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Cognitive addition: On the convergence of statistical and conceptual models

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Thirty-five children were individually videotaped as they solved simple addition problems. Following Siegler and Robinson's (1982) method, reaction time (RT) and the strategies used for problem solution were recorded. Structural variables representing counting and memory-retrieval conceptual models for cognitive addition were correlated with RT for trials on which an observable counting strategy or memory-retrieval strategy was invoked for problem solution. A strong convergence between a structural variable representing a counting model and RT for counting trials was found. However, no single variable, representing alternative memory-retrieval models, proved to be the best predictor of RT for memory-retrieval trials. Implications for the interpretation of statistical models for cognitive addition are discussed.

Conceptual models describing the processes used in the solution of addition problems have included analog (Restle, 1970), counting (Groen & Parkman, 1972), and memory-retrieval models (Ashcraft & Battaglia, 1978; Geary, Widaman, & Little, 1986; Siegler & Shrager, 1984; Widaman, Geary, Cormier, & Little, 1987). Associated with each conceptual approach is a mathematical or statistical model. Statistical models are used to represent mathematically the processes described within each conceptual model, often by means of regression equations. Embodied within a regression equation are structural variables representing the component processes specified in the conceptual model (Widaman et al., 1987).

To illustrate, consider a computation model for simple addition (Groen & Parkman, 1972). According to Groen and Parkman, all counting models involve the manipulation of an internal incrementing device. Conceptually, each model differs as a function of the value to which the incrementing device is initially set. One such model sets the internal counter to the cardinal value of the larger addend and then increments the counter by ones a number of times equal to the value of the smaller addend until a sum is obtained. If this model accurately reflects the cognitive process used in simple addition, then reaction time (RT) should increase linearly with the value of the smaller, or minimum (min), addend. Furthermore, the min structural variable should provide a statistically better representation of RT than structural variables representing alternative models.

Alternative models include memory retrieval of addition facts. Several conceptual models, each with an accompanying statistical model, have been proposed to represent the process invoked in the retrieval of arithmetic facts from long-term memory. The structural variables representing memory retrieval include the correct sum squared (sum^2; see Ashcraft & Battaglia, 1978), the correct product (prod; see Geary & Widaman, 1987; Geary et al., 1986; Miller, Perlmutter, & Keating, 1984), and various indices of associative strength between a given problem and its correct answer (e.g., Wheeler, 1939).

The use of structural variables to make inferences about cognitive processes poses an important problem. Specifically, the actual processes are often not directly observable or cannot be readily assessed by means of an independent methodology. As a result, the substantive interpretation of the process represented by any given variable is open to question (see Campbell & Graham, 1985; Svenson, 1985). The foregoing problem could be largely circumvented if there were a way, in addition to RT, to independently observe the process used for problem solving. In this case, structural variables representing RT should provide results (in terms of which variable best represents RT) that are consistent with results provided by this alternative method.

The procedures described by Siegler and Robinson (1982) may be one such alternative method for cognitive arithmetic. During the early acquisition of arithmetic skill, the processes used in solving problems (e.g., finger counting) are often readily observable. For these trials, the time required to count fingers should be best predicted by a variable representing a counting model (e.g., min or sum; see Groen & Parkman, 1972), and this outcome would support the conceptual interpretation of the structural variable.

The present study used the Siegler and Robinson (1982) procedures to record the observable processes, and the associated RT, involved in solving simple addition problems. Alternative structural variables were then fit to the

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RT data, and the convergence of the process inferred by the best-fitting structural variable and the actual observed process was assessed.

METHOD

Subjects

Thirty-five (19 male, 16 female) preschool and kindergarten children, between the ages of 4 and 6 years, served as subjects. These children were students at a university-based kindergarten or a private preschool/kindergarten.

Stimuli and Apparatus

Stimuli consisted of the 25 simple addition problems defined by the Cartesian product of the integers 1 to 5. Five random orders of problem presentation were generated.

Equipment included a Panasonic WV-1150A television camera, and a Panasonic NV-8050 time-lapse video recorder. The experimenter controlled a timer, which recorded RTs with an accuracy of $\pm.1$ sec. The timer was started with the presentation of the addend and stopped when the child verbally produced an answer.

Procedure

Each subject was tested individually and videotaped as each problem was solved. The subjects were tested in a partitioned section of their classroom, or in an adjacent room. The subjects were presented with instructions identical to those used by Siegler and Robinson (1982):

> I want you to imagine that you have a pile of oranges. I'll give you more oranges to add to your pile; then you need to tell me how many oranges you have altogether. Okay? You have *m* oranges, and I'm going to give you *n* to add to your pile. How many do you have altogether? (p. 290)

The majority of children preferred to have the problems presented in the form of "How much is $m+n$?" The majority of problems were, therefore, presented in this form. The subjects were given a rest period after the 13th problem, or if they appeared to be fatigued or distracted. At the end of the testing session, each subject was questioned regarding the strategies used for problem solution.

During the test period, the strategy used to solve each problem was recorded. Each was classified as one of the four strategies described by Siegler and Robinson (1982): (1) counting fingers, (2) fingers, (3) verbal counting, or (4) no visible strategy. For problems on which a counting strategy was invoked, the specific algorithm (min or sum) used for problem solution was recorded. Tapes for each subject's session were later reviewed, and strategy choice was rescored for each problem. Reliability was 96.8%.

RESULTS

For the present report only correct responses that were classified as the sum strategy for the "verbal counting" and "counting fingers" trials, and correct responses for "no visible strategy" trials were analyzed. A memory-retrieval strategy appears to be invoked for the "no visible strategy" trials (Siegler & Shrager, 1984). Of the 875 (25×35) problems, 83 involved the correct use of the sum strategy and 211 involved correct memory retrieval. Reaction times for each of the two strategies were independently averaged for each of the 25 stimuli, providing a mean of 3.45 and 8.44 RTs for each problem for the sum and memory-retrieval strategies, respectively.[1] Inspection of these data indicated that average RT to tie (e.g., 1+1 or 3+3) problems on memory-retrieval trials was rather flat, as found for older children (see Groen & Parkman, 1972). For the memory-retrieval data, tie and nontie problems were, therefore, initially analyzed separately, and then in a combined data set.

Average RT for sum trials, and tie and nontie memory-retrieval trials were correlated with alternative search/compute structural variables, representing the five counting-based models proposed by Groen and Parkman (1972), sum² (Ashcraft & Battaglia, 1978), and prod (Miller et al., 1984). Reaction time for memory-retrieval trials was also correlated with various indices that may represent the associative strength in long-term memory between any given simple addition problem and its correct answer. Specifically, memory-retrieval RT was correlated with an index reflecting the ranked difficulty of simple addition problems for early elementary school children (Hamann & Ashcraft, 1985; Wheeler, 1939), the percentage of children who mastered each problem on a learning task (Wheeler, 1939), and a variable reflecting the frequency with which simple addition problems are presented in kindergarten mathematics texts (Hamann & Ashcraft, in press). A final index represented the associative strength between a simple addition problem and its correct answer, measured by the probability of retrieving a correct answer (Siegler & Shrager, 1984).

Reaction time for trials on which the sum strategy was invoked showed the highest zero-order correlation with the sum structural variable ($r = .60, p < .002$), followed by the sum² variable ($r = .546, p < .01$). To determine if the level of fit comparing the sum and sum² variables was significantly different, both variables were, first, fit to RT data within a single regression equation. Next, the sum² variable was dropped from the equation and the decrease in R^2 associated with dropping this variable was tested using an incremental F test (Cohen & Cohen, 1983). The sum variable was then dropped from the equation (sum² was added back into the equation) and the decrease in R^2 associated with dropping the sum variable was also tested using an incremental F test. The significance of the F test indicated the importance of the dropped variable "above and beyond" the alternative variable in explaining RT variance. Dropping the sum² variable resulted in a nonsignificant decrease in the model R^2 [$F(1,22) = 2.21, p > .05$]. The decrease in the model R^2 associated with dropping the sum variable was, however, significant [$F(1,22) = 4.46, p < .05$].

The resulting regression equation provided a slope estimate of 1,010 msec for the sum variable. This estimate may be interpreted as indicating an explicit counting rate of slightly greater than 1,000 msec per incrementation. This counting rate is highly similar to the implicit counting rate of 933 msec (for small numbers) per incrementation (by ones) for first-grade children (Ashcraft, Fierman, & Bartolotta, 1984).

In all, RT for problems solved with the use of the sum strategy was better represented by the sum structural variable than by any alternative search/compute variable. Furthermore, the magnitude of the slope estimate for the sum variable is consistent with estimates of the temporal dura-

Table 1
Correlations Among Search/Compute Indices and Average Reaction Time for Nontie Memory-Retrieval Trials

	1	2	3	4	5	6	7	8
1 AVRT	—							
2 Min	.587†	—						
3 Sum²	.513*	.898‡	—					
4 Prod	.565†	.967‡	.977‡	—				
5 AS	−.512*	−.828‡	−.846‡	−.851‡	—			
6 Per	−.449*	−.850‡	−.842‡	−.862‡	.806‡	—		
7 Rank	.473*	.838‡	.842‡	.852‡	−.801‡	−.992‡	—	
8 FK	−.430	−.641†	−.748‡	−.715‡	.728‡	.547*	−.524*	—

Note—$n = 20$; AVRT = average RT for nontie problems on which the retrieved answer was correct; Min = smaller of the addend or augend; Sum² = (addend + augend)²; Prod = addend × augend; AS = associative strength, or the probability of retrieving a correct addition answer, from Siegler and Shrager (1984, p. 240); Per = percentage of children mastering each simple addition problem (Wheeler, 1939); Rank = ranked difficulty of simple addition problems for early elementary school children (Wheeler, 1939); FK = frequency of presentation in mathematics texts of simple addition problems in kindergarten (Hamann & Ashcraft, in press). *$p < .05$. †$p < .01$. ‡$p < .001$.

tion of the process this variable theoretically represents (Groen & Parkman, 1972).

With regard to memory-retrieval trials, Table 1 presents the correlations among each of the alternative search/compute indices and average RT to nontie problems. Inspection of the first column in the table indicates that each of the indices, with the exception of the FK variable, shows a significant zero-order correlation with RT. The min variable shows the highest correlation with RT ($r = .587, p < .01$), followed by prod ($r = .565, p < .01$). Procedures identical to those described earlier indicated that the level of fit comparing min and prod could not be statistically differentiated. The level of fit comparing min with PC, and sum² also did not differ significantly (all ps $> .05$).

From these results, it might be argued that trials scored as "no visible strategy" actually involve an implicit incrementing strategy, consistent with the conceptual model for min, rather than memory retrieval. However, such a conclusion is contradicted by two findings. First, the subjects generally reported that for these trials an answer was obtained, for example, by "just thinking," and the subjects stated that they did not count "in their head" for these trials. Second, the magnitude of the slope estimate ($b = 450$ msec) for the min variable is inconsistent with the temporal duration of the process (implicit/explicit counting) that the variable presumably represents.

With regard to tie problems, no index was significantly correlated with average RT (all ps $> .10$). However, for the combined data (tie and nontie), the two Wheeler (1939) indices, rank order and percentage, showed the highest zero-order correlations with RT ($r = .548, p < .01$, and $r = -.572, p < .01$, respectively). One additional variable, AS, significantly correlated with the combined RT ($r = -.449$, $p < .05$). None of the remaining search/compute variables showed a significant zero-order correlation with the combined RT (all ps $> .15$).

Again, procedures identical to those described earlier were used to determine if the level of fit of the alternative indices differed significantly. The results indicated that the level of fit comparing the rank order and percentage variables did not differ significantly. However, the level of fit for the Wheeler (1939) percentage variable was significantly better than the level of fit for the AS variable [$F(1,23) = 4.30, p < .05$].

DISCUSSION

A strong convergence between the conceptual model and the statistical model was found for the counting trials on which subjects used the sum strategy. The sum structural variable provided the best representation of RT for these counting trials. Furthermore, the magnitude of the sum slope estimate was consistent with the temporal duration of the process (implicit/explicit counting) that the variable theoretically represents.

On the other hand, the analyses correlating various structural variables representing conceptual models for the long-term memory representation of addition facts with RT for memory-retrieval trials were contradictory. Reaction time to nontie problems was most strongly correlated with the min (counting) and prod (memory-retrieval) variables. No search/compute index was significantly correlated with RT to tie problems, but the combined (nontie and tie) RT was best predicted by the two Wheeler (1939) indices. Nevertheless, it must be concluded that at least for nontie problems, variables representing memory-retrieval models (e.g., prod, sum²) do represent some form of memory-retrieval process, even though none of the alternative models was definitive. For the overall data, the Wheeler variables best represented memory-retrieval RT. Although values for the Wheeler indices are not directly conformable to a memory-retrieval process, the Wheeler variables most likely represent an associative strength model (Hamann & Ashcraft, 1985). In this case, the associative strength between any given problem and its correct answer would be inversely and linearly related to the temporal duration of the memory-retrieval process.

The strong correlation between the min (counting) variable and memory-retrieval RT for nontie problems is consistent with previous findings (Groen & Parkman, 1972; Svenson, 1985). In these cases, the slope estimate for the min variable was of a magnitude inconsistent with an implicit/explicit counting process. These results argue that both the level of fit of the structural variable and the consistency of the slope estimate with the temporal duration of the process represented by the variable need to be considered when inferences are made regarding the cognitive process presumably represented by a structural variable.

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NOTE

1. The problem 5+5 was excluded from the analyses of sum trials, because of the small number of correct sum trials for this problem.

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