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Homotopy and Path Integrals in the Time-dependent Aharonov-Bohm Effect

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Abstract For time-*independent* fields the Aharonov-Bohm effect has been obtained by idealizing the coordinate space as multiply-connected and using representations of its fundamental homotopy group to provide information on what is physically identified as the magnetic flux. With a time-*dependent* field, multiple-connectedness introduces the same degree of ambiguity; by taking into account electromagnetic fields induced by the time dependence, full physical behavior is again recovered once a representation is selected. The selection depends on a single arbitrary time (hence the so-called holonomies are not unique), although no physical effects depend on the value of that particular time. These features can also be phrased in terms of the selection of self-adjoint extensions, thereby involving yet another question that has come up in this context, namely, boundary conditions for the wave function.

Keywords Aharonov-Bohm effect · Time-dependence · Homotopy · Path integral · Essential self-adjointness · Gauge theory · Vector potential · Holonomies

1 Introduction

We consider two aspects of the Aharonov-Bohm (AB) [1–3] effect: the first is the idealization in which the solenoid is removed from the configuration space. The sec-

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ond is the introduction of time dependence into the field of the solenoid. For the time-*independent* case the enrichment of the homotopic character of the configuration space allows the replacement of the vector potential by a single number, a wave-function boundary condition that can also be looked upon as the selection of a self-adjoint extension of the Hamiltonian [4, 5]. That number corresponds physically to the flux through the solenoid, a region inaccessible—in this idealization—to the particle. In the course of dealing with this question we confront one of the vexing problems of the AB effect: should the wave function be single-valued or not? This was discussed at length by Roy and Singh [6]. We hope that our development will cast further light on this issue. For the time-*dependent* case, fields (not just potentials) appear in the accessible region and the motion of the particle is more complicated. We show however that the ambiguity induced by the idealized removal of the solenoid is still resolved through a single number, in this case the value of the flux at an arbitrary time. Changes in the flux can be deduced from fields in the accessible region.

In the philosophical literature gauge theories have recently attracted considerable attention [7, 8], in particular because of that nagging notion, "reality," which we remark, has not been out of place in the writings of physicists [9, 10]. Gauge invariance is a particularly blatant way of *not* fixing a quantity, which has led to doubts about the vector potential as a real, causally efficacious object, as opposed to a mere mathematical artifact. The AB effect sharpens the controversy and together with the work of Wu and Yang [11] reduces the physical impact of the potential to holonomy terms alone, i.e., factors of the form $\exp(i\,\text{const}\cdot\text{Flux})$. It has been further argued [7] that these holonomies, evaluated on a particular subset of loops, constitute the entire physical reality that one can associate with the vector potential. We will see however that in the time-dependent case there is ambiguity in these holonomies in that one can choose to evaluate them at different times, with different values for them as well as for the potentials seen outside the solenoid. To obtain all physical effects one must appeal to further information. In particular, the values of the time-dependent fields, evaluated most conveniently from gauge potentials, are needed—in addition to the holonomies at any particular time. Moreover, as remarked, since that time is arbitrary, the holonomies themselves may vary according to the arbitrarily chosen time.

In Sect. 2 we review the relevant portions of the time-independent AB effect. Following that, we present modifications that enter with time dependence. The last section is a discussion.

2 A One-parameter Ambiguity and Its Resolution. The Time-independent Case

A solenoid is on the z-axis.[1] It has radius r_0 and is surrounded by a barrier that is impenetrable, or nearly impenetrable, to the particle. For simplicity we assume

[1] To deal with issues arising from requiring an infinite solenoid, one can imagine that instead one has a ring. Like the solenoid, excising this object will change the homotopy. The definitive confirmation of the AB effect was performed experimentally in this way [24].

a uniform field $\boldsymbol{B} = \hat{z}F/\pi r_0^2$ exists within the solenoid, with F the total flux. The vector potential outside the solenoid is

$$\boldsymbol{A} = \frac{F}{2\pi}\left(-\frac{y}{r^2}, \frac{x}{r^2}, 0\right) = \frac{F}{2\pi r}\hat{\theta} = \frac{F}{2\pi}\nabla\theta, \tag{1}$$

using polar coordinates (r, θ, z). Physical results, such as interference patterns, depend only on $\exp(ieF/\hbar c)$.

For quantization we employ the Feynman path integral. With this approach, dealing with the homotopic consequences of removing the solenoid is natural. The path integral construction of the propagator for a non-relativistic charged particle in an electromagnetic field is based on the Lagrangian

$$L = \frac{1}{2}mv^2 + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A} - V(\boldsymbol{\xi}) - e\Phi(\boldsymbol{\xi}), \tag{2}$$

with $\boldsymbol{\xi} \in \mathbb{R}^3$. $\boldsymbol{A}$ and Φ represent the external electromagnetic potentials, while V arises from other sources and includes the barrier around the solenoid that makes it (nearly) inaccessible. The propagator is

$$G(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t') = \mathcal{N}\sum_{\boldsymbol{\xi}(\cdot)}\exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L\,dt\right), \tag{3}$$

where the sum is over "all" paths from the initial to the final points and $\mathcal{N}$ is a normalization.

We solve this problem in two ways: first, and more fundamentally, assuming small but finite penetrability of the solenoid. Second, and more conveniently, we idealize the solenoid as impenetrable.

2.1 Penetrable Solenoid

Combining (2) and (3) and taking $\Phi = 0$,

$$G(\boldsymbol{\xi}'', t'; \boldsymbol{\xi}', t') = \mathcal{N}\sum_{\boldsymbol{\xi}(\cdot)}\exp\frac{i}{\hbar}\left(\int_{t'}^{t''} dt\left(\frac{1}{2}mv^2 - V(\boldsymbol{\xi})\right) + \frac{e}{c}\int \boldsymbol{A}\cdot d\boldsymbol{\xi}\right). \tag{4}$$

Call the region of the solenoid $\mathcal{S}$ and assume that V provides a steep but finite barrier against the particle's entry. The path integral can be written

$$G(\boldsymbol{\xi}'', t'; \boldsymbol{\xi}', t') = \mathcal{N}\left\{\sum_n\sum_{\boldsymbol{\xi}(\cdot)\in\Gamma_n}\exp\frac{i}{\hbar}\left(\int_{t'}^{t''} dt\left(\frac{1}{2}mv^2 - V(\boldsymbol{\xi})\right) + \frac{eF}{2\pi c}\int d\theta\right) + \sum_{\Gamma_{\mathcal{S}}} e^{iS/\hbar}\right\}, \tag{5}$$

where we have used $\boldsymbol{A} = (F/2\pi)\nabla\theta$ in the region $\mathcal{Q} \equiv \mathbb{R}^3/\mathcal{S}$ (the exterior of the solenoid). The set of paths, Γ_n, consists of those with winding number n and which remain at all times out of the solenoid; $\Gamma_{\mathcal{S}}$ corresponds to paths that at *any* intermediate time enter the solenoid.

Suppose $\boldsymbol{\xi}''$ and $\boldsymbol{\xi}'$ are outside $\mathcal{S}$. Then the contribution of the paths in $\Gamma_{\mathcal{S}}$ is exponentially small (roughly in $\ell\sqrt{(V/m)}/\hbar$, where ℓ is the penetration depth) and will henceforth be neglected. Note that this quantitative approximation does not change qualitative features, such as the fact that the wave function should be single-valued on $\mathbb{R}^3$.

We introduce the notation $\bar{\theta}$ to mean θ modulo 2π, i.e., its value in the interval $[0, 2\pi)$ when an appropriate multiple of 2π is added or subtracted. $\bar{\theta}$ will be called the principal value of θ. Now note that the integral $\int d\theta$ appearing in (5) is independent of details of the path and only depends on its initial and final θ values. Moreover, in a given Γ_n it will take the value $\bar{\theta}'' - \bar{\theta}' + 2n\pi$. As a result, to a good approximation the propagator becomes

$$G(\boldsymbol{\xi}'', t'; \boldsymbol{\xi}', t') = \mathcal{N} e^{i\frac{eF(\bar{\theta}''-\bar{\theta}')}{\hbar c}} \sum_n e^{i\frac{neF}{\hbar c}} \sum_{\boldsymbol{\xi}(\cdot)\in\Gamma_n} \exp\frac{i}{\hbar}\int_{t'}^{t''} dt \left(\frac{1}{2}mv^2 - V(\boldsymbol{\xi})\right). \quad (6)$$

Note in particular how the interplay of the two discontinuous functions, $\bar{\theta}$ and n, combine to produce a continuous, albeit multivalued, function. The wave function produced by exponentiating this multivalued function, however, is single valued. It is convenient to define propagators G_n as

$$G_n(\boldsymbol{\xi}'', t'; \boldsymbol{\xi}', t') = \mathcal{N} \sum_{\boldsymbol{\xi}(\cdot)\in\Gamma_n} \exp\frac{i}{\hbar}\int_{t'}^{t''} dt \left(\frac{1}{2}mv^2 - V(\boldsymbol{\xi})\right), \quad (7)$$

which can be thought of as free propagators on the covering space of $\mathcal{Q}$ to a particular preimage (labeled by n) of $\boldsymbol{\xi}''$. With this notation

$$G^P(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t') = e^{i\frac{eF(\bar{\theta}''-\bar{\theta}')}{\hbar c}} \sum_n e^{i\frac{neF}{\hbar c}} G_n(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t'), \quad (8)$$

where a superscript "P" has been placed on G to emphasize that this is the form for the penetrable solenoid (recalling of course that it is only approximate).

2.2 Impenetrable Solenoid

$\mathcal{S}$ is now idealized as fully impenetrable, and paths are restricted to the multiply connected space $\mathcal{Q}$. Paths having different winding numbers (number of loops around the solenoid) cannot be continuously[2] deformed into one another. The propagator becomes

$$G(\boldsymbol{\xi}'', t'; \boldsymbol{\xi}', t') = \mathcal{N} \sum_n e^{i\phi_n} \sum_{\boldsymbol{\xi}(\cdot)\in\Gamma_n} \exp\frac{i}{\hbar}\int_{t'}^{t''} dt \left(\frac{1}{2}mv^2 - V(\boldsymbol{\xi})\right). \quad (9)$$

[2] We adopt the usual mathematical fiction of the path integrator: the paths summed over are continuous and nowhere differentiable, as for Wiener measure.

This expression differs in two significant ways from the previous forms. The vector potential has been dropped—it's no longer part of the "world." And (following [4, 12, 13]) a new set of phase factors, $e^{i\phi_n}$, has been introduced. The idea is that since different homotopy classes of paths cannot be deformed into one another, they can have phase factors relative to one another. As shown in the articles just cited these phases are not arbitrary and form a commutative representation (for a scalar wave function) of the fundamental homotopy group of the underlying space,[3] which in this case means $\phi_n = n\mu$ for some real number μ. The propagator thus becomes

$$G^I(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t') = \sum_n e^{in\mu} G_n(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t'), \tag{10}$$

with the "I" recalling that this is the form for the impenetrable solenoid.

On $\mathcal{Q}$ there is no requirement of single-valuedness for wave functions. To find the correct boundary condition we consider the role of G as propagator for a wave function, ψ:

$$\psi(\boldsymbol{\xi}'', t'') = \sum_n e^{in\mu} \int d\boldsymbol{\xi}' G_n(\boldsymbol{\xi}'', t''; \boldsymbol{\xi}', t') \psi(\boldsymbol{\xi}', t'). \tag{11}$$

Suppose that for fixed t' and t'' the endpoint $\boldsymbol{\xi}''$ is moved. This is not a movement in time; we only consider the values of $\psi(\cdot, t'')$ as different points $\boldsymbol{\xi}''$ are contemplated. Let $\boldsymbol{\xi}''$ move on a loop of winding number 1 around the solenoid, returning to its original location. Then the same has happened to the partial propagators G_n. But $G_n(\theta'' + 2\pi, \theta')$ is the same as $G_{n+1}(\theta'', \theta')$ (where all variables but θ'' and θ' are suppressed). This assertion is most easily visualized on the covering space. If the summation index is changed, the propagation reads $\psi(\theta'' + 2\pi) = \sum_m e^{i(m-1)\mu} \int G_m(\theta'', \theta')\psi(\theta') = e^{-i\mu} \sum_m e^{im\mu} \int G_m(\theta'', \theta')\psi(\theta')$. Thus the wave function picks up a factor $e^{-i\mu}$ as a result of the rotation. The freedom to fix this boundary condition on ψ arises because with the solenoid cut out of $\mathbb{R}^3$ the Hamiltonian is no longer essentially self-adjoint and its domain must be further specified. The propagator in effect demands such specification since, as (the kernel of) a bounded operator, it must be self adjoint. This also means that the energy levels are not fixed a priori and may depend on the self-adjoint extension selected, which in this case means they could depend on the number μ. The same argument can be applied to G itself (or one can take $\psi(\boldsymbol{\xi}', t')$ to be a δ-function), showing that $\mu = -\frac{eF}{\hbar c}$.

2.3 Energy Eigenstates

For the penetrable solenoid, taking a Hamiltonian approach, one can examine the effect of $\boldsymbol{A}$ on the eigenvalues. As above, we take the perspective that the solenoid is

[3] The essence of the argument is to let (say) the final point, $\boldsymbol{\xi}''$ travel on a loop, which rearranges the sum for G. (This "travel" is not in time, but follows the wave function's values on this loop.) This rearrangement implies a relation between the phases ϕ_n, namely that they form a representation of the fundamental homotopy group of the underlying configuration space. See Chap. 23 in [5].

penetrable, but that its effects are negligible. For the field we have been considering the Schrödinger equation takes the form

$$\begin{aligned}E\psi &= \frac{(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})^2}{2m}\psi + V\psi \\ &= -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2eF}{2\pi c\hbar r^2}i\frac{\partial\psi}{\partial\theta} - \left(\frac{eF}{2\pi c\hbar r}\right)^2\psi\right] \\ &\quad + V\psi \end{aligned} \tag{12}$$

(z dependence is ignored in this subsection). Writing $\psi = R(r)\Theta(\theta)$, we see that Θ must satisfy

$$\frac{\partial^2\Theta}{\partial\theta^2} - \frac{2eF}{2\pi c\hbar}i\frac{\partial\Theta}{\partial\theta} = \lambda, \tag{13}$$

for some constant λ. Taking $\Theta = \exp(i\alpha\theta)$ *and imposing the periodicity requirement* (which implies $\alpha = n/2\pi$ for some (unrestricted) integer n), we get

$$\lambda = -\frac{n^2}{4\pi^2} + \frac{n}{2\pi}\left(\frac{2eF}{2\pi c\hbar}\right). \tag{14}$$

For R this implies

$$\begin{aligned}ER &= -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{\frac{n^2}{4\pi^2} - \frac{n}{2\pi}\left(\frac{2eF}{2\pi c\hbar}\right) + \left(\frac{eF}{2\pi c\hbar}\right)^2}{r^2}R\right] + VR \\ &= -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{1}{(2\pi)^2}\frac{\left(n-\frac{eF}{c\hbar}\right)^2}{r^2}R\right] + VR. \end{aligned} \tag{15}$$

For each n one seeks eigenvalues of this equation. For appropriate V there will be bound states *whose eigenvalues would appear to depend on F, not only on* $\exp(\frac{ieF}{c\hbar})$. This seems to contradict the thesis [7, 11] that all physical effects depend only on this quantity. But in fact it all works out: as F is varied, the eigenvalues will change, but when the argument of $\exp(\frac{ieF}{c\hbar})$ increases by an integer times 2π one has simply changed "n," and the set of eigenvalues is the same as before.

In the idealization of an impenetrable solenoid, the boundary conditions in θ and the appropriate extension of the operator $\frac{1}{i}\frac{\partial}{\partial\theta}$ provide the same dependence of the energy on F.

3 Time-dependent Fields

Time dependent fields for the AB effect have been considered for a variety of reasons [6, 14–16]. Our purpose is to see to what extent the homotopy ideas described in Sect. 2 continue to hold.

Let Γ be a circle of radius r in the x–y plane surrounding the solenoid and let $\bar{\Gamma}$ be a surface capping this curve. Use Maxwell's equations to obtain

$$\begin{aligned}\int_{\bar{\Gamma}} \nabla \times \boldsymbol{E} \cdot d\boldsymbol{\sigma} &= \int_{\Gamma} \boldsymbol{E} \cdot d\boldsymbol{\xi} = 2\pi r E_\theta \\ &= -\int_{\bar{\Gamma}} \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\dot{F}(t)}{c},\end{aligned} \tag{16}$$

with $F(t)$ the flux at time t. Thus

$$\boldsymbol{E} = \frac{1}{r}\frac{\dot{F}}{2\pi c}\hat{\boldsymbol{\theta}}. \tag{17}$$

Now take an arbitrary time t_0. Let the flux through the solenoid at that time be given by F_0. It follows that any flux changes from the value at time-t_0 can be known from field values *outside* the solenoid. Consequently, one can continue to solve this problem on $\mathcal{Q}$ by supplying a single number, the flux through the solenoid at t_0. Another way to say this is to write the vector potential as $\boldsymbol{A} = \boldsymbol{A}_0 + \widetilde{\boldsymbol{A}}$, with $\boldsymbol{A}_0$ the vector potential at time-t_0. Then, *on the space* $\mathcal{Q}$, one can write the Lagrangian (as in (2)) using $\widetilde{\boldsymbol{A}}$. As for the constant-field case, the solution is indeterminate (H is not essentially self-adjoint) on $\mathcal{Q}$, and requires[4] the single number, F_0, to determine (the domain of) the operator.

To see how this is implemented, we work out a specific example. Suppose that it is the current that is controlled. A constant magnetic field inside the solenoid, abruptly terminating on the outside of the coils, is generated by the current $\boldsymbol{J} = -(cF_0/4\pi^2 r_0^2)\delta(r - r_0)\hat{\boldsymbol{\theta}}$. Let us therefore consider the case of $\boldsymbol{J} = -(cF(t)/4\pi^2 r_0^2)\delta(r - r_0)\hat{\boldsymbol{\theta}}$, $\rho = 0$ and seek a solution for fields generated by this source. For convenience we solve for $\boldsymbol{A}$ and Φ in the Lorentz gauge [17]. Since there are no charges, Φ satisfies the free wave equation and we take it to be zero. This leaves us with $\nabla^2 \boldsymbol{A} - \ddot{\boldsymbol{A}}/c = -4\pi \boldsymbol{J}/c$. By symmetry, $\boldsymbol{A}$ cannot depend on either z or θ, leaving only an r-dependence. Moreover, it is consistent to take $A_r = 0$, so that we need only solve $\frac{\partial^2}{\partial r^2}A_\theta + \frac{1}{r}\frac{\partial}{\partial r}A_\theta - \frac{A_\theta}{r^2} - \frac{1}{c}\ddot{A}_\theta = -\frac{4\pi}{c}J_\theta$. For the moment we restrict ourselves to the case $\ddot{F} = 0$, which allows us to take $\ddot{A}_\theta = 0$. This leads to the following tremendous simplification: the vector potential can have exactly the form we wrote down for the constant current, but with time-dependent coefficients. See [16] for a related result. This means that the magnetic field is zero outside the solenoid. The electric field of course is not, and is given in (17). The magnetic potential is exactly as we have previously written it, but with time-dependent (total) flux.

With this perspective one can continue to work with no (magnetic) vector potential.[5] The flux at some particular time is given and other physical features can be

[4] As we showed when evaluating energy levels, it requires even less than that single number—only its value exponentiated in a phase factor.

[5] It still may be most convenient to use the vector and scalar potentials to calculate phase shifts for interference, since in principle $\boldsymbol{A}$ and Φ can be calculated from the external fields, and from them the interference terms calculated. There would nevertheless be an overall constant flux that is undetermined.

deduced from the external $\boldsymbol{E}$ (in particular the values of flux as a function of time). Thus one can have a complete theory on $\mathcal{Q}$ alone with no mention of $\boldsymbol{A}$. Changed values of the flux relative to the single given number can be deduced from what may be considered the more directly observable electric fields.

Let us recapitulate. We take our space to be $\mathcal{Q}$, 3-space *minus* the solenoid, a space of non-trivial homotopy. We are given electric and (perhaps) magnetic fields. From them we construct a vector potential that gives those fields. This is not enough to solve the problem since the field information will still be missing a constant, namely the flux through the solenoid at some particular time. A full proof of this consists of showing that with zero fields (electric and magnetic) the only vector potential possible is that corresponding to a constant flux. That proof is in the Appendix.

4 More Elaborate Time Dependence

From Sect. 3, the only component of $\boldsymbol{A}$ to compute for the case $\boldsymbol{J} = -(cF(t)/4\pi^2 r_0^2)\,\delta(r - r_0)\hat{\boldsymbol{\theta}}$, $\rho = 0$ is A_θ, which must satisfy

$$\frac{\partial^2}{\partial r^2}A_\theta + \frac{1}{r}\frac{\partial}{\partial r}A_\theta - \frac{A_\theta}{r^2} - \frac{1}{c}\ddot{A}_\theta = -\frac{4\pi}{c}J_\theta. \tag{18}$$

As the first step, we take

$$f(t) \equiv -\frac{4\pi J_\theta}{c} = f_0 e^{i\omega t}. \tag{19}$$

Defining $F(r) \equiv A_\theta e^{-i\omega t}/f_0$, we get

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{F}{r^2} + \frac{\omega^2 F}{c^2} = \delta(r - r_0). \tag{20}$$

With the substitution $\xi = \frac{\omega}{c}r$, we see that F is the Green's function for the Bessel equation,

$$\xi^2 G'' + \xi G' + (\xi^2 - 1)G = \lambda\delta(\xi - \xi_0), \tag{21}$$

and we make the correspondence

$$\xi = \frac{\omega}{c}r, \quad G(\xi) = F(r), \quad \xi_0 = \frac{\omega}{c}r_0, \quad G' = \frac{dG}{d\xi} = \frac{c}{\omega}\frac{dF}{dr}, \quad \lambda = \xi_0^2\frac{c}{\omega}. \tag{22}$$

This Green's function can be expressed in terms of the Bessel functions $J_1(\xi)$ and $Y_1(\xi)$ [18]. It is

$$G(\xi, \xi_0) = aJ_1(\xi) + \begin{cases} 0 & \text{for } \xi < \xi_0, \\ a_1 J_1(\xi) + a_2 Y_1(\xi) & \text{for } \xi > \xi_0, \end{cases} \tag{23}$$

with a arbitrary (not determined by the indicated conditions) and a_1 and a_2 determined by

$$a_1 J_1(\xi_0) + a_2 Y_1(\xi_0) = 0, \tag{24}$$

$$a_1 J_1'(\xi_0) + a_2 Y_1'(\xi_0) = \frac{c}{\omega}. \tag{25}$$

The magnetic field is

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = f_0 e^{i\omega t} \left(\frac{F(r)}{r} + \frac{dF(r)}{dr} \right) \hat{z}, \tag{26}$$

and outside the solenoid becomes

$$\boldsymbol{B} = f_0 e^{i\omega t} \hat{z} \frac{\omega}{c} \left[(a + a_1) \left(\frac{J_1(\xi)}{\xi} + \frac{dJ_1(\xi)}{d\xi} \right) + a_2 \left(\frac{Y_1(\xi)}{\xi} + \frac{dY_1(\xi)}{d\xi} \right) \right]. \tag{27}$$

Making use of the Bessel function identities $\mathcal{C}_\nu = \mathcal{C}'_{\nu+1} + \nu \mathcal{C}_{\nu+1}/\xi$ (where $\mathcal{C} = J, Y$, etc.) we get

$$\boldsymbol{B} = f_0 e^{i\omega t} \hat{z} \frac{\omega}{c} \left[(a + a_1)(J_0(\xi)) + a_2 (Y_0(\xi)) \right]. \tag{28}$$

This gives the asymptotic form

$$(a + a_1) J_0(\xi) + a_2 Y_0(\xi) \sim \sqrt{\frac{2}{\pi \xi}} \left[(a + a_1) \cos(\xi - \pi/4) + a_2 \sin(\xi - \pi/4)) + \mathrm{O}\left(\frac{1}{\xi} \right) \right]. \tag{29}$$

4.1 A Special Case

If r_0 and ω are such that ξ_0 is a zero of J_1, then from (24) it follows that $a_2 = 0$, which, using (25), gives a_1 as a fixed, non-zero number (since J_1 and its derivative cannot vanish at the same point). Recognizing that the coefficient "a" represents a free-field contribution, we see that for an appropriate value of the free field (in particular, such that $a = -a_1$) there is *no* external field, hence no interference pattern—notwithstanding the current in the solenoid.

5 Discussion

For the case of a time-independent magnetic field the AB effect can be obtained without use of a vector potential by going to a multiply connected space on which it is natural to introduce an additional number [4].[6] Physically speaking (i.e., sans the impenetrability idealization) this number is the flux through the solenoid. When the

[6] In fact it's only *a single number modulo* 2π.

field in the solenoid is varied in time physical effects depend on more than this single number. However, with the time-variation there appears an inevitable variation in electromagnetic fields—not just potentials—outside the solenoid. These fields supply information on the variation of the flux within the solenoid. Thus, even with time variation, the idealization of excluding the solenoid and going to a multiply connected space leaves the same degree of ambiguity—a single number—for the recovery of physical effects. The actual value of this number can vary according to the time chosen for evaluating the flux, and the potentials from which full physical results can be deduced will vary accordingly.

Finally, we point out an interesting parallel. For classical electromagnetism on $\mathbb{R}^3$ the vector potential is only felt through the magnetic field calculated from it. In quantum mechanics, Aharonov and Bohm showed that things were different. The same is true—with different wording—when one idealizes an impenetrable solenoid and works on the space $\mathcal{Q} \equiv \mathbb{R}^3/\mathcal{S}$, with $\mathcal{S}$ a tube (or even a line) representing a solenoid. Now, in the quantum context, dynamics is not fixed until the Hamiltonian is made self-adjoint, because of the multiple connectedness of $\mathcal{Q}$. On the other hand, classically one can get a variety of solutions for the vector potential beyond the usual gauge freedom. In particular, one can take the vector potential to be the gradient of a multi-valued function, something not permitted on $\mathbb{R}^3$. However—it doesn't matter. The extra freedom has no physical consequences.

Bearing in mind the relation between non-trivial homotopy (i.e., the fundamental homotopy group has more than one element) and the absence of essential self adjointness [4, 5, 13] it is not surprising that quantum mechanics has a greater richness than classical mechanics (see also [19]). Other examples come to mind. The need for "statistics" (Fermi or Bose) can be thought of in the same way [5, 20], that is, using both homotopy and functional analysis (i.e., self-adjointness) perspectives. There is also the difference in the way that Noether's theorem is applied in the two domains [21, 22]. Finally, we mention the parity quantum number, particular to quantum mechanics, but which is not usually thought of in this way. Nevertheless, through an appropriate topological construct (like that mentioned earlier for statistics), the need for a parity quantum number can be phrased either as a homotopy consequence or as an extension of self-adjointness.

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Appendix: Proof of Uniqueness

Suppose you have observed fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ in $\mathcal{Q}$ (which is $\mathbb{R}^3$ minus a cylinder, that cylinder being the solenoid). Suppose further that you have some vector and scalar potentials, $\boldsymbol{A}$ and Φ that are associated with them. We show that $\boldsymbol{A}$ and Φ are unique up to (1) a gauge transformation and (2) the addition of a field $\boldsymbol{A}_0 \equiv \frac{F_0}{2\pi}\nabla\theta$ with F_0 an arbitrary constant.

If they are not unique, there is another pair, a vector potential and a scalar potential, that give rise to the same fields, so that their difference is a vector and scalar potential for zero field on $\mathcal{Q}$. We need to show that such a field is gauge equivalent to $\boldsymbol{A}_0 \equiv \frac{F_0}{2\pi}\nabla\theta$, for some constant (in time) F_0.

We thus have both $\boldsymbol{E}(\boldsymbol{r}) = 0$ and $\boldsymbol{B}(\boldsymbol{r}) = 0$ for $r \equiv \sqrt{x^2 + y^2} > 0$ (for now, idealizing the solenoid as the z-axis). From $\boldsymbol{B} = 0$ we have $\nabla \times \boldsymbol{A} = 0$. Then *locally* $\boldsymbol{A}$ can be written as the gradient of a scalar function, $\boldsymbol{A} = \nabla\chi$. On $\mathcal{Q}$, χ can be multiple valued, but even so its multivaluedness is restricted. At any point the values that χ can take can only differ from one another by a multiple of some constant. Call that constant α. This "constant" is also constant in space: it takes the same value for all points in $\mathcal{Q}$. (You can see this by integrating $\int \boldsymbol{B} \cdot d\boldsymbol{r}$ along loops.)

Thus $\boldsymbol{A} = \nabla\chi$, with χ a multivalued function on $\mathcal{Q}$ and in general a function of time. The electric field is given by

$$\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\left[\Phi + \frac{1}{c}\frac{\partial\chi}{\partial t}\right]. \tag{30}$$

Since $\boldsymbol{E} = 0$ (for all time), we must have

$$\Phi + \frac{1}{c}\frac{\partial\chi}{\partial t} = g(t), \tag{31}$$

for some function $g(t)$ (independent of $\boldsymbol{r}$). Consider the function $f \equiv \alpha\theta/2\pi$. This is multivalued and has the increment α when f is followed around the solenoid a single time, returning to the same point. Therefore the function

$$\tilde{\chi} \equiv \chi - f \tag{32}$$

is single-valued on $\mathcal{Q}$ and can be gauged away, which we implement as follows. For a function $\Lambda(\boldsymbol{r}, t)$ a gauge transformation has the following form (using the units and conventions of [17])

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda, \tag{33}$$

$$\Phi \to \Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}. \tag{34}$$

Take $\Lambda = f - \chi + G$, where $G(t) \equiv \int^t g(t')dt'$. Λ is a single-valued function on $\mathcal{Q}$. The new potentials are

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla(f - \chi + G) = \nabla f, \tag{35}$$

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial(f - \chi + G)}{\partial t} = -\frac{1}{c}\frac{\partial f}{\partial t}, \tag{36}$$

where we have used $\boldsymbol{A} = \nabla\chi$ and (31).

At this point we have *almost* proved our assertion. If f were independent of time, we'd be through. We next show that giving f time dependence leads to unphysical fields within the solenoid.

We therefore further impose the condition that the potentials we've found externally represent the effect of physical fields. So we go inside the solenoid (which has current flowing at radius r_0) to see what $\dot{f} \neq 0$ implies. Since there is no charge on the surface of the solenoid, $\boldsymbol{E}$ is continuous across it. Inside, from Maxwell's equations we have

$$\frac{\partial^2 A_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2} - \frac{1}{c}\ddot{A}_\theta = -\frac{4\pi}{c} J_\theta, \tag{37}$$

for A_θ (which is its only non-zero component, for the current and charge distribution considered). Consider a short time interval, during which the time dependence of $\boldsymbol{J}$ may be considered linear. Then as observed earlier, for $\boldsymbol{J} = -(cF/4\pi^2 r_0^2)\delta(r - r_0)\hat{\theta}$ we can take $\boldsymbol{A} = (F/2\pi r_0^2)(-y, x, 0) = (F/2\pi r_0^2)r\hat{\theta} = (Fr^2/2\pi r_0^2)\nabla\theta$ inside the solenoid. This immediately yields an expression for $\boldsymbol{E}$:

$$\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\dot{F}}{2\pi r c}\hat{\theta} - \frac{1}{c}\frac{\dot{F}}{2\pi r_0^2} r\hat{\theta} = \frac{\dot{F}\hat{\theta}}{2\pi c}\left[\frac{1}{r} - \frac{r}{r_0^2}\right]. \tag{38}$$

The energy density of the field is proportional to $\boldsymbol{E}^2$ and the volume differential is $dz\, r dr\, d\theta$ (in cylindrical coordinates), so that, per unit length this field has infinite energy. The way to avoid this is to set $\dot{F} = 0$. This shows that Φ must be zero outside the solenoid and $\dot{f} = 0$.

We remark that this kind of field has been shown to have peculiar properties with respect to radiation as well [23].

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