frege (1906, 303) and the discussion in May (2006).

signs. In fact, this move would amount to a change of *language*. And this, again, would threaten, at the very least, the claim that logicism is subject-matter preserving.

Thirdly, if senses are the only locus of normative guidance, and if we have two competing senses with respect to what looks like a single fundamental notion, then it becomes unclear what provides normative guidance when it comes to adjudicating between those two senses (or when it comes to settling *whether* they are co-referential, or intensionally equivalent, or free from contradiction once admitted to the same language). And indeed, in considering issues of this kind, Frege could do no better than appeal to self-evidence to settle the question of whether or not two senses are equivalent.⁵⁸

There is also a fourth problem with this response. As I have already pointed out, Frege (1914a, 221) is quite explicit that in incomplete understanding what one lacks is *the provision of a system*—the system is the genuine *mist-buster*. That is to say, partial grasp of sense is indeed, by Frege's lights, grasp that *misses out* (or misconstrues) some crucial component. The target one hits is genuinely wrong, defective, incomplete (or unfocused). So, the two senses *cannot* be sufficiently similar: for logicism to have any significance, the "received sense" must be *gravely defective*. Indeed, Frege's chief point of pride was to have shown the sort of *substantial* addition to current mathematical practice that was required to make it scientifically respectable (namely, the construction of a system that would allow full specification of the senses of the fundamental notions).

Now, one might complain that I am overlooking another way of looking at our predicament. Why can't we say that C-STABILITY *and* S-STABILITY both hold, that the *language* we use stably expresses senses that stably pick stable concepts even if *we* lack proper access to those senses? Nothing I have said so far has ruled out this option. After all, the problems I raised were all triggered by the attempt to hold on to G-STABILITY in some way or other. The complaint is, I think, well-taken. But the picture that would emerge is one whereby users and the language they employ would be wholly disconnected. If the considerations I rehearsed in this section and the previous one are

⁵⁸Frege (1914a, 210–211). As I said earlier, one could argue that something like reflective equilibrium would guide us in picking the correct sense. But given that the senses involved are located at the foundational level of logicity, the familiar problem is that it is unclear which background logic one *could* use to adjudicate between the competing options.

sound, it seems as if there can be *no engagement at all* between expressed and grasped senses as long as the former are taken to be stable. And if so, then the pressing question is: would there be any *point* to jointly holding on to the two weaker forms of STABILITY other than as a blind commitment to an article of unsupported dogma? Sense was supposed to give an answer to the problem of normative guidance, the problem of specifying how we can access inference-guiding reasons. And unless there is *some* connection between the content of the expressed senses and our grasp thereof, mathematical reasoning cannot be deemed to be rationally grounded. Accordingly, I think that the complaint, while strictly speaking correct, can be dismissed.

With this wrinkle out of the way, it thus seems to me that neither an appeal to the richer notion of sense provided by implicit conceptions, nor to the only Frege-admissible weakening of G-STABILITY would succeed in making available a satisfactory escape from the tension.

Implicit conceptions, however, had one promising feature, namely, they seemed to provide a useful theoretical gloss on Frege's (1914b, 239) remarks concerning Weierstrass's confused grasp of number:

He had an inkling of what number is, but it was a very unclear one; and drawing on that inkling he improved and completed what should have instead followed from his definitions.

In the next section I wrap things up by considering another way of glossing what Frege may have meant by inkling (*Ahnung*) here. It is a way that is revealed by looking at what Frege says regarding how we can define concepts. And it allows us to move away from what I think is the key problem faced by all the suggestions I considered in this section, namely, that while we have a fairly clear purchase on the idea of object-tracking (i.e. keeping track of objects whose properties may be opaque to us) it instead seems extremely hard to make stable the idea of concept-tracking (i.e. keeping track of *concepts* whose defining marks may be opaque to us).

4 Saving STABILITY?

How, then, do we define concepts? Frege (1893/2013, xiv) tells us that a concept is defined “by stating what property an object must have in order to fall under it”. There is, thus, *no direct way* to talk about a

concept. We always need to go via the detour of talking about properties of objects, and thereby giving what Frege calls the characteristic marks of the concept.⁵⁹ In the case of the concept *number*, for example, we say what *it* is by saying what numbers are, what essential properties *they* have. That's how we give the reference of the concept-word 'number(ξ)'. The reference itself is a function that is "sensitive"—just as we, *qua* appropriately primed rational thinkers, must be—to the properties as given by the definition (the conception). It is a function that maps numbers to the True and everything else to the False. And how do we give the *sense* of that concept-word, then, how do we spell out our conception of the concept? That is, how is that function given to us? Well, again, in the same way in which we have specified the concept: by listing the properties objects have to satisfy so as to be mapped to the True. So, the sense, the conception, is nothing but the definition of the concept in the sense above (it is the structured list of the relevant properties: no more, no less).

With that in mind, we could now try to see whether there is a version of STABILITY available to the Fregean formulated solely in terms of what I want to call SUBSUMPTION-STABILITY (SUB-STABILITY) that might ensure that logicism remains, to some extent at least, a subject-preserving enterprise. What remains stable is neither the conceptions nor the concepts, but rather the objects subsumed under varying concepts.⁶⁰ The idea behind SUB-STABILITY, that is, is that pre-logicist thinkers got hold of the right objects under the wrong guise, that they were thinking of *them* all right but by means of the wrong concepts. Those thinkers were wrong about which properties to predicate of those objects, and *hence* they thereby determined the wrong concept via the wrong conception. Or rather, the conception of *that* concept

⁵⁹Frege (1884/1987, §53; 1980, 93; 1997, 190; 1892c, 179; 1893/2013, xiv). At the time of *Begriffsschrift*, Frege also distinguished a different type of concept formation: rather than via definition, some concepts are attained via *extraction* from the content of a judgement (1880/1881a, 33–35)—see Sullivan (2004, §3.3.4) for discussion. I concentrate on the procedure given in *Grundgesetze* for a number of reasons. First, it represents Frege's actual practice in his mature work (concept-extraction is a form of implicit definition, that is, rather than his preferred method of giving explicit definitions for the fundamental notions). Secondly, although some concepts and properties may best be attained via concept-extraction, the concept and properties thus generated can then enter into an explicit definition as *analysans* of pre-existing concepts. Thirdly, even in the case of concept-extraction we start from a judgement that is *about* objects characterised as falling under the concept (or concepts subsumed under the so extracted concept).

⁶⁰Yes, I am now espousing what earlier on (fn. 50) I treated as an unacceptably radical approach to the problem. The difference is that one is now insisting on the stability of the target objects.

(the defective one) was correct, as far as it went, but as a conception of *the* concept that supposedly characterises the target objects it was the wrong conception since it left out essential properties of those objects. Accordingly, if different thinkers attribute different properties to numbers then *both* their respective conceptions and concepts *must* diverge.⁶¹

Given Frege's realism, what *can* instead plausibly be said to remain stable is *the objects being talked about*, the numbers themselves. And these are the objects subsumed under the different concepts employed by different thinkers.⁶² The commonality of subject-matter is thus preserved, but at the level of object-reference, as it were, not of sense or of concept-reference. In fact, what differs is precisely the *conceptual filter* through which those objects are apprehended.⁶³ But still, speakers with a different conceptual repertoire can nevertheless be said to share a common domain of discourse, although they may use different words (and languages) to talk about it—the incapacitating plunge into intellectual confusion that so worried Frege is thus prevented.

One useful way of putting the contrast between pre-logicist (incomplete) understanding and logicist (complete) understanding is to say that while e.g. Weierstrass was right about the *computational* properties of numbers, he was missing out the *justificatory* properties of *number* (the properties which determine under what conditions thoughts about numbers are justified). Weierstrass had a sound inkling, as Frege said, of what numbers were (without question, he

⁶¹One might dig one's Fregean heels and insist that Weierstrass's inkling was directed at the concept-as-function that Frege was to make precise. But how can attributing to numbers the property of being made up of trains and books (as Frege (1914a, 217–218) claims would follow from Weierstrass's definition) be taken to show that Weierstrass had the *same* concept as Frege in his intellectual crosshair? How can one be *so wrong* in one's conception of the *concept* that one is allegedly tracking?

⁶²For the case of *function* and its cognates the account needs a slight modification. It's not the objects subsumed under the first-order concept, but the concepts that fall *within* other higher-level concepts. See e.g. Frege (1892a, 190; 1914a, 213). The modified account would, I think, be parasitic on the one given for first-level concepts.

⁶³As Burge (1993, 296) would say, the speakers had "other epistemic hooks on the entities" to keep them on the right track despite the wrong explications they gave themselves. Note also that the move to an object-based conception of normative guidance would tally with Frege (1892b, 158) remark that "comprehensive knowledge of *the reference* would require us to be able to say *immediately* whether any given sense belongs to it" (my emphases). In other words, and as Kripke (2001) has argued, there *is* a backward road from reference to sense, and I think that road is actually what can give us a true grip on normative constraints.

was thinking about *them*), but still in an inchoate state. In fact, we could say that he was unable to articulate (or indeed apprehend) the justificatory properties because of an uncertain grasp of the representational properties of the concept *number* (an uncertain grasp of what numbers really are, of what having representationally correct thoughts about them would involve). Nevertheless he had *some* grasp of these latter properties, enough to allow him to *fix reference* correctly for any occurrence of ‘number’ as a noun-word, but, crucially, not as a concept-word.

By contrast, logicism enables full grasp of justificatory properties by making available all (or more) representational properties, by disclosing the real nature of numbers, by revealing what they are and how they are given to us: they are logical objects apprehended through a logical source of cognition. It was this that Weierstrass had missed and it is this, and this alone, that can finally and appropriately complete our understanding.⁶⁴ And the missing element that was unavailable to Weierstrass, Frege (1914a, 221, 242) tells us, was the construction of a system. In the absence of a system of proof where all assumptions, axioms and rules of inference are clearly laid out, there was no way to test one’s “logical convictions” (1893/2013, xxvi), no way to “bring to light any flaw in concept formation” (1880/1881a, 35); there was no way, that is, of vindicating one’s mathematical claims as fully justified in the light of the normative guidance made available and *exemplified* by the system.⁶⁵

The rational intelligibility of the practice is then preserved *to the extent* (but no more than to *that* extent) that the various conceptions determining various concepts all pick out the same objects as subsumed under those concepts. What the proofs to be found in Weierstrass’ and Frege’s efforts show is that only numbers can satisfy what both Frege and Weierstrass establish about them in their different ways.⁶⁶

⁶⁴I think that there is a natural way of explaining the significance of the absence of justificatory properties in pre-logicist thinking along externalist lines. On that proposal we view the content of the fundamental notions as being partly *about* the system to be developed in the appropriate *Begriffsschrift*, as partly determined by the structural properties of that system. So construed, then, mathematical content is anti-individualist in the desired manner (thinker-opaque and partly determined by external factors). I develop this proposal in an unpublished manuscript still in progress.

⁶⁵As should be clear, under the current proposal, that guidance is *channeled* through the provision of a system but is in the first instance provided by the objects themselves, not by the received senses of the fundamental notions.

⁶⁶This is assuming categoricity, which Frege’s system does satisfy. Note that SUB-STABILITY is

And so, on the suggestion that we are exploring, we should say that properly speaking the normative guidance is provided by the objects themselves, by the demand that one be faithful to *their* properties, as one progressively uncovers them. In short, we are normatively bound by objects, not concepts. It's objects that keep us honest, even though we may be using different concepts to keep track of those objects across different temporal stages of the practice.⁶⁷

In this respect, it was Leibniz who was on the right track. As he tells us in *Discourse on Metaphysics*, §24, confused understanding is the ability to single out the correct objects “without being able to say what [their] properties consist in”. The transition to mature understanding occurs when one “can explain the evidence [one] is using” in so singling out the correct objects, when one is in a position to supplement representational properties with justificatory ones.⁶⁸ Or equivalently: when one is able to *see* the (adequately framed) representational properties *as* justificatory ones.⁶⁹ To begin with, however, our

NOT EXTENSION-STABILITY. EXTENSION-STABILITY would be another way to go. Neither senses, nor concepts are stable, all that is required is that extensions be stable. And one might insist that for Frege the concepts would then be the same. But that is precisely what Frege (1895, 122) denies that one can *say*: “sameness” has no applicability to concepts. The most we can say, according to Frege is that two concept-words are *co-referential* if their two extensions coincide, but even this is very heavily hedged indeed (it is all right to say so under a very peculiar understanding of ‘refer’, Frege tells us). At any rate, what is important in terms of normative guidance is that it be the *objects* that guide our reasoning, and not the mistaken concept adopted (even if co-extensional with the correct one). With SUB-STABILITY we get something like sameness of concepts *from below*, as it were; we get there because of stability at the level of the objects, that is. And hence, I think SUB-STABILITY is not C-STABILITY—certainly it isn't if we take the latter thesis in an intensional sense (see e.g. Frege 1895, 118).

⁶⁷It hardly needs saying that this is mere hand-waving. One way to make it less suggestive would be to say that throughout mathematical history we compare attributions of different ranges of properties to objects and we verify which properties fit the objects best. This is the sense in which objects are the only stable element in conceptual history. I doubt we can say anything much more precise than this.

⁶⁸This, I think, is the correct way of spelling out Leibniz's notion of grasp of a clear but confused idea or rather of an idea that is known clearly but confusedly. There is no doubt as to which idea it is, but there is a doubt about the sort of idea that it is. Within a Fregean framework, however, the notion of a clear but confused sense/concept is, if I am right, wholly incoherent. More generally, I believe that *any* theory of concepts faces a problem about making sense of conceptual opacity that one doesn't face when trying to make sense of objectual opacity in the sense discussed in the text.

⁶⁹A further historical analogy could be with Kant's contrast, drawn in the *Critique of Judgment*, between determinative (i.e. subsumptive) judgement and reflective judgement, judgement that goes in search of a universal starting from a particular. Determinative judgements are thus concept-based, while reflective ones are object-based. What I'm sketching in the text is, in effect, a reflective epistemology of the basic notions of arithmetic. We start with the objects and eventually

recognitional capacities with respect to those objects are rudimentary, defective, but still aimed at those objects, and they are defective precisely because, as Gödel (1951, 321) suggested, our knowledge of what I want to call the *deeply analytic* content of the fundamental notions of arithmetic “may be as limited and incomplete as that of the world of things”.

Now, the question that inevitably will arise at this point is: how can SUB-STABILITY be made to cohere with the passages from Frege that I discussed at the end of §1, the passages where, as I insisted then, Frege claimed that it was senses that had to remain stable as a condition for the rational intelligibility of mathematical practice, that any weaker form of stability would not have sufficed for that purpose? Haven't we simply given up on that demand? Haven't we not so much *solved* the tension as *dodged* it altogether by throwing away any sufficiently robust version of STABILITY? And the answer is, yes, I'm afraid that we have. However, if the considerations I brought to bear in §3 are as decisive as I think they are then SUB-STABILITY is really the best we can hope for on behalf of Frege. We have to admit, that is, that concepts *do* change across history not just in the empirical domain but in the mathematical domain too. But that doesn't have to condemn our practice to outright incoherence. Perhaps we should conclude that this is just one more respect in which our practice can merely *approximate* the stringent requirements of logic, no doubt much to the further despair of Frege.⁷⁰

Still, despite these concessions (or maybe because of them) one might want to raise the complaint that e.g. SUB-STABILITY gives rise to further, insurmountable problems. For instance, how can SUB-STABILITY account for empty concepts, concepts under which no objects fall? How do we track those object-less concepts, since all that we have are the (unsatisfiable) specifications contained in our definitions?⁷¹ The worry here, clearly, is that SUB-STABILITY lacks full generality, that it is a purely *ad hoc* answer, an awkwardly implemented patch that would lack justification in other cases.

find the concepts. I'm only too aware that this would reverse Frege oft-repeated dictum that we generate objects from concepts, but I think that a commitment to a concept-based epistemology is incompatible with *any* form of STABILITY.

⁷⁰Tarski (1935, 267; 1944, 347) reached the same conclusion with respect to the concept of truth.

⁷¹I am grateful to an anonymous referee for raising this objection and for expressing more general doubts regarding SUB-STABILITY.

The appropriate response to this objection, I think, is to appeal to what Frege says in *Grundlagen* §74: we know *of* any object that it doesn't fall under any empty concept. So SUB-STABILITY is in fact vacuously satisfied by every object with respect to empty concepts. To track *any* object is to know, of that object, that it doesn't fall under any empty concept. Concept-tracking is based on object-tracking even in the case of empty concepts, that is. I think the answer is technically sound but I grant that it is unlikely to pacify the objector who found SUB-STABILITY *ad hoc* in the first place.

Another complaint could be that all that SUB-STABILITY provides is a different way of drawing the familiar distinction between concept-possession and concept-mastery, and hence a different way of explaining the transition between incomplete and complete understanding. With SUB-STABILITY on hand, that is, one could just say that to (merely) possess a concept is to have a clear idea as to the objects that fall under it, while having only an indistinct idea as to *why* they so fall.⁷² To master a concept is instead to have knowledge (either explicit or inexplicit) of its internal articulation, or as Frege would put, of its *organic* composition, and *thereby* know why certain objects, and no others, fall under it. If this account could somehow be made stable, we would then have a means of arguing for STABILITY *sans phrase* after all—SUB-STABILITY would just be an epistemic stepping stone towards establishing that the concepts themselves have remained stable. I have however argued that if I am right about sense-individuation, any unclarity with regard to the *why*-properties of a concept will entail that one is in fact grasping a different concept. I therefore think that the *attempt* to save STABILITY via SUB-STABILITY fails too. Nonetheless, I would also insist that reflection on SUB-STABILITY brings out important features of our apprehension of abstract objects and concepts.

⁷²Of course, strictly speaking I must deny that here 'it' refers back to the same concept! To those familiar with the social externalist picture, it will be obvious that I also have to deny that Putnam possesses the concept BEECH. My view is that Putnam is *linguistically* competent with that concept, in the sense of knowing certain true *propositions* involving the concept, while not possessing the concept itself.

Conclusion

To sum matters up: I have argued that by Frege's own lights there can be no such thing as misty grasp (or partial understanding) of senses and concepts because both types of (unsaturated) entities (conceptions and functions) are individuated *solely* in terms of the conceptual material that constitutes them. Any mistake regarding the individuating marks of an entity of that kind will entail getting hold of something else instead (or of nothing at all, or at least not of an entity of *that* specific, and austere, kind). By contrast, there is no difficulty in the idea that we may succeed in thinking of objects of a certain kind even when we do so in a tentative manner; in the case of objects, that is, being mistaken about some of their properties won't impair our capacity to think about them.

From a Fregean perspective, then, the phenomenon of incomplete understanding cannot be explained in terms of the stable targeting of senses and concepts because the individuation conditions for those entities do not admit of degrees of understanding.

Accordingly, it seems to follow that the rationality of mathematical progress cannot be traced back to (un)reflective observance of the normative dictates encoded in senses, for such senses could *not* remain stable under variations in the degree of understanding attained by thinkers (since any such variation, assuming that it *could* occur, would affect the *content* of those senses).

What can plausibly be said to remain stable is the objects we talk about. We are right about which objects populate the mathematical domain, but we are wrong, in our pre-logicist state, about their justificatory properties, the properties that can ground our statements about those objects. We use senses and concepts that are defective in terms of fully capturing those properties but adequate for fixing reference for the singular terms involved.

I also argued that SUB-STABILITY is not as problematic as G-STABILITY because the notion of object-tracking is not as problematic as that of concept-tracking. This is so because a deictic model of sorts is available even for the case of abstract objects (we can hook onto them through their behaviour in proof and computation). By contrast, no model involving deixis of any kind can coherently be ap-

plied to Fregean senses and concepts.⁷³

I have granted that there are ample reasons to be dissatisfied with SUB-STABILITY however. Above all, although we would be preserving one aspect that is very dear to Frege, namely, his commitment to a realist conception of mathematical objects, it is unquestionable that by adopting SUB-STABILITY we would also be jettisoning away much that was of value to Frege's highly distinctive epistemology of logic, in particular his insistence on the central epistemological role of concepts.

More awkwardly still, if STABILITY with respect to sense is rejected, it then becomes wholly unclear what the proper role of sense is within Frege's framework. If senses *cannot* provide normative guidance, that is, what is the *point* of their introduction? Also, if senses cannot provide normative guidance, what does? Without relying on senses, what can guide the transition from partial to full understanding? There is nothing, at least nothing *concept-like*, that can rationally guide that transition, nothing that can provide accessible reasons to justify our inferential moves. All that we have is objects. But objects, at least within a Fregean framework, are *always* given to us as falling under some concept or other and hence are always mediated by some sense or other. And yet senses cannot remain stable across mathematical progress. We seem stuck in a circle.

Unfortunately, I have no answer to offer. Fregean senses, as far as I can tell, cannot be the *repositories* of normativity. They can do no more than exhibit *post facto* the normative strictures that are constrained by the objects themselves and that we progressively uncover as we refine the logico-mathematical system that we employ to study mathematical structures. Unless we can make sense of the notion of tracking abstract objects without the direct deployment of concepts, then, the problem of normative guidance is here to stay.

I started by quoting the introduction to *Grundlagen*. Let me close by returning to that passage. The misty understanding predicament

⁷³It is rather intriguing that in the first mist passage, the one in the introduction to *Grundlagen*, Frege compares the case of the bumbling mathematician to that of Columbus catching the first "dubious glimpse" of what he mistakenly thought was India. The idea that we can similarly be mistaken about which concept we got hold of but that we nonetheless could *already* be tracking the correct one is precisely what I am resisting in this paper. The model of object-tracking does not transfer seamlessly to that of concept-tracking, at least with respect to *a priori* concepts. It is thus a mistake to think that the model of incomplete understanding that Putnam and Burge have argued for with respect to empirical concepts can seamlessly transfer to the logico-mathematical domain.

that Frege sketched there was said to arise *as we approach the sources* of the concepts under scrutiny. Those sources can be nothing other than the logical objects that are epistemically accessible to us *as the objects they are* only through reflection on properly structured concepts. And the proper articulation of those concepts is, to paraphrase the foreword to *Grundgesetze*, one more test of our logical convictions. That is to say, the proper articulation of the fundamental notions of arithmetic provides the proper articulation of our conception of the logical objects themselves. But until we have articulated the concepts clearly *and* distinctly all that we have is an uncertain grasp of the *objects*. Properly speaking, there is no grasping *the* correct concept until our grasp of the objects is itself fully firmed up. And on the way to the sources, to that proper grasp, we have no other *stable* help than what the objects themselves can provide.

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References

- Beaney, Michael 1996. *Frege: Making Sense*. London: Duckworth.
- Bell, David 1987. "Thoughts." *Notre Dame Journal of Formal Logic* 28 (1), 36–50.
- Blanchette, Patricia A. 2012. *Frege's Conception of Logic*. Oxford: Oxford University Press.
- Boghossian, Paul 2000. "Knowledge of Logic." In: *New Essays on the A Priori*, edited by Paul Boghossian & Christopher Peacocke, Oxford: Clarendon Press, 229–254.
- 2001. "How are Objective Epistemic Reasons Possible?" In: Boghossian (2008), 235–266.
- 2003. "Blind Reasoning." In: Boghossian (2008), 267–287.
- 2008. *Content and Justification*. Oxford: Clarendon Press.
- Buehler, Denis 2014. "Incomplete Understanding of Complex Numbers. Girolamo Cardano: A Case Study in the Acquisition of Mathematical Concepts." *Synthese* 191, 4231–52.
- Burge, Tyler 1979. "Sinning Against Frege." In: Burge (2005), 213–241.
- 1984. "Frege on Extensions of Concepts, From 1884 to 1903." In: Burge (2005), 273–298.
- 1989. "Wherein is Language Social?" In: Burge (2007), 275–290.

- 1990. “Frege on Sense and Linguistic Meaning.” In: Burge (2005), 242–269.
- 1993. “Concepts, Definitions, and Meaning.” In: Burge (2007), 291–306.
- 1998. “Frege on Knowing the Foundation.” In: Burge (2005), 317–355.
- 2003a. “Concepts, Conceptions, Reflective Understanding: Reply to Peacocke.” In: *Reflections and Replies. Essays on the Philosophy of Tyler Burge*, edited by Martin Hahn & Bjørn Ramberg, Cambridge, MA: The MIT Press, 383–396.
- 2003b. “Concepts, Conceptions, Reflective Understanding: Reply to Peacocke.” In: Burge (2013), 521–533.
- 2003c. “Reply to Benjamen: ‘Thought Experiment and Semantic Competence’.” In: Burge (2013), 516–520.
- 2005. *Truth, Thought, Reason. Essays on Frege*. Oxford: Clarendon Press.
- 2006. “Postscript to ‘Individualism and the Mental’.” In: Burge (2007), 151–181.
- 2007. *Foundations of Mind. Philosophical Essays, Volume 2*. Oxford: Clarendon Press.
- 2012. “Living Wages of Sinn.” In: Burge (2013), 556–593.
- 2013. *Cognition Through Understanding. Self-knowledge, Interlocution, Reasoning, Reflection*. Oxford: Oxford University Press.
- Diamond, Cora 1984. “What Does a Concept Script Do?” In: Wright (1984), 158–183.
- Dummett, Michael 1973. *Frege. Philosophy of Language*. London: Duckworth, 2nd edition.
- 1981. *The Interpretation of Frege’s Philosophy*. London: Duckworth.
- 1986. “Frege’s Myth of the Third Realm.” In: Dummett (1991a), 249–262.
- 1989. “More about Thoughts.” In: Dummett (1991a), 289–314.
- 1990a. “Frege and Husserl on Reference.” In: *The Seas of Language*, Oxford: Clarendon Press, 224–229.
- 1990b. “Thought and Perception: The Views of Two Philosophical Innovators.” In: Dummett (1991a), 262–288.

- 1991a. *Frege and Other Philosophers*. Oxford: Clarendon Press.
- 1991b. *Frege. Philosophy of Mathematics*. London: Duckworth.
- Eklund, Matti 2007. “Meaning-Constitutivity.” *Inquiry* 50 (6), 559–574.
- Evans, Gareth 1981. “Semantic Theory and Tacit Knowledge.” In: *Collected Papers*, Oxford: Clarendon Press, 322–342.
- Frege, Gottlob 1879. “*Begriffsschrift*.” In: *From Frege to Gödel*, edited by Jean van Heijenoort, Cambridge, MA: Harvard University Press, 5–82.
- 1879/1891. “Logic.” In: Frege (1979), 1–8.
- 1880/1881a. “Boole’s logical Calculus and the Concept-Script.” In: Frege (1979), 9–52.
- 1880/1881b. “Booles rechnende Logik und die Begriffsschrift.” In: Frege (1983), 9–52.
- 1884. *The Foundations of Arithmetic*. Oxford: Blackwell, (1950) 2nd rev. edition.
- 1884/1987. *Die Grundlagen der Arithmetik*. Stuttgart: Philipp Reclam.
- 1891a. “Function and Concept.” In: Frege (1984), 137–156.
- 1891b. “On the Law of Inertia.” In: Frege (1984), 123–136.
- 1891/1892. “On the Concept of Number.” In: Frege (1979), 72–86.
- 1892a. “On Concept and Object.” In: Frege (1984), 182–194.
- 1892b. “On Sense and Meaning.” In: Frege (1984), 157–177.
- 1892c. “Review of Georg Cantor, *Zur Lehre vom Transfiniten: Gesammelte Abhandlungen aus der Zeitschrift für Philosophie und philosophische Kritik*.” In: Frege (1984), 178–181.
- 1892/1895. “Ausführungen über Sinn und Bedeutung.” In: Frege (1983), 128–136.
- 1893/2013. *Basic Laws of Arithmetic*. Oxford: Oxford University Press.
- 1895. “Comments on *Sinn und Bedeutung*.” In: Frege (1979), 118–125.
- 1897. “Logic.” In: Frege (1979), 118–151.
- 1898/1899. “Logical Defects in Mathematics.” In: Frege (1979), 157–166.
- 1899–1906. “On Euclidean Geometry.” In: Frege (1979), 167–169.

- 1906. “On the Foundations of Geometry: Second Series.” In: Frege (1984), 293–340.
- 1906a. “Introduction to Logic.” In: Frege (1979), 185–196.
- 1906b. “A Brief Survey of my Logical Doctrines.” In: Frege (1979), 197–202.
- 1914a. “Logic in Mathematics.” In: Frege (1979), 203–250.
- 1914b. “Logik in der Mathematik.” In: Frege (1983), 219–270.
- 1918-19a. “Negation.” In: Frege (1984), 373–389.
- 1918-19b. “Thought.” In: Frege (1984), 351–372.
- 1919. “Notes for Ludwig Darmstaedter.” In: Frege (1979), 253–257.
- 1923. “Compound Thoughts.” In: Frege (1984), 390–406.
- 1924/25. “Sources of Knowledge of Mathematics and the mathematical natural Sciences.” In: Frege (1979), 267–274.
- 1979. *Posthumous Writings*. Oxford: Blackwell.
- 1980. *Philosophical and Mathematical Correspondence*. Oxford: Blackwell.
- 1983. *Nachgelassene Schriften*. Hamburg: Felix Meiner Verlag.
- 1984. *Collected Papers on Mathematics, Logic, and Philosophy*. Oxford: Blackwell.
- 1997. “On Concept and Object.” In: *The Frege Reader*, edited by Michael Beaney, Oxford: Blackwell, 181–193.
- Gödel, Kurt 1951. “Some Basic Theorems on the Foundations of Mathematics and their Implications.” In: *Collected Works*, edited by Solomon Feferman, Oxford University Press, volume III. Unpublished Essays and Lectures, 304–323.
- Hart, H.L.A. 1961/2012. *The Concept of Law*. Clarendon Law Series, Oxford University Press, 3rd edition.
- Higginbotham, James 1998. “Conceptual Competence.” In: Villanueva (1998), 149–162.
- Horwich, Paul 1998. *Meaning*. Oxford: Clarendon Press.
- Kremer, Michael 2010. “Sense and Reference: the Origins and Development of the Distinction.” In: Potter & Ricketts (2010), 220–292.
- Kripke, Saul 2001. “Frege’s Theory of Sense and Reference. Some Exegetical Notes.” In: *Philosophical Troubles. Philosophical Papers Vol. 1*, Oxford: Oxford University Press, 254–291.

- May, Robert 2006. "The Invariance of Sense." *Journal of Philosophy* 103 (3), 111–144.
- McDowell, John 2005. "Evan's Frege." In: *Having the World in View. Essays on Kant, Hegel, and Sellars*, Cambridge, MA: Harvard University Press, 163–185.
- McGee, Vann 1985. "A Counterexample to Modus Ponens." *Journal of Philosophy* 82 (9), 462–471.
- Peacocke, Christopher 1992. *A Study of Concepts*. Cambridge, MA: The MIT Press.
- 1998a. "Implicit Conceptions, the *A Priori*, and the Identity of Concepts." In: Villanueva (1998), 121–148.
- 1998b. "Implicit Conceptions, Understanding and Rationality." In: Villanueva (1998), 43–88.
- Pedriali, Walter B. 2012. "Solvitur Ambulando. Meaning-constitutive Principles and the Inscrutability of Inference." In: *Insolubles and Consequences. Essays in Honour of Stephen Read*, edited by Catarina Dutilh Novaes & Ole Thomassen Hjortland, London: College Publications, 163–185.
- 2017. "When Logic Gives Out. Frege on Basic Logical Laws." In: *Essays on Frege's Basic Laws of Arithmetic*, edited by Marcus Rossberg & Philip A. Ebert, Oxford: Oxford University Press.
- Pettit, Dean 2002. "Why Knowledge is Unnecessary for Understanding Language." *Mind* 111 (443), 519–49.
- Potter, Michael & Ricketts, Tom, editors 2010. *The Cambridge Companion to Frege*. Cambridge: Cambridge University Press.
- Quine, Willard V. 1970. *Philosophy of Logic*. Cambridge, Mass.: Harvard University Press, 2nd (1986) edition.
- 1972. "Methodological Reflections on Current Linguistic Theory." In: *Semantics of Natural Language*, edited by Donald Davidson & Gilbert Harman, Dordrecht: D. Reidel Publishing Company, 442–454.
- Rawls, John 1971/1999. *A Theory of Justice*. Oxford: Oxford University Press, revised edition.
- Rey, Georges 1998. "What Implicit Conceptions are Unlikely to Do." In: Villanueva (1998), 93–104.
- Russell, Bertrand 1914/1993. *Our Knowledge of the External World*. Routledge.

- Schiffer, Stephen 1998. "Doubts about Implicit Conceptions." In: Villanueva (1998), 89–91.
- Smith, Sheldon R. 2015. "Incomplete Understanding of Concepts: The Case of the Derivative." *Mind* 124.
- Sullivan, Peter 2004. "Frege's Logic." In: *Handbook of the History of Logic*, edited by Dov M. Gabbay & John Woods, Amsterdam: Elsevier, volume 3. The Rise of Modern Logic: From Leibniz to Frege, 671–762.
- Tarski, Alfred 1935. "The Concept of Truth in Formalized Languages." In: *Logic, Semantics, Metamathematics*, Indianapolis, Indiana: Hackett. 2nd (1983) edition, 152–278.
- 1944. "The Semantic Conception of Truth and the Foundation of Semantics." *Philosophy and Phenomenological Research* 4 (3), 341–376.
- Textor, Mark 2011. *Routledge Philosophy Guidebook to Frege on Sense and Reference*. London and New York: Routledge.
- Villanueva, Enrique, editor 1998. *Concepts*, volume 9 of *Philosophical Issues*. Atascadero, CA: Ridgeview.
- Weiner, Joan 2001. "Theory and Elucidation. The End of the Age of Innocence." In: *Future Pasts: The Analytic Tradition in Twentieth Century Philosophy*, edited by Juliet Floyd & Sanford Shieh, Oxford: Oxford University Press, 43–65.
- 2010. "Understanding Frege's Project." In: Potter & Ricketts (2010), 32–62.
- Wiggins, David 1980/2001. *Sameness and Substance Renewed*. Cambridge: Cambridge University Press.
- 1984. "The Sense and Reference of Predicates: A Running Repair to Frege's Doctrine and A Plea for the Copula." In: Wright (1984), 126–143.
- 1994. "Putnam's Doctrine of Natural Kind Words and Frege's Doctrines of Sense, Reference, and Extension: Can They Cohere?" In: *Reading Putnam*, edited by Peter Clark & Bob Hale, Oxford: Blackwell, 201–215.
- Wright, Crispin, editor 1984. *Frege. Tradition & Influence*. Oxford: Blackwell.
- Wright, Crispin 1986. "Theories of Meaning and Speakers' Knowledge." In: *Realism, Meaning and Truth*, Oxford: Blackwell. 1st edition, 204–

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