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Content & Watkins's Account of Natural Axiomatizations

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Abstract: This paper briefly recounts the importance of the notion of natural axiomatizations for explicating hypothetico-deductivism, empirical significance, theoretical reduction, and organic fertility. Problems for the account of natural axiomatizations developed by John Watkins in *Science and Scepticism* and the revised account developed by Elie Zahar are demonstrated. It is then shown that Watkins's account can be salvaged from various counter-examples in a principled way by adding the demand that every axiom of a natural axiomatization should be part of the content of the theory being axiomatized. The crucial point here is that content cannot be simply identified with the set of logical consequences of a theory but must be restricted to a proper subset of the consequence set. It is concluded that the revised Watkins account has certain advantages over the account of natural axiomatizations offered in Gemes (1993).

1. The Importance of Natural Axiomatizations

In *Theory and Evidence* Clark Glymour argued that hypothetico-deductivism fails as a theory of confirmation because it distributes confirmation too broadly (Glymour, 1980, p. 31). For instance, according to classical accounts of H-D, e confirms h iff h $\vdash$e, so 'Fa' confirms not only '(x)Fx' but also '(x)Fx &(x)Gx'. Glymour observed that if there were a viable account of the notion of natural axiomatizations this problem could be solved (Glymour, 1980, p. 171). Before determining confirmation relations between a given theory and some evidence we would first construct a natural axiomatization of the theory and then see what axioms of that natural axiomatization are needed to deduce the given evidence. Only the axioms needed for such a deduction would be confirmed by the given evidence.

The notion of natural axiomatization is also vital for explicating the notion of empirical significance. A.J. Ayer, in his canonical attempts in *Language, Truth and Logic*, to define empirical significance, could not ward of the challenge generated by the fact that through a clever reformulation of a theory any consequence of the theory could be shown to be empirically significant within that theory. Consider a simple theory consisting of the two claims that Sydney has a harbour bridge, 'Bs', and the nothing nothings, 'Nn'. It is natural to say that the claim that the nothing nothings is not empirically significant within this theory because removing that claim from the theory does not change the empirical consequences of the theory. But now consider the following reformulation of that theory; Axiom 1: Nn; Axiom 2: Nn$\rightarrow$Bs. In this case removing Axiom 1 leads to a resultant theory that no longer has the empirical consequence 'Bs'. The natural response here is to say that a claim is empirically significant within a theory if it occurs in a natural axiomatization of the theory and its removal from that natural axiomatization leads to a new theory with less empirical consequences than the original theory. In our simple case the natural axiomatization is Axiom 1: Nn; Axiom 2: Bs, and here removing Axiom 1 does not lead to a resultant theory with any less empirical consequences than the original theory.

The notion of a natural axiomatization is also vital for explaining the notion of theoretical reduction. It is natural to say that one theory reduces another if it has all the consequences of, but has less axioms than, the other theory. But, as Friedman, (1974, p. 16), has observed, we would not count the conjunction of the laws of a theory, say the conjunction of Kepler's three laws of planetary motions, as a reduction of the formulation that treats each law as a separate axiom. Again it seems that we need to construct natural axiomatizations of theories before addressing question of reduction. Presumably, conjoining all the axioms of a given theory into one long axiom would not count as a natural axiomatization.

2. Watkins's Account and Its Problems; Zahar's Revision and Its Problems

In *Science and Skepticism*, John Watkins argued that the notion of a natural axiomatization is needed to explicate what he called the organic fertility requirement. Organic fertility is for Watkins a desideratum for good theories.[1] Intuitively, a theory is organically fertile if the testable consequences of the theory as a whole are greater than the sum of the testable consequences of each of its parts. To make sense of this idea one has to take great care in determining exactly what the parts of a theory are. To this end Watkins introduced the notion of natural axiomatization. The idea being that the parts of a theory are just those axioms that occur in a natural axiomatization of the theory. Watkins 1984, 208-9, canvassed the following set of conditions for defining what counts as a natural axiomatization of a theory:

1. Independence requirement: each axiom in the axiom set must be logically independent of the conjunction of the others.
2. Non-redundancy requirement: no predicate or individual constant may occur inessentially in the axiom set [or in any axiom of the axiom set.][2]

[1] In fact, Watkins rejects the general casual assumption that any set of sentences constitutes a theory. Organic fertility is for Watkins "a condition for theoryhood". Cf. Watkins,1984, p. 205.
[2] The bracketed insertion is a clarificational addition to Watkins's text to make explicit that, for instance, {((x)Fx & (x)(Gx v ~Gx)), (x)Gx} is not a natural axiomatization of the theory

3. Segregation requirement: if axioms containing only theoretical predicates can be separately stated, without violating other rules, they should be.
4. Wajsberg's Requirement: an axiom is impermissible if it contains a (proper) component that is a theorem of the axiom set, or becomes one when its variables are bound by the quantifiers that bind them in the axiom.
5. Decomposition requirement: if the axiom set can be replaced by an equivalent one that is more numerous (though still finite) without violating the preceding rules, it should be.

To see the import of Wajsberg's requirement consider that theory (set of sentences) which can be axiomatized by

A1*: $(x)(Px \rightarrow Tx)$
A2*: $(x)(Px \rightarrow Tx) \rightarrow (x)(Tx \rightarrow Qx)$.

Wajsberg's requirement stops the set {A1*, A2*} counting as a natural axiomatization since A2* contains as a proper component, namely '$(x)(Px \rightarrow Tx)$', which is a theorem of the relevant theory. The desired natural axiomatization here is the set containing the two members

A1: $(x)(Px \rightarrow Tx)$
A2: $(x)(Tx \rightarrow Qx)$.

There are two objections to Wajsberg's requirement. The first is that it seems ad hoc. Where each of the other requirements answers to some of our intuitions about the nature of natural axiomatizations, the only reason for Wajsberg's requirement is that it rules out certain counter-examples. The second is that it is not effective in ruling out all that needs

{(x)Fx, (x)(Gx}. In this case 'G' occurs essentially in the axiom set, but not essentially in every axiom in which it occurs.

to be ruled out. To see this consider the following reaxiomatization, due to Graham Oddie, 1989;[3]

A1+: (x)((Px&Qx) →Tx)

A2+: (x)((Px&~Qx)→~Tx)

A3+: (x)((~Px&~Qx) →~Tx)

A4+: (x)((Px&~Qx) →Tx)

We stipulate that 'P', 'Q' and 'T', are all theoretical predicates. Then, since {A1+, A2+, A3+, A4+} is logically equivalent to {A1, A2}, and satisfies all of the requirements 1-4, requirement 5 rules out {A1, A2} as a natural axiomatization. Furthermore, where 'F' and 'G' are theoretical predicates Wajsberg's requirement does not rule out the following as a natural axiomatization

A1*: (x)(Fx → Gx)

A2*: (x)~Gx

Yet, intuitively the natural axiomatization here is

A1**: (x)~Fx

A2**: (x)~Gx

Elie Zahar, 1991, has suggested a revision of Wajsberg's requirement that effectively blocks these counterexamples. However Zahar's revision involves an extraordinary level of complexity, moreover, it has the same ad hoc air as does the original Wajsberg's requirement. Perhaps more importantly, where Watkins's conditions include Zahar's revision in place of Wajsberg's requirement we get the unacceptable result that no law statement of the form (x)(…..≡___) can be part of a natural axiomatization. This follows from the fact that where Zaher's revision allows the axiom (x)(…..≡___) it will equally allow both (x)(….→___) and (x)(___→….), and so, by Watkins's decomposition requirement, a natural axiomatization would have to include both

[3] Oddie,1998, p. 347, gives a propositional version of this case.

of these rather than (x)(…..≡___) .[4] Yet clearly laws statements asserting the equivalence of certain quantities can be part of natural axiomatizations. To demand their decomposition into laws statements with a conditional form is like insisting that ‘f=ma’, the claim that force equals mass times acceleration, cannot be part of a natural axiomatization of Newtonian mechanics since any such axiomatization should rather contain two separate axioms ‘f≥ma’ and ‘f≤ma’. Furthermore, to demand such decomposition would lead to bizarre results when trying to asses which axioms of a natural axiomatization of a theory are true. To see this let T be some medical theory which includes the axiom ‘(x)(Ax≡Bx)’, say, that anybody has aids if and only they have virus B. Then, according to Watkins supplemented with Zahar’s revision this axiom could not occur in a natural axiomatization of T. Rather, a natural axiomatization of T would have the two separate axioms ‘(x)Ax → Bx)’ and ‘(x)(Bx → Ax)’. In case one is tempted to think this is an acceptable result, note that this entails that if we find that everybody in the relevant domain of discourse has aids but none of them have virus B then we would have to conclude that at least one axiom of a natural axiomatization of T, a theory which claims that one has aids iff they have virus B, has been conclusively confirmed. This is bizarre as saying that if we found that force is always greater than mass times acceleration we would have confirmed one axiom of a natural axiomatization of Newtonian mechanics.

3. A New Account of Content

Before proposing an alternative to 4, it is helpful to consider exactly what counts as part of the content of a theory. Consider again the theory that may be axiomatized by

A1*: (x)(Px→Tx)

A2*: (x)(Px→Tx) → (x)(Tx→Qx).

Note, that if we count A2* as a content part of this theory then on the evidence of 'Pa&~Ta' we would have to say that part of the theory has been conclusively confirmed since

[4] In fact this problem arises equally for the version of Watkins’s account which includes Wajsberg’s requirement

'Pa&~Ta' deductively entails A2*. In other words, the theory that is naturally axiomatized by

A1: $(x)(Px \rightarrow Tx)$
A2: $(x)(Tx \rightarrow Qx)$

is partially confirmed by 'Pa&~Ta'. This result seems simply monstrous. The lesson here is that not every consequence of a theory should count as part of the theory's content. The content parts of the theory that may be naturally axiomatized by A1 and A2 (or unnaturally axiomatized by A1* and A2*) should include A1 and A2 but not A2*. In other words, not every axiom of any axiomatization of a theory should count as part of the theory's content.

In fact, most philosophers of science, including Carnap and Popper, endorse the claim that α is part of the content of ß iff ß $\vdash$α.[5] Gemes, 1994 and 1997 showed that this notion yields a number of highly counter-intuitive results. For instance, where a hypothesis/theory is defined as being partially true where some content part of it is true, the above notion of content yields the result that any claim is partially true. This follows since for any claim/theory T, where p is some truth, the disjunction of T and p will be a true content part of T. So for instance, 'There are no black ravens is partially true since it entails the truth statement 'There are no black ravens or Sydney has a Harbour bridge. Given the traditional notion of content and the plausible claim that if e entails h, e conclusively confirms h, it follows that for any e and any h there will always be a content part of h such that e conclusively confirms that content part of h, namely the disjunction of h and e. So, for instance, 'a is a white raven' conclusively confirms part of the content of 'All ravens are black' since the later claim entails 'All ravens are black or a is a white raven'.

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[5] In Popper,1972, p. 385, we read

> By logical content (or the consequence class of a) we mean the class of all statements that follow from a.

Similarly, Cf. Carnap, 1935, p. 42.

Gemes,1994, and 1997, developed a new account of (logical) content according to which not every consequence of a claim/theory counts as part of the content of the claim/theory. Besides avoiding the negative results mentioned above Gemes, 2005, 1998, 1997, 1994, 1994a, and 1993, showed that the new notion of content can be used to solve various canonical problems in the philosophy of science, including the construction of versions of hypothetico-deductivism, empirical significance, theoretical unification, and bootstrapping confirmation not subject to canonical counter-examples that bedeviled earlier versions.

According to the account of content in Gemes, 1997, where α is a variable over wffs and ß is variable over wffs and sets of wffs,

NCT α is part of the content of ß iff α and ß are contingent, ß $\vdash \alpha$, and every relevant model of α has an extension which is a relevant model of ß.

A relevant model of arbitrary wff α is a model of α that assigns values to all and only those atomic wffs relevant to α. An atomic wff α is relevant to wff β iff there is some model m of β such that where m’ differs from m only in the value it assigns α, m’ is not a model of β. In the case of quantificational wffs the quantifiers are treated substitutionally in order to determine relevant atomic wffs and content parts. So ‘Fa’ is relevant to ‘(x)Fx’ since were m is a model of ‘(x)Fx’ and m’ differs from m only in the value it assigns ‘Fa’, m must assign the value T to ‘Fa’ and hence m’ assigns the value F to ‘Fa’ and hence m’ is not a model of ‘(x)Fx'. According to NCT, ‘GbvFa’ is not part of the content of ‘(x)Fx’ since that relevant model of ‘GbvFa’ that assigns ‘Fa’ the value F and ‘Gb’ the value T cannot be extended to a model of ‘(x)Fx. ‘Fa’ is a content part of ‘(x)Fx’, since the sole relevant model of ‘Fa, namely, that which makes the single assignment of T to ‘Fa’, can clearly be extended to a relevant model of ‘(x)Fx’, by adding the assignment of T to ‘Fb’, ‘Fc’, ‘Fd’, etc.

4. <u>Using the New Account of Content to Fix Watkins's Account of Natural Axiomatizations</u>

I propose we replace Wajsberg's requirement with the following,

4a Content requirement: Every axiom must be a content part of theory determined by the axiom set.

4a eliminates Oddie's counter-example since none of A1+ - A4+ count as content parts of the relevant theory. For instance, A1+, that is, '(x)((Px&Q)x→Tx)',does not count as part of the content of any theory that has '(x)(Px→Tx)' as a consequence. To see this consider any relevant model of '(x)((Px&Qx)→Tx)' that assigns 'Ta' and 'Qa' the value F and 'Pa' the value T. Clearly any such model cannot be extended to a model of '(x)(Px→Tx)'. It also precludes '(x)(Fx → Gx)' as being an axiom in a natural axiomatization of any theory that entails '(x)~Gx' since '(x)(Fx → Gx)' cannot be a content part of any theory that entails '(x)~Gx'. This last claim follows because those relevant models of '(x)(Fx → Gx)' which assign 'Fa' and 'Ga' the value T cannot be extended to a model of '(x)~Gx'. Finally, where a theory has the consequence '(x)(Ax≡Bx)', 4a precludes '(x)Bx→Ax)' being an axiom in a natural axiomatization of the theory because '(x)Bx→Ax)' cannot be a content part of such a theory. This follows because those relevant models of '(x)Bx→Ax)' which assign 'Ba' F and 'Aa' T cannot be extended to a model of '(x)(Bx≡Ax)'.

The revised version of Watkins's account of natural axiomatization which results from replacing his orignal clause 4 with clause 4a above yields a workable notion of natural axiomatization. At least, it is not open to the objections that have been raised against the original version. Furthermore, clause 4a is not ad hoc in the way that 4 is; the requirement that a natural axiomatization of a theory should only contain axioms that are actually part of the content of the theory axiomatized seems perfectly sensible. Hereafter I shall use the phrase ' revised Watkins natural axiomatization' and various cognates to refer to this revised version.

Gemes, 1993, offered the following less stringent account of natural axiomatizations

> T' is a natural axiomatization of T iff (i) T' is a finite set of wffs such that T' is logically equivalent to T, (ii) every member of T' is a content part of T' and (iii) no content part of any member of T' is entailed by the set of the remaining members of T'. (Gemes, 1993, p.483)

I conjecture that every axiomatization that counts as a revised Watkins natural axiomatization counts as a Gemes natural axiomatization. The crucial assumption behind this conjecture is that Watkins's stipulations that each axiom in the axiom set must be logically independent of the conjunction of the others ensures the satisfaction of Gemes' requirement that no content part of any axiom is entail by the remaining axioms. The revised version of Watkins's account is demonstrably more demanding in the following three ways: (1) it prohibits the occurrence of inessential vocabulary in an axiom – thus were Gemes, 1993, allows that '(x)(Fx & (Gxv~Gx)' might occur as an axiom in a natural axiomatization of some theory, the revised Watkins forbids this since 'G' occurs non-essentially in '(x)(Fx & (Gxv~Gx) '; (2) it requires that were possible axioms containing theoretical predicates should be free of non-theoretical predicates and (3) it requires that axioms should be maximized – thus were Gemes would allow, for instance '(x)(Rx→Bx) & (x)Sx→Wx)' to count as an axiom in a natural axiomatization revised Watkins insists that '(x)(Rx→Bx)' and '(x)Sx→Wx)' count as separate axioms. All three of these requirements, and especially (1) and (3) are intuitively plausible. Furthermore, when it comes to applying the notion of natural axiomatizations to certain canonical problems, for instance, the problem of theory reduction and explaining the organic fertility requirement, Watkins's requirements present certain clear advantages. A relevant consideration here is that Gemes, 1993, was narrowly concerned with the problems of devising an account of natural axiomatization sufficient for the construction of viable accounts of hypothetico-deductivism and empirical significance. Where we are more concerned with capturing the intuitive notion of natural axiomatizations and for various canonical projects such as the definition of reduction and organic fertility the revised Watkins' account has demonstrable advantages, bought at the usual price of added complexity.[6]

[6] This piece was greatly improved by comments from an unnamed referee from this journal.

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