

The $\lambda\mu^{\mathbf{T}}$ -calculus

Herman Geuvers^{1,2}, Robbert Krebbers¹ and James McKinna¹

¹Radboud University Nijmegen

²Eindhoven University of Technology

July 9, 2012

Abstract

Calculi with control operators have been studied as extensions of simple type theory. Real programming languages contain datatypes, so to really understand control operators, one should also include these in the calculus. As a first step in that direction, we introduce $\lambda\mu^{\mathbf{T}}$, a combination of Parigot's $\lambda\mu$ -calculus and Gödel's \mathbf{T} , to extend a calculus with control operators with a datatype of natural numbers with a primitive recursor.

We consider the problem of confluence on raw terms, and that of strong normalization for the well-typed terms. Observing some problems with extending the proofs of Baba *et al.* and Parigot's original confluence proof, we provide new, and improved, proofs of confluence (by complete developments) and strong normalization (by reducibility and a postponement argument) for our system.

We conclude with some remarks about extensions, choices, and prospects for an improved presentation.

1 Introduction

In pursuit, on the one hand, of a satisfactory equational theory of call-by-value λ -calculus, and on the other, of a means to interpret the computational content of classical proofs, a variety of calculi with control operators have been proposed. Few of these systems address the problem of how to incorporate primitive datatypes in direct style, preferring instead to consider the usual Church encoding of datatypes or else to analyze computation over datatypes via CPS-translations.

In part this appears to arise because of the technical difficulty in getting standard results such as confluence or strong normalization, and their proof methods, either for classical calculi, or for simply-typed calculi with datatypes, to extend to their combination.

This paper introduces a new λ -calculus with control, $\lambda\mu^{\mathbf{T}}$, in which for example constructs for `catch` and `throw` may be represented, which moreover has

a basic datatype of natural numbers with a primitive recursor, in the style of Gödel’s \mathbf{T} . We demonstrate that it is possible to achieve a synthesis of classical computation with datatypes with a conventional metatheory of typing and reduction. To show how the system can be used in programming, we give a simple example in 3.16, where we define a function that multiplies the first n values of $f : \mathbb{N} \rightarrow \mathbb{N}$ and throws an exception as soon as it encounters the value 0.

1.1 Our approach

Since Lafont’s counterexample [GTL89], it is well known that a calculus providing a general content to classical logic cannot be confluent. It only may become confluent if one adds an evaluation strategy (call-by-name or call-by-value). To define a calculus with control operators and datatypes we have therefore observed a tension between the call-by-name features taken directly from Parigot’s $\lambda\mu$ -calculus, and the need to add certain call-by-value features to obtain a system that is confluent and satisfies a normal form theorem (each closed term of type \mathbb{N} is convertible to a numeral). The $\lambda\mu^{\mathbf{T}}$ -calculus is therefore a call-by-name system with strict evaluation on datatypes. To avoid losing a normal form theorem, we could not make it a full call-by-name system, and to avoid losing confluence we had to restrict the primitive recursor to only allow conversion when the numerical argument is a numeral.

Given these technical considerations, we were able to prove that $\lambda\mu^{\mathbf{T}}$ satisfies subject reduction, has a normal form theorem, is confluent and strongly normalizing. The last two proofs are non-trivial because various niceties are required to make the standard proof methods work.

Our confluence proof uses the notion of parallel reduction and defines a complete development for each term. Surprisingly, it was difficult to find a confluence proof for the original untyped $\lambda\mu$ -calculus. Baba, Hirokawa and Fujita [BHF01] have given a confluence proof for $\lambda\mu$ without the $\rightarrow_{\mu\eta}$ -rule ($\mu\alpha.[\alpha]t \rightarrow t$ provided that $\alpha \notin \text{FCV}(t)$). Although they suggest how to extend parallel reduction for the $\rightarrow_{\mu\eta}$ -rule, they do not provide a formal definition of the complete development nor a proof. Nakazawa [Nak03] has successfully carried out their suggestion for a call-by-value variant of $\lambda\mu$, but does not use the notion of complete development. Walter Py’s PhD thesis [Py98] was the only place where we have found a complete proof of confluence for $\lambda\mu$. It uses Aczel’s generalization of parallel reduction [Acz78] and a number of postponement arguments. In the present paper we extend the methodology of [BHF01] to the case of $\lambda\mu^{\mathbf{T}}$, which also includes the $\rightarrow_{\mu\eta}$ -rule.

Our strong normalization proof proceeds by defining relations \rightarrow_A and \rightarrow_B such that $\rightarrow = \rightarrow_{AB} := \rightarrow_A \cup \rightarrow_B$. First we prove that \rightarrow_A is strongly normalizing by the reducibility method. Secondly, we prove that \rightarrow_B is strongly normalizing and that both reductions commute in a way that we can obtain strong normalization for \rightarrow_{AB} . The first phase is inspired by Parigot’s proof of strong normalization for the $\lambda\mu$ -calculus [Par97].

1.2 Related work

The extension of simply typed lambda calculus with control operators and the observation that these operators can be typed using the rules of classical logic is originally due to Griffin [Gri90] and has led to a lot of research [Par92, Par93, dG94, RS94, BS95, Coq96, BB96, AH03, vBLL05], by considering variations on the control operators, the underlying calculus or the computation rules, or by studying concrete examples of the computational content of proofs in classical logic. The $\lambda\mu$ -calculus of Parigot [Par92] has become a central starting point for much research in this area.

The extension with datatypes, to make the calculus into a real programming language with control operators, has not received so much attention. We briefly summarize the research done in this direction and compare it with our work.

Murthy has defined a system with control operators, arithmetic, products and sums in his PhD thesis [Mur90]. His system uses the control operators \mathcal{C} and \mathcal{A} (originally due to [Gri90]) and the semantics of these operators is specified by evaluation contexts rather than local reduction rules, as we do. So his system does not really describe a *calculus* for datatypes and control. Furthermore, Murthy mainly considers CPS-translations to give an operational semantics of his system and did not prove properties like confluence or strong normalization.

Crolard and Polonowski have considered a version of Gödel's \mathbf{T} with products and `call/cc` [CP11]. As with Murthy, the semantics is presented by CPS-translations instead of a direct specification via a calculus. Therefore properties like confluence and strong normalization are trivial because they hold for the target system already.

Barthe and Uustalu have worked on CPS-translations for inductive and coinductive types [BU02]. Their work includes a system with a primitive for iteration over the natural numbers and the control operator Δ . Unfortunately only some properties of CPS-translations are proven.

Rehof and Sørensen have described an extension of the λ_{Δ} -calculus with basic constants and functions [RS94]. Unfortunately their extension is quite limited. For example the primitive recursor `nrec` takes terms, rather than basic constants, as its arguments. Their extension does not allow this, making it impossible to define `nrec`.

Parigot has described a second-order variant of his $\lambda\mu$ -calculus [Par92]. This system is very powerful, because it includes all the well-known second-order representable datatypes. However, it suffers from the same weakness as System \mathbf{F} , namely poor computational efficiency (for example, an $O(n)$ -predecessor function). Also, as observed in [Par92, Par93], this system does not ensure *unique representation of datatypes*. For example, there is no one-to-one correspondence between natural numbers and closed normal forms of the type of Church numerals.

There have been various investigations into concrete examples of computational content of classical proofs. Coquand gives an overview in his notes [Coq96]. An earlier example is [BS95], where a binpacking problem is analyzed using proof transformations. More recent work is by Makarov [Mak06], who takes Griffin's

calculus and adds various rules to optimize the extracted program.

If we look in particular at Gödel’s \mathbf{T} , Berger, Buchholz and Schwichtenberg have described a form of program extraction from classical proofs [BBS00]. Their method extracts a term from a classical proof in which all computationally irrelevant parts are removed. To prove the correctness of their approach they give a realizability interpretation. However, since their target language is Gödel’s \mathbf{T} , extracted programs do not contain control mechanisms.

Caldwell, Gent and Underwood have considered program extraction from classical proofs in the proof assistant NuPrl [CGU00]. In their work they extend NuPrl with a proof rule for Peirce’s law and they associate `call/cc` to the extraction of Peirce’s law. Now, program extraction indeed results in a program with control. The main focus of their work is on using program extraction to obtain efficient search algorithms. The authors do not prove any meta theoretical results so it is unclear whether their approach is correct for arbitrary classical proofs.

1.3 Outline

The paper is organized as follows:

- Section 2 recapitulates Gödel’s \mathbf{T} , fixing notation and conventions, together with the key normal form property.
- Section 3 introduces $\lambda\mu^{\mathbf{T}}$, our Gödel’s \mathbf{T} variant of Parigot’s $\lambda\mu$ -calculus extended with a datatype of natural numbers with primitive recursor `nrec`. We define the basic reduction rules, whose compatible closure defines computation in $\lambda\mu^{\mathbf{T}}$. We show how to represent rules for a statically bound `catch` and `throw` mechanism. We prove subject reduction, and the extended analogue of the normal form property.
- In Section 4, we develop the corresponding CPS-translation for $\lambda\mu^{\mathbf{T}}$, and show it preserves typing and conversion.
- Section 5 contains one of our two principal technical contributions: a direct proof of confluence on the raw terms of $\lambda\mu^{\mathbf{T}}$, based on a novel analysis of complete developments.
- In Section 6, our second technical contribution is to prove SN for our calculus, using the reducibility method and a postponement argument.
- We close with some conclusions and indications for further work, both in extending our system with a richer type system, and in investigating a fully-fledged call-by-value version.

2 Gödel’s \mathbf{T}

Gödel’s \mathbf{T} (henceforth $\lambda^{\mathbf{T}}$) was introduced by Gödel to prove the consistency of Peano Arithmetic [SU06]. It arises from $\lambda\rightarrow$ by addition of a base type for

natural numbers and a construct for primitive recursion.

Definition 2.1. *The types of $\lambda^{\mathbf{T}}$ are built from a basic type (the natural numbers) and a function type (\rightarrow) as follows.*

$$\rho, \sigma, \tau ::= \mathbb{N} \mid \sigma \rightarrow \tau$$

Definition 2.2. *The terms of the $\lambda^{\mathbf{T}}$ are inductively defined over an infinite set of λ -variables (x, y, \dots) as follows.*

$$t, r, s ::= x \mid \lambda x : \rho. r \mid ts \mid 0 \mid \mathbf{St} \mid \mathbf{nrec}_\rho r s t$$

Here, ρ ranges over $\lambda^{\mathbf{T}}$ -types.

As one would imagine, the terms 0 , \mathbf{S} and \mathbf{nrec} denote zero, the successor function and primitive recursion over the natural numbers, respectively. We let $\text{FV}(t)$ denote the set of free variables of t and we define the operation of capture avoiding substitution $t[x := r]$ of r for x in t in the usual way.

Convention 2.3. *Although a λ -abstraction and \mathbf{nrec} construct are annotated by a type, we omit these type annotations when they are obvious or not relevant. Furthermore, we use the Barendregt convention. That is, given an expression, we may assume that bound variables are distinct from free variables and that all bound variables are distinct.*

Definition 2.4. *The derivation rules for $\lambda^{\mathbf{T}}$ are as shown in Figure 1.*

$$\begin{array}{c} \frac{x : \rho \in \Gamma}{\Gamma \vdash x : \rho} \quad \frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x : \sigma. t : \sigma \rightarrow \tau} \quad \frac{\Gamma \vdash t : \sigma \rightarrow \tau \quad \Gamma \vdash s : \sigma}{\Gamma \vdash ts : \tau} \\ \text{(a) var} \qquad \text{(b) lambda} \qquad \text{(c) app} \\ \\ \Gamma \vdash 0 : \mathbb{N} \quad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathbf{St} : \mathbb{N}} \quad \frac{\Gamma \vdash r : \rho \quad \Gamma \vdash s : \mathbb{N} \rightarrow \rho \rightarrow \rho \quad \Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathbf{nrec}_\rho r s t : \rho} \\ \text{(d) zero} \qquad \text{(e) suc} \qquad \text{(f) nrec} \end{array}$$

Figure 1: The rules for typing judgments in $\lambda^{\mathbf{T}}$.

Definition 2.5. *Reduction $t \rightarrow t'$ is defined as the compatible closure of the following rules.*

$$\begin{array}{ll} (\lambda x. t)r \rightarrow t[x := r] & (\beta) \\ \mathbf{nrec} r s 0 \rightarrow r & (0) \\ \mathbf{nrec} r s (\mathbf{St}) \rightarrow s t \ (\mathbf{nrec} r s t) & (\mathbf{S}) \end{array}$$

As usual, \rightarrow denotes the reflexive/transitive closure and $=$ denotes the reflexive/symmetric/transitive closure.

Although we do not specify a deterministic reduction strategy it is obviously possible to create a call-by-name and call-by-value version of $\lambda^{\mathbf{T}}$. Yet it is interesting to remark that in a call-by-value version of $\lambda^{\mathbf{T}}$ calculating the predecessor takes at least linear time while in a call-by-name version the predecessor can be calculated in constant time [CF98].

Fortunately, despite the additional features of $\lambda^{\mathbf{T}}$, the important properties of $\lambda \rightarrow$, subject reduction, confluence and strong normalization, are preserved [Ste72, GTL89].

Because it is convenient to be able to talk about a term representing an actual natural number we introduce the following notation.

Notation 2.6. $\underline{n} := S^n 0$

Definition 2.7. Values are inductively defined as follows.

$$v, w ::= 0 \mid Sv \mid \lambda x.r$$

Theorem 2.8. Given a term t that is in normal form and such that $\vdash t : \rho$:

1. If $\rho = \mathbb{N}$, then $t \equiv \underline{n}$ for some $n \in \mathbb{N}$.
2. If $\rho = \sigma \rightarrow \tau$, then $t \equiv \lambda x.r$ for a variable x and term r .

As the following indicates, the system $\lambda^{\mathbf{T}}$ has quite some expressive power.

Definition 2.9. A function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ is representable in $\lambda^{\mathbf{T}}$ if there is a term t with $\vdash t : \mathbb{N}^n \rightarrow \mathbb{N}$ such that:

$$t \underline{m}_1 \dots \underline{m}_n = \underline{f(m_1, \dots, m_n)}$$

Theorem 2.10. The functions representable in $\lambda^{\mathbf{T}}$ are exactly the functions that are provably recursive in first-order arithmetic¹.

Proof. This is proven in [SU06]. □

3 The $\lambda\mu^{\mathbf{T}}$ -calculus

In this section we present our Gödel's \mathbf{T} extension of Parigot's $\lambda\mu$ -calculus (henceforth $\lambda\mu^{\mathbf{T}}$).

Definition 3.1. The terms and commands of $\lambda\mu^{\mathbf{T}}$ are mutually inductively defined over an infinite set of λ -variables (x, y, \dots) and μ -variables (α, β, \dots) as follows.

$$\begin{aligned} t, r, s &::= x \mid \lambda x : \rho.r \mid ts \mid \mu\alpha : \rho.c \mid 0 \mid St \mid \mathbf{nrec}_\rho r s t \\ c, d &::= [\alpha]t \end{aligned}$$

Here, ρ ranges over $\lambda^{\mathbf{T}}$ -types (Definition 2.1). We give $[\alpha]t$ lower precedence than sr , allowing us to write $[\alpha]sr$ instead of $[\alpha](sr)$.

¹Here we are allowed to say either Peano Arithmetic (PA) or Heyting Arithmetic (HA), because a function is provably recursive in PA iff it is provably recursive in HA [SU06].

As usual, we let $\text{FV}(t)$ and $\text{FCV}(t)$ denote the set of free λ -variables and μ -variables of t , respectively. Moreover, we define substitution $t[x := r]$ of r for x in t , which is capture avoiding for both λ - and μ -variables, in the obvious way. Similar to Convention 2.3, we will often omit type annotations for μ -binders.

Notation 3.2. $\mu.c := \mu\gamma.c$ provided that $\gamma \notin \text{FCV}(c)$.

Definition 3.3. The typing rules for $\lambda\mu^{\mathbf{T}}$ are as shown in Figure 2.

$$\begin{array}{c}
\frac{x : \rho \in \Gamma}{\Gamma; \Delta \vdash x : \rho} \quad \frac{\Gamma, x : \sigma; \Delta \vdash t : \tau}{\Gamma; \Delta \vdash \lambda x : \sigma. t : \sigma \rightarrow \tau} \quad \frac{\Gamma; \Delta \vdash t : \sigma \rightarrow \tau \quad \Gamma; \Delta \vdash s : \sigma}{\Gamma; \Delta \vdash ts : \tau} \\
\text{(a) axiom} \qquad \text{(b) lambda} \qquad \text{(c) app} \\
\Gamma; \Delta \vdash 0 : \mathbb{N} \quad \frac{\Gamma; \Delta \vdash t : \mathbb{N}}{\Gamma; \Delta \vdash \mathbf{S}t : \mathbb{N}} \\
\text{(d) zero} \qquad \text{(e) suc} \\
\frac{\Gamma; \Delta \vdash r : \rho \quad \Gamma; \Delta \vdash s : \mathbb{N} \rightarrow \rho \rightarrow \rho \quad \Gamma; \Delta \vdash t : \mathbb{N}}{\Gamma; \Delta \vdash \mathbf{nrec}_\rho r s t : \rho} \\
\text{(f) nrec} \\
\frac{\Gamma; \Delta, \alpha : \rho \vdash c : \perp\!\!\!\perp}{\Gamma; \Delta \vdash \mu\alpha : \rho. c : \rho} \quad \frac{\Gamma; \Delta \vdash t : \rho \quad \alpha : \rho \in \Delta}{\Gamma; \Delta \vdash [\alpha]t : \perp\!\!\!\perp} \\
\text{(g) activate} \qquad \text{(h) passivate}
\end{array}$$

Figure 2: The rules for typing judgments in $\lambda\mu^{\mathbf{T}}$.

A typing judgment $\Gamma; \Delta \vdash t : \rho$ is *derivable in $\lambda\mu^{\mathbf{T}}$* in case it is the conclusion of a derivation tree that uses the rules of Definition 3.3. We say “term t has type ρ in environment of λ -variables Γ and environment of μ -variables Δ ”.

Similarly, a typing judgment $\Gamma; \Delta \vdash c : \perp\!\!\!\perp$ is *derivable in $\lambda\mu^{\mathbf{T}}$* in case it is the conclusion of a derivation tree that uses the rules of Definition 3.3. We say “command c is typable in environment of λ -variables Γ and environment of μ -variables Δ ”.

Fact 3.4. *The typing judgment is closed under weakening of both environments. That is, if $\Gamma; \Delta \vdash t : \rho$, $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma'; \Delta' \vdash t : \rho$.*

In order to define the reduction rules we first define the notions of *contexts* and *structural substitution*. Although the reduction rules merely require contexts of a restricted shape (those that are *singular*) we define contexts of a more general shape so we can reuse these definitions in our proof of confluence (Section 5) and strong normalization (Section 6).

Definition 3.5. A $\lambda\mu^{\mathbf{T}}$ -context is defined as follows.

$$E ::= \square \mid Et \mid \mathbf{S}E \mid \mathbf{nrec} r s E$$

A context is singular if it is the following shape.

$$E^s ::= \square t \mid \mathbf{S}\square \mid \mathbf{nrec} \ r \ s \ \square$$

Definition 3.6. Given a context E and a term s , substitution of s for the hole in E , notation $E[s]$, is defined as follows.

$$\begin{aligned} \square[s] &:= s \\ (Et)[s] &:= E[s]t \\ (\mathbf{S}E)[s] &:= \mathbf{S}E[s] \\ (\mathbf{nrec} \ r \ s \ E)[s] &:= \mathbf{nrec} \ r \ s \ E[s] \end{aligned}$$

Definition 3.7. Given contexts E and F , the context EF is defined by:

$$\begin{aligned} \square F &:= F \\ (Et)F &:= (EF)t \\ (\mathbf{S}E)F &:= \mathbf{S}(EF) \\ (\mathbf{nrec} \ r \ s \ E)F &:= \mathbf{nrec} \ r \ s \ (EF) \end{aligned}$$

Fact 3.8. $E[F[t]] \equiv EF[t]$

Using contexts we can now define *structural substitution*. Structural substitution of a μ -variable β and a context E for a μ -variable α in t , notation $t[\alpha := \beta E]$, recursively replaces each command $[\alpha]q$ in t by $[\beta]E[q']$ where $q' \equiv q[\alpha := \beta E]$. Our notion of structural substitution is more general than Parigot's original presentation [Par92]. He defines $t[\beta := \alpha]$, which renames each μ -variable β in t into α , and $t[\alpha := s]$, which replaces each command $[\alpha]q$ in t by $[\alpha]q's$ where $q' \equiv q[\alpha := s]$. Of course, his notions are just instances of our definition, namely, the former corresponds to $t[\beta := \alpha \ \square]$ and the latter to $t[\alpha := \alpha \ (\square s)]$. Parigot's presentation suffices for the definition of the reduction rules, but our presentation allows us to prove properties like confluence (Section 5) and strong normalization (Section 6) in a more streamlined way.

Definition 3.9. Structural substitution $t[\alpha := \beta E]$ of a μ -variable β and a context E for a μ -variable α is defined as follows.

$$\begin{aligned} x[\alpha := \beta E] &:= x \\ (\lambda x.r)[\alpha := \beta E] &:= \lambda x.r[\alpha := \beta E] \\ (ts)[\alpha := \beta E] &:= t[\alpha := \beta E]s[\alpha := \beta E] \\ 0[\alpha := \beta E] &:= 0 \\ (\mathbf{S}t)[\alpha := \beta E] &:= \mathbf{S}(t[\alpha := \beta E]) \\ (\mathbf{nrec} \ r \ s \ t)[\alpha := \beta E] &:= \mathbf{nrec} \ (r[\alpha := \beta E]) \ (s[\alpha := \beta E]) \ (t[\alpha := \beta E]) \\ (\mu\gamma.c)[\alpha := \beta E] &:= \mu\gamma.c[\alpha := \beta E] \\ ([\alpha]t)[\alpha := \beta E] &:= [\beta]E[t[\alpha := \beta E]] \\ ([\gamma]t)[\alpha := \beta E] &:= [\gamma]t[\alpha := \beta E] \quad \text{provided that } \gamma \neq \alpha \end{aligned}$$

Structural substitution is capture avoiding for both λ - and μ -variables.

Definition 3.10. Reduction $t \rightarrow t'$ is defined as the compatible closure of the following rules.

$$\begin{array}{ll}
(\lambda x.t)r \rightarrow t[x := r] & (\beta) \\
\mathbf{S}(\mu\alpha.c) \rightarrow \mu\alpha.c[\alpha := \alpha \ (\mathbf{S}\square)] & (\mu\mathbf{S}) \\
(\mu\alpha.c)s \rightarrow \mu\alpha.c[\alpha := \alpha \ (\square s)] & (\mu R) \\
\mu\alpha.[\alpha]t \rightarrow t \quad \text{provided that } \alpha \notin \text{FCV}(t) & (\mu\eta) \\
[\alpha]\mu\beta.c \rightarrow c[\beta := \alpha \ \square] & (\mu i) \\
\mathbf{nrec} \ r \ s \ 0 \rightarrow r & (0) \\
\mathbf{nrec} \ r \ s \ (\mathbf{S}\underline{n}) \rightarrow s \ \underline{n} \ (\mathbf{nrec} \ r \ s \ \underline{n}) & (\mathbf{S}) \\
\mathbf{nrec} \ r \ s \ (\mu\alpha.c) \rightarrow \mu\alpha.c[\alpha := \alpha \ (\mathbf{nrec} \ r \ s \ \square)] & (\mu\mathbf{N})
\end{array}$$

As usual, \rightarrow^+ denotes the transitive closure, \twoheadrightarrow denotes the reflexive/transitive closure and $=$ denotes the reflexive/symmetric/transitive closure of \rightarrow .

Fact 3.11. As in [FH92], the notion of a singular context allows us to replace the reduction rules $\rightarrow_{\mu\mathbf{S}}$, $\rightarrow_{\mu R}$ and $\rightarrow_{\mu\mathbf{N}}$ by the following single rule.

$$E^s[\mu\alpha.c] \rightarrow \mu\alpha.c[\alpha := \alpha E^s]$$

Fact 3.12. $E[\mu\alpha.c] \twoheadrightarrow \mu\alpha.c[\alpha := \alpha E]$

From a computational point of view one should think of $\mu\alpha.[\beta]t$ as a combined operation that catches exceptions labeled α in t and throws the results of t to β . Following Crolard [Cro99], we define the operators **catch** and **throw**.

Definition 3.13. The terms **catch** $_{\alpha} t$ and **throw** $_{\beta} s$ are defined as follows.

$$\begin{array}{l}
\mathbf{catch}_{\alpha} t := \mu\alpha.[\alpha]t \\
\mathbf{throw}_{\beta} s := \mu_.[\beta]s
\end{array}$$

Similar to commands, we give **catch** $_{\alpha} t$ and **throw** $_{\beta} s$ lower precedence than sr , allowing us to write **catch** $_{\alpha} sr$ instead of **catch** $_{\alpha} (sr)$.

Crolard [Cro99] moreover defines a system with **catch** and **throw** as primitives and proves a correspondence with the $\lambda\mu$ -calculus. We prove that the above simulation of **catch** and **throw** satisfies a generalization of Crolard's rules.

Lemma 3.14. We have the following reductions for **catch** and **throw**.

1. $E[\mathbf{catch}_{\alpha} t] \twoheadrightarrow \mathbf{catch}_{\alpha} E[t[\alpha := \alpha E]]$
2. $E[\mathbf{throw}_{\alpha} t] \twoheadrightarrow \mathbf{throw}_{\alpha} t$
3. $\mathbf{catch}_{\alpha} \mathbf{catch}_{\beta} t \rightarrow \mathbf{catch}_{\alpha} t[\beta := \alpha \ \square]$
4. $\mathbf{throw}_{\alpha} \mathbf{throw}_{\beta} t \rightarrow \mathbf{throw}_{\beta} t$
5. $\mathbf{throw}_{\alpha} \mathbf{catch}_{\beta} t \rightarrow \mathbf{throw}_{\alpha} t[\beta := \alpha \ \square]$

6. $\text{catch}_\alpha \text{throw}_\alpha t \rightarrow \text{catch}_\alpha t$

7. $\text{catch}_\alpha t \rightarrow t$ provided that $\alpha \notin \text{FCV}(t)$

Proof. These reductions follow directly from the reduction rules of $\lambda\mu^{\mathbf{T}}$, except for (1) and (2) where we need Fact 3.12. \square

The **catch** and **throw** as defined above give rise to a system with *statically bound* exceptions. This is different from exceptions in for example **Lisp**, where they are *dynamically bound*. In a system with dynamically bound exceptions, substitution is not capture avoiding for exception names.

Example 3.15. Consider the following term:

$$\text{catch}_\alpha \mathbf{S}((\lambda f : \mathbb{N} \rightarrow \mathbb{N}. \text{catch}_\alpha f \ 0) (\lambda x : \mathbb{N}. \text{throw}_\alpha x)).$$

Here, both occurrences of **catch** bind different occurrences α . So after two β -reduction steps we obtain $\text{catch}_\alpha \mathbf{S}(\text{catch}_\beta \text{throw}_\alpha 0)$ and hence its normal form is 0. In systems with dynamically bound exceptions this term would reduce to **S0** because the **throw** would get caught by the innermost **catch**.

Example 3.16. We consider a simple $\lambda\mu^{\mathbf{T}}$ -program F that, given $f : \mathbb{N} \rightarrow \mathbb{N}$, computes the product of the first n values of f , that is $F \underline{n} = f \ 0 * \dots * f \ \underline{n}$ for $n \in \mathbb{N}$. The interest of this program is that it uses the exception mechanism to stop multiplying once a zero is encountered. First we define addition and multiplication in the usual way in $\lambda\mu^{\mathbf{T}}$.

$$\begin{aligned} (+) &:= \lambda nm. \text{nrec } m (\lambda xy. \mathbf{S}y) \ n \\ (*) &:= \lambda nm. \text{nrec } 0 (\lambda xy. m + y) \ n \end{aligned}$$

Now, given $f : \mathbb{N} \rightarrow \mathbb{N}$, we define the term $F : \mathbb{N} \rightarrow \mathbb{N}$, using a ‘helper function’ H , which does a case analysis on the value of $f y$, as follows.

$$\begin{aligned} F &:= \lambda x. \text{catch}_\alpha \text{nrec } \underline{1} \ H (\mathbf{S}x) \\ H &:= \lambda y m. \text{nrec } (\text{throw}_\alpha 0) (\lambda z \dots m * \mathbf{S}z) (f y). \end{aligned}$$

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be some term that satisfies $f \ 0 = \underline{3}$, $f \ 1 = \underline{0}$ and $f \ 2 = \underline{5}$. We show a computation of $F \ 2$.

$$\begin{aligned} F \ 2 &\rightarrow \text{catch}_\alpha \text{nrec } \underline{1} \ H \ \underline{3} \\ &\rightarrow \text{catch}_\alpha H \ \underline{2} (\text{nrec } \underline{1} \ H \ \underline{2}) \\ &\rightarrow \text{catch}_\alpha \text{nrec } (\text{throw}_\alpha 0) (\lambda z \dots (\text{nrec } \underline{1} \ H \ \underline{2}) * \mathbf{S}z) (f \ \underline{2}) \\ &\rightarrow \text{catch}_\alpha (\text{nrec } \underline{1} \ H \ \underline{2}) * \underline{5} \\ &\rightarrow \text{catch}_\alpha (H \ \underline{1} (\text{nrec } \underline{1} \ H \ \underline{1})) * \underline{5} \\ &\rightarrow \text{catch}_\alpha (\text{nrec } (\text{throw}_\alpha 0) (\lambda z \dots (\text{nrec } \underline{1} \ H \ \underline{1}) * \mathbf{S}z) (f \ \underline{1})) * \underline{5} \\ &\rightarrow \text{catch}_\alpha \text{throw}_\alpha 0 * \underline{5} \\ &\rightarrow \text{catch}_\alpha \text{nrec } 0 (\lambda xy. \underline{5} + y) (\text{throw}_\alpha 0) \\ &\rightarrow \text{catch}_\alpha \text{throw}_\alpha 0 \\ &\rightarrow 0 \end{aligned}$$

In order to prove that $\lambda\mu^{\mathbf{T}}$ satisfies subject reduction we have to prove that each reduction rule preserves typing. Because some of the reduction rules involve structural substitution it is convenient to prove an auxiliary result that structural substitution preserves typing. To express this property we introduce the notion of a *contextual typing judgment*, notation $\Gamma; \Delta \vdash E : \sigma \Leftarrow \rho$, which expresses that $\Gamma; \Delta \vdash t : \rho$ implies $\Gamma; \Delta \vdash E[t] : \sigma$.

Definition 3.17. *The derivation rules for the contextual typing judgment $\Gamma; \Delta \vdash E : \sigma \Leftarrow \rho$ are as shown in Figure 3.*

$$\begin{array}{c}
\Gamma; \Delta \vdash \square : \rho \Leftarrow \rho \quad \frac{\Gamma; \Delta \vdash E : \sigma \rightarrow \tau \Leftarrow \rho \quad \Gamma; \Delta \vdash t : \sigma}{\Gamma; \Delta \vdash Et : \tau \Leftarrow \rho} \quad \frac{\Gamma; \Delta \vdash E : \mathbf{N} \Leftarrow \rho}{\Gamma; \Delta \vdash \mathbf{SE} : \mathbf{N} \Leftarrow \rho} \\
\text{(a) hole} \qquad \qquad \qquad \text{(b) app} \qquad \qquad \qquad \text{(c) suc} \\
\\
\frac{\Gamma; \Delta \vdash r : \sigma \quad \Gamma; \Delta \vdash s : \mathbf{N} \rightarrow \sigma \rightarrow \sigma \quad \Gamma; \Delta \vdash E : \mathbf{N} \Leftarrow \rho}{\Gamma; \Delta \vdash \mathbf{nrec} \ r \ s \ E : \sigma \Leftarrow \rho} \\
\text{(d) nrec}
\end{array}$$

Figure 3: The rules for contextual typing judgments in $\lambda\mu^{\mathbf{T}}$.

Fact 3.18. *Contextual typing judgments do indeed enjoy the intended behavior. That is, we have $\Gamma; \Delta \vdash E[t] : \sigma$ iff there is a type ρ such that $\Gamma; \Delta \vdash E : \sigma \Leftarrow \rho$ and $\Gamma; \Delta \vdash t : \rho$.*

Fact 3.19. *Typing is preserved under (structural) substitution.*

1. If $\Gamma, x : \rho; \Delta \vdash t : \tau$ and $\Gamma; \Delta \vdash r : \rho$, then $\Gamma; \Delta \vdash t[x := r] : \tau$.
2. If $\Gamma; \Delta, \alpha : \rho \vdash t : \tau$ and $\Gamma; \Delta \vdash E : \sigma \Leftarrow \rho$, then $\Gamma; \Delta, \beta : \sigma \vdash t[\alpha := \beta E] : \tau$.

We have corresponding results for commands.

Proof. The first property is proven by mutual induction on the derivations of $\Gamma, x : \rho; \Delta \vdash t : \tau$ and $\Gamma, x : \rho; \Delta \vdash c : \perp\!\!\!\perp$. All cases are straightforward. The second property is proven by induction on the derivations of $\Gamma; \Delta, \alpha : \rho \vdash t : \tau$ and $\Gamma; \Delta, \alpha : \rho \vdash c : \perp\!\!\!\perp$. Most cases are straightforward, so we only treat the passive case. Let $\Gamma; \Delta, \alpha : \rho \vdash [\alpha]t : \perp\!\!\!\perp$ with $\Gamma; \Delta, \alpha : \rho \vdash t : \rho$. By the induction hypothesis we have $\Gamma; \Delta, \beta : \sigma \vdash t[\alpha := \beta E] : \rho$. This leaves us to prove that $\Gamma; \Delta, \beta : \sigma \vdash ([\alpha]t)[\alpha := \beta E] : \perp\!\!\!\perp$. Since $([\alpha]t)[\alpha := \beta E] \equiv [\beta]E[t[\alpha := \beta E]]$, the result follows from Fact 3.18 and the induction hypothesis. \square

Theorem 3.20. *The $\lambda\mu^{\mathbf{T}}$ -calculus satisfies subject reduction.*

Proof. We have to prove that all reduction rules preserve typing.

1. Proving that the result holds for the \rightarrow_{β} , \rightarrow_0 and $\rightarrow_{\mathbf{S}}$ -rule is straightforward, so we omit that.

2. To prove that the result holds for the $\rightarrow_{\mu R}$, $\rightarrow_{\mu S}$ and $\rightarrow_{\mu N}$ -rule it is sufficient to show that the result holds for $E^s[\mu\alpha.c] \rightarrow \mu\beta.c[\alpha := \beta E^s]$ by Fact 3.11. Given $\Gamma; \Delta \vdash E^s[\mu\alpha.c] : \tau$ we use Fact 3.18 to obtain a type σ such that $\Gamma; \Delta \vdash \mu\alpha.c : \sigma$ and $\Gamma; \Delta \vdash E : \tau \Leftarrow \sigma$.

$$\frac{\frac{\Gamma; \Delta, \alpha : \sigma \vdash c : \perp}{\Gamma; \Delta \vdash \mu\alpha.c : \sigma} \quad \Gamma; \Delta \vdash E^s : \tau \Leftarrow \sigma}{\Gamma; \Delta \vdash E^s[\mu\alpha.c] : \tau} \rightarrow \frac{\Gamma; \Delta, \beta : \tau \vdash c[\alpha := \beta E^s] : \perp}{\Gamma; \Delta \vdash \mu\beta.c[\alpha := \beta E^s] : \tau}$$

Here we have $\Gamma; \Delta, \beta : \tau \vdash c[\alpha := \beta E^s] : \perp$ by Fact 3.19.

3. For the $\rightarrow_{\mu\eta}$ -rule we have the following.

$$\frac{\frac{\Gamma; \Delta, \alpha : \rho \vdash t : \rho}{\Gamma; \Delta \vdash [\alpha]t : \perp}}{\Gamma; \Delta \vdash \mu\alpha.[\alpha]t : \rho} \rightarrow \Gamma; \Delta \vdash t : \rho$$

Since $\alpha \notin \text{FV}(t)$, we have $\Gamma; \Delta \vdash t : \rho$ by strengthening.

4. For the $\rightarrow_{\mu i}$ -rule we have the following.

$$\frac{\frac{\Gamma; \Delta, \alpha : \rho, \beta : \rho \vdash c : \perp}{\Gamma; \Delta, \alpha : \rho \vdash \mu\beta.c : \rho}}{\Gamma; \Delta, \alpha : \rho \vdash [\alpha]\mu\beta.c : \perp} \rightarrow \Gamma; \Delta, \alpha : \rho \vdash c[\beta := \alpha \square] : \perp$$

Here we have $\Gamma; \Delta, \alpha : \rho \vdash c[\beta := \alpha \square] : \perp$ by Fact 3.19 and the fact that $\Gamma; \Delta, \alpha : \rho \vdash \square : \rho \Leftarrow \rho$. \square

The \rightarrow_S -rule, in contrast to the corresponding rule of $\lambda^{\mathbf{T}}$ (Definition 2.5), only allows conversion when the numerical argument is a numeral. This restriction ensures that primitive recursion is not performed on terms that might reduce to a term of the shape $\mu\alpha.c$. If we omit this restriction we lose confluence.

Example 3.21. *We illustrate this by considering a variant of our system with the following rule instead.*

$$\mathbf{nrec} \ r \ s \ (\mathbf{S}t) \rightarrow s \ t \ (\mathbf{nrec} \ r \ s \ t) \quad (\mathbf{S}')$$

Now we can reduce the term $t \equiv \mu\alpha.[\alpha]\mathbf{nrec} \ 0 \ (\lambda xh.2) \ (\mathbf{S}\mu_{-}.[\alpha]\underline{4})$ to two distinct normal forms:

$$\begin{aligned} t &\equiv \mu\alpha.[\alpha]\mathbf{nrec} \ 0 \ (\lambda xh.2) \ (\mathbf{S}\mu_{-}.[\alpha]\underline{4}) \\ &\rightarrow \mu\alpha.[\alpha]\mathbf{nrec} \ 0 \ (\lambda xh.2) \ (\mu_{-}.[\alpha]\underline{4}) && (\mu\mathbf{S}) \\ &\rightarrow \mu\alpha.[\alpha]\mu_{-}.[\alpha]\underline{4} && (\mu\mathbf{N}) \\ &\rightarrow \mu\alpha.[\alpha]\underline{4} && (\mu i) \\ &\rightarrow \underline{4} && (\mu\eta) \end{aligned}$$

and:

$$\begin{aligned}
t &\equiv \mu\alpha.[\alpha]\mathbf{nrec}\ 0\ (\lambda x h.\underline{2})\ (\mathbf{S}\mu_{-}.[\alpha]\underline{4}) \\
&\rightarrow \mu\alpha.[\alpha](\lambda x h.\underline{2})\ (\mu_{-}.[\alpha]\underline{4})\ (\mathbf{nrec}\ 0\ (\lambda x h.\underline{2})\ (\mu_{-}.[\alpha]\underline{4})) && (\mathbf{S}') \\
&\rightarrow \mu\alpha.[\alpha]\underline{2} && (\beta) \\
&\rightarrow \underline{2} && (\mu\eta)
\end{aligned}$$

Alternatively, in order to obtain a confluent system, it is possible to remove the $\rightarrow_{\mathbf{S}}$ -rule while retaining the unrestricted $\rightarrow_{\mu\mathbf{S}'}$ -rule. However, then we can construct closed terms $t : \mathbb{N}$ that are in normal form but are not a numeral. An example of such a term is $\mu\alpha.[\alpha]\mathbf{S}\mu\beta.[\alpha]0$.

Lemma 3.22. *Given a value v such that $;\Delta \vdash v : \rho$, we have:*

1. *If $\rho = \mathbb{N}$, then $v \equiv \underline{n}$.*
2. *If $\rho = \sigma \rightarrow \tau$, then $t \equiv \lambda x.r$ for some variable x and term r .*

Proof. This result is proven by induction on the structure of values. \square

Lemma 3.23. *Given a term t that is in normal and such that $;\Delta \vdash t : \rho$, then t is a value or $t \equiv \mu\alpha.[\beta]v$ for some value v .*

Proof. By induction on the derivation $;\Delta \vdash t : \rho$.

- (var) Let $;\Delta \vdash x : \rho$ with $x : \rho \in \emptyset$. Now we obtain a contradiction since $x : \rho \notin \emptyset$.
- (λ) Let $;\Delta \vdash \lambda x.r : \sigma \rightarrow \tau$. Now we are immediately done.
- (app) Let $;\Delta \vdash rs : \tau$ with $;\Delta \vdash r : \sigma \rightarrow \tau$ and $;\Delta \vdash s : \sigma$. Now by the induction hypothesis and Lemma 3.22 we have $r \equiv \lambda x.r'$ or $r \equiv \mu\alpha.[\beta]v$. But since rs should be in normal form we obtain a contradiction.
- (zero) Let $;\Delta \vdash 0 : \mathbb{N}$. Now we are immediately done.
- (suc) Let $;\Delta \vdash St : \mathbb{N}$ with $;\Delta \vdash t : \mathbb{N}$. Now we have $t \equiv \underline{n}$ or $t \equiv \mu\alpha.[\beta]v$ by the induction hypothesis and Lemma 3.22. In the former case we are immediately done, in the latter case we obtain a contradiction because the $\rightarrow_{\mu\mathbf{S}}$ -rule can be applied.
- (nrec) Let $;\Delta \vdash \mathbf{nrec}\ r\ s\ t : \rho$ with $;\Delta \vdash t : \mathbb{N}$. Now we have $t \equiv \underline{n}$ or $t \equiv \mu\alpha.[\beta]v$ by the induction hypothesis and Lemma 3.22. But in both cases we obtain a contradiction because the reduction rules $\rightarrow_{\mu 0}$, $\rightarrow_{\mu\mathbf{S}}$ and $\rightarrow_{\mu\mathbb{N}}$ can be applied, respectively.
- (act/pas) Let $;\Delta \vdash \mu\alpha.[\beta]t : \rho$ with $;\Delta, \alpha : \rho \vdash t : \tau$ and $\beta : \tau \in (\Delta, \alpha : \rho)$. Now we have that t is a value or $t \equiv \mu\alpha.[\beta]v$ by the induction hypothesis. In the former case we are immediately done, in the latter case we obtain a contradiction because the $\rightarrow_{\mu i}$ -rule can be applied. \square

Theorem 3.24. *Given a term t that is in normal form and such that $\vdash t : \mathbb{N}$, then $t \equiv \underline{n}$ for some $n \in \mathbb{N}$.*

Proof. By Lemma 3.23 we obtain that $t \equiv v$ or $t \equiv \mu\alpha.[\beta]v$ for some value v . In the former case we have $t \equiv \underline{n}$ by Lemma 3.22. In the latter case we have $\beta = \alpha$ since t is closed for μ -variables, so $t \equiv \mu\alpha.[\alpha]\underline{n}$ by Lemma 3.22. But now we obtain a contradiction because we can apply the $\rightarrow_{\mu\eta}$ -rule. \square

4 CPS-translation of $\lambda\mu^{\mathbf{T}}$ into $\lambda^{\mathbf{T}}$

In this section we will present a CPS-translation from $\lambda\mu^{\mathbf{T}}$ into $\lambda^{\mathbf{T}}$. We will use this CPS-translation to prove the main result of this section: the functions that are representable in $\lambda\mu^{\mathbf{T}}$ are exactly the functions that are provably recursive in first-order arithmetic.

Definition 4.1. *Let $\neg\rho$ denote $\rho \rightarrow \perp$ for a fixed type \perp . Given a type ρ , the negative translation ρ° of ρ is mutually inductively defined with ρ^\bullet as follows.*

$$\begin{aligned}\rho^\circ &:= \neg\neg\rho^\bullet \\ \mathbb{N}^\circ &:= \mathbb{N} \\ (\sigma \rightarrow \tau)^\circ &:= \sigma^\circ \rightarrow \tau^\circ\end{aligned}$$

Definition 4.2. *Given $\lambda^{\mathbf{T}}$ -terms t and r , the CPS-application $t \bullet r$ of t and r is defined as follows.*

$$t \bullet r := \lambda k.t(\lambda l.lrk)$$

Definition 4.3. *Given a $\lambda^{\mathbf{T}}$ -term t , the negative of t is defined as follows.*

$$\bar{t} := \lambda k.kt$$

Fact 4.4. *If $\Gamma \vdash t : (\sigma \rightarrow \tau)^\circ$ and $\Gamma \vdash r : \sigma^\circ$, then $\Gamma \vdash t \bullet r : \tau^\circ$.*

Definition 4.5. *Given a $\lambda\mu^{\mathbf{T}}$ -term t , then the CPS-translation t° of t into $\lambda^{\mathbf{T}}$ is inductively defined as follows.*

$$\begin{aligned}x^\circ &:= \lambda k.xk \\ (\lambda x.t)^\circ &:= \lambda k.k(\lambda x.t^\circ) \\ (tr)^\circ &:= t^\circ \bullet r^\circ \\ 0^\circ &:= \bar{0} \\ (\mathbf{St})^\circ &:= \lambda k.t^\circ(\lambda l.k(\mathbf{Sl})) \\ (\mathbf{nrec}_\rho r s t)^\circ &:= \lambda k.t^\circ(\lambda l.\mathbf{nrec} r^\circ s' l k) \\ &\quad \text{where } s' := \lambda xp.s^\circ \bullet \bar{x} \bullet p \\ (\mu\alpha.c)^\circ &:= \lambda k_\alpha.c^\circ \\ ([\alpha]t)^\circ &:= t^\circ k_\alpha\end{aligned}$$

Here k_α is a fresh λ -variable for each μ -variable α .

In the translation of $\mathbf{nrec}_\rho r s t$ we see that we are required to evaluate t first, simply because it is the only way to obtain a numeral from t .

Fact 4.6. *If $\Gamma \vdash t : \mathbb{N}$, then $\Gamma \vdash \bar{t} : \mathbb{N}^\circ$.*

Theorem 4.7. *The translation from $\lambda\mu^{\mathbf{T}}$ into $\lambda^{\mathbf{T}}$ preserves typing. That is:*

$$\Gamma; \Delta \vdash t : \rho \text{ in } \lambda\mu^{\mathbf{T}} \quad \Longrightarrow \quad \Gamma^\circ, \Delta^\circ \vdash t^\circ : \rho^\circ \text{ in } \lambda^{\mathbf{T}}$$

where $\Gamma^\circ = \{x : \rho^\circ \mid x : \rho \in \Gamma\}$ and $\Delta^\circ = \{k_\alpha : \neg\rho^\bullet \mid \alpha : \rho \in \Delta\}$.

Proof. We prove that we have $\Gamma; \Delta \vdash t : \rho$ and $\Gamma; \Delta \vdash c : \perp$ by mutual induction on the derivations $\Gamma^\circ, \Delta^\circ \vdash t^\circ : \rho^\circ$ and $\Gamma^\circ, \Delta^\circ \vdash t^\circ : \perp$, respectively. Most of the cases are straightforward, so we treat just one interesting case.

(nrec) Let $\Gamma; \Delta \vdash \mathbf{nrec}_\rho r s t : \rho$ with $\Gamma; \Delta \vdash r : \rho$, $\Gamma; \Delta \vdash s : \mathbb{N} \rightarrow \rho \rightarrow \rho$ and $\Gamma; \Delta \vdash t : \mathbb{N}$. Now we have $\Gamma^\circ, \Delta^\circ \vdash r^\circ : \rho^\circ$, $\Gamma^\circ, \Delta^\circ \vdash s^\circ : (\mathbb{N} \rightarrow \rho \rightarrow \rho)^\circ$ and $\Gamma^\circ, \Delta^\circ \vdash t^\circ : \mathbb{N}^\circ$ by the induction hypothesis. Furthermore we have $s' \equiv \lambda x p. s^\circ \bullet \bar{x} \bullet p : \mathbb{N} \rightarrow \rho^\circ \rightarrow \rho^\circ$ as shown below.

$$\frac{\frac{\frac{s^\circ : (\mathbb{N} \rightarrow \rho \rightarrow \rho)^\circ}{s^\circ \bullet \bar{x} : (\rho \rightarrow \rho)^\circ} \quad \frac{x : \mathbb{N}}{\bar{x} : \mathbb{N}^\circ} \text{ (a)}}{\text{ (b)}} \quad p : \rho^\circ \text{ (c)}}{s^\circ \bullet \bar{x} \bullet p : \rho^\circ} \quad \lambda x p. s^\circ \bullet \bar{x} \bullet p : \mathbb{N} \rightarrow \rho^\circ \rightarrow \rho^\circ$$

Here, step (a) follows from Fact 4.6 and step (b) and (c) follow from Fact 4.4. So $\Gamma^\circ, \Delta^\circ \vdash (\mathbf{nrec}_\rho r s t)^\circ : \rho^\circ$ as shown below.

$$\frac{\frac{\frac{r^\circ : \rho^\circ \quad s' : \mathbb{N} \rightarrow \rho^\circ \rightarrow \rho^\circ \quad l : \mathbb{N}}{\mathbf{nrec} r^\circ s' l : \rho^\circ} \quad k : \neg\rho^\bullet}{\frac{\mathbf{nrec} r^\circ s' l k : \perp}{\lambda l. \mathbf{nrec} r^\circ s' l k : \neg\mathbb{N}}} \quad t^\circ : \mathbb{N}^\circ}{\frac{t^\circ(\lambda l. \mathbf{nrec} r^\circ s' l k) : \perp}{\lambda k. t^\circ(\lambda l. \mathbf{nrec} r^\circ s' l k) : \rho^\circ}} \quad \square$$

Fact 4.8. *For each $n \in \mathbb{N}$ we have $\underline{n}^\circ \rightarrow \bar{n}$.*

Proof. By induction on n .

1. Let $n = 0$. We have $\underline{0}^\circ \equiv \bar{0}$ by Definition 4.5.
2. Let $n > 0$. We have $\underline{n}^\circ \rightarrow \bar{n}$ by the induction hypothesis and hence:

$$\begin{aligned} \underline{n+1}^\circ &\equiv \lambda k. \underline{n}^\circ(\lambda l. k(Sl)) \\ &\rightarrow \lambda k. (\lambda q. q\underline{n})(\lambda l. k(Sl)) \\ &\rightarrow \lambda k. k(\underline{Sn}) \\ &\equiv \bar{n+1} \end{aligned} \quad \square$$

Lemma 4.9. For each term t we have $\lambda k.t^\circ k \rightarrow t^\circ$.

Proof. This follows immediately from the Definition 4.5 since the translation t° of a term t is of the shape $\lambda l.t'$, so $\lambda k.(\lambda l.t')k \rightarrow \lambda k.t'[l := k] \equiv t^\circ$. \square

Lemma 4.10. We have $\lambda k.\mathbf{nrec} \ r^\circ \ s' \ \underline{n} \ k = \mathbf{nrec} \ r^\circ \ s' \ \underline{n}$ for $s' \equiv \lambda x p.s^\circ \bullet \bar{x} \bullet p$.

Proof. We distinguish the following cases.

1. Let $n = 0$. The result follows from Lemma 4.9.
2. Let $n > 0$. Now we have the following.

$$\begin{aligned}
\lambda k.\mathbf{nrec} \ r^\circ \ s' \ \underline{n} \ k &\rightarrow \lambda k.s' \ \underline{n-1} \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n-1}) \ k \\
&\rightarrow \lambda k.(s^\circ \bullet \overline{n-1} \bullet \mathbf{nrec} \ r^\circ \ s' \ \underline{n-1}) \ k \\
&\equiv \lambda k.(\lambda k_2.(s^\circ \bullet \overline{n-1}) \ (\lambda l.l \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n-1}) \ k_2)) \ k \\
&\rightarrow \lambda k.(s^\circ \bullet \overline{n-1}) \ (\lambda l.l \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n-1}) \ k) \\
&\equiv s^\circ \bullet \overline{n-1} \bullet \mathbf{nrec} \ r^\circ \ s' \ \underline{n-1} \\
&= s' \ \underline{n-1} \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n-1}) \\
&= \mathbf{nrec} \ r^\circ \ s' \ \underline{n}
\end{aligned}$$

\square

Lemma 4.11. The translation from $\lambda\mu^{\mathbf{T}}$ into $\lambda^{\mathbf{T}}$ preserves (structural) substitution. That is:

1. $t^\circ[x := r^\circ] \rightarrow (t[x := r])^\circ$
2. $(t[\alpha := \beta \ \square])^\circ \equiv t^\circ[k_\alpha := k_\beta]$
3. $(t[\alpha := \beta \ (\mathbf{S}\square)])^\circ \rightarrow t^\circ[k_\alpha := \lambda l.k_\beta(\mathbf{S}l)]$
4. $(t[\alpha := \beta \ (\square s)])^\circ \rightarrow t^\circ[k_\alpha := \lambda l.l s^\circ k_\beta]$
5. $(t[\alpha := \beta \ (\mathbf{nrec} \ r \ s \ \square)])^\circ \rightarrow t^\circ[k_\alpha := \lambda l.\mathbf{nrec} \ r^\circ \ s' \ l \ k_\beta]$

Proof. These results are proven by induction on the structure of t . \square

Lemma 4.12. The translation from $\lambda\mu^{\mathbf{T}}$ into $\lambda^{\mathbf{T}}$ preserves convertibility. That is, if $t_1 = t_2$, then $t_1^\circ = t_2^\circ$.

Proof. By induction on the derivation of $t_1 \rightarrow t_2$. Most of the cases are straightforward, so we treat just one interesting case.

1. Let $\mathbf{nrec} \ r \ s \ (\mathbf{S}\underline{n}) \rightarrow s \ \underline{n} \ (\mathbf{nrec} \ r \ s \ \underline{n})$. Now:

$$\begin{aligned}
(\mathbf{nrec} \ r \ s \ (\mathbf{S}\underline{n}))^\circ &\equiv \lambda k.(\mathbf{S}\underline{n})^\circ \ (\lambda l.\mathbf{nrec} \ r^\circ \ s' \ l \ k) \\
&\rightarrow \lambda k.\overline{\mathbf{S}\underline{n}} \ (\lambda l.\mathbf{nrec} \ r^\circ \ s' \ l \ k) \\
&\rightarrow \lambda k.\mathbf{nrec} \ r^\circ \ s' \ (\mathbf{S}\underline{n}) \ k \\
&\rightarrow \lambda k.s' \ \underline{n} \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n}) \ k \\
&\rightarrow \lambda k.(s^\circ \bullet \overline{\underline{n}} \bullet \mathbf{nrec} \ r^\circ \ s' \ \underline{n}) \ k \\
&= \lambda k.(\lambda k_2.(s^\circ \bullet \overline{\underline{n}}) \ (\lambda l.l \ (\mathbf{nrec} \ r^\circ \ s' \ \underline{n}) \ k_2)) \ k
\end{aligned} \tag{a}$$

$$\begin{aligned}
&= \lambda k.(s^\circ \bullet \bar{n}) (\lambda l.l (\mathbf{nrec} r^\circ 's \underline{n}) k) \\
&= s^\circ \bullet \bar{n} \bullet \mathbf{nrec} r^\circ s' \underline{n} \\
&= s^\circ \bullet \bar{n} \bullet \lambda k_2.\mathbf{nrec} r^\circ s' \underline{n} k_2 & (b) \\
&= s^\circ \bullet \bar{n} \bullet \lambda k_2.\bar{n} (\lambda l.\mathbf{nrec} r^\circ s' l k_2) \\
&= s^\circ \bullet \bar{n} \bullet \lambda k_2.\underline{n}^\circ (\lambda l.\mathbf{nrec} r^\circ s' l k_2) & (c) \\
&\equiv (s \underline{n} (\mathbf{nrec} r s \underline{n}))^\circ
\end{aligned}$$

Here, step (a) holds by Fact 4.8, step (b) holds by Lemma 4.10 and step (c) holds by Fact 4.8. \square

Theorem 4.13. *Each function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ that is representable in $\lambda\mu^{\mathbf{T}}$ is representable in $\lambda^{\mathbf{T}}$. That is, if a term t with $;\vdash t : \mathbb{N}^n \rightarrow \mathbb{N}$ represents the function f in $\lambda\mu^{\mathbf{T}}$, then there exists a term t' with $\vdash t' : \mathbb{N}^n \rightarrow \mathbb{N}$ that represents the function f in $\lambda^{\mathbf{T}}$.*

Proof. Suppose that $t : \mathbb{N}^n \rightarrow \mathbb{N}$ represents $f : \mathbb{N}^n \rightarrow \mathbb{N}$ in $\lambda\mu^{\mathbf{T}}$. That means that $\underline{f(m_1, \dots, m_n)} = t \underline{m_1} \dots \underline{m_n}$. Now define a term t' as follows.

$$t' := \lambda x_1 : \mathbb{N} \dots \lambda x_n : \mathbb{N} . (t^\circ \bullet \bar{x}_1 \bullet \dots \bullet \bar{x}_n) (\lambda x : \mathbb{N} . x)$$

Now we have $t^\circ : (\mathbb{N}^n \rightarrow \mathbb{N})^\circ$ by Theorem 4.7, $x_i : \mathbb{N}^\circ$ by Fact 4.6 and therefore $t^\circ \bullet \bar{x}_1 \bullet \dots \bullet \bar{x}_n : \mathbb{N}^\circ$ by Fact 4.4. Hence by setting $\perp = \mathbb{N}$ we have $t' : \mathbb{N}$. Now it remains to prove that $\underline{f(m_1, \dots, m_n)} = t' \underline{m_1} \dots \underline{m_n}$.

$$\begin{aligned}
t' \underline{m_1} \dots \underline{m_n} &= (t^\circ \bullet \bar{m_1} \bullet \dots \bullet \bar{m_n}) \lambda x.x \\
&= (t^\circ \bullet \underline{m_1}^\circ \bullet \dots \bullet \underline{m_n}^\circ) \lambda x.x & (a) \\
&= (t \underline{m_1} \dots \underline{m_n})^\circ \lambda x.x \\
&= (\underline{f(m_1, \dots, m_n)})^\circ \lambda x.x & (b) \\
&= \underline{f(m_1, \dots, m_n)} \lambda x.x & (c) \\
&= \underline{f(m_1, \dots, m_n)}
\end{aligned}$$

Here, step (a) holds by Fact 4.8, step (b) holds by Lemma 4.12 and step (c) holds by Fact 4.8. \square

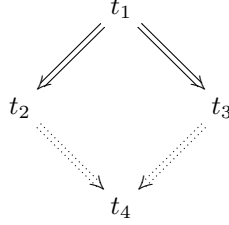
Corollary 4.14. *The functions representable in $\lambda\mu^{\mathbf{T}}$ are exactly those that are provably recursive in first-order arithmetic.*

Proof. This result follows immediately from Theorem 2.10 and 4.13. \square

5 Confluence of $\lambda\mu^{\mathbf{T}}$

To prove confluence one typically uses the notion of *parallel reduction*, as introduced by Tait and Martin-Löf. Intuitively, a parallel reduction relation \Rightarrow allows to contract a number of redexes in a term simultaneously. Following Takahashi [Tak95], \Rightarrow can be defined by induction over the term structure, making it easy to prove that it is preserved under substitution. Then one proves that \Rightarrow satisfies:

- The *diamond property*: if $t_1 \Rightarrow t_2$ and $t_1 \Rightarrow t_3$, then there exists a t_4 such that $t_2 \Rightarrow t_4$ and $t_3 \Rightarrow t_4$, in a diagram:



- $\Rightarrow \subset \Rightarrow$: if $t_1 \Rightarrow t_2$, then $t_1 \Rightarrow t_2$.
- $\Rightarrow \subset \Rightarrow^*$: if $t_1 \Rightarrow t_2$, then $t_1 \Rightarrow^* t_2$.

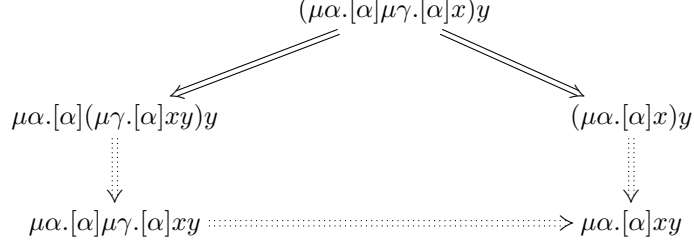
Thus one obtains confluence of \Rightarrow . To streamline proving the diamond property of \Rightarrow one can define the *complete development* of a term t , notation t^\diamond , which is obtained by contracting all redexes in t . Now it suffices to prove that $t_1 \Rightarrow t_2$ implies $t_2 \Rightarrow t_1^\diamond$. Unfortunately, as observed in [Fuj97, BHF01], adopting the notion of parallel reduction in a standard way does not work for $\lambda\mu$. The resulting parallel reduction relation will only be weakly confluent and not confluent.

In this section we will focus on resolving this problem for $\lambda\mu^{\mathbf{T}}$. For an extensive discussion of parallel reduction and its application to various systems we refer to [Tak95]. A simple-minded parallel reduction relation, obtained by extending Parigot's parallel reduction [Par92] to $\lambda\mu^{\mathbf{T}}$, would have the follow rules:

- (t6.1) If $c \Rightarrow c'$, then $\mu\alpha.c \Rightarrow \mu\alpha.c'$.
- (t6.2) If $c \Rightarrow c'$ and $s \Rightarrow s'$, then $(\mu\alpha.c)s \Rightarrow \mu\alpha.c'[\alpha := \alpha (\Box s')]$.
- (t6.3) If $c \Rightarrow c'$, then $\mathbf{S}(\mu\alpha.c) \Rightarrow \mu\alpha.c'[\alpha := \alpha (\mathbf{S}\Box)]$.
- (t6.4) If $r \Rightarrow r'$, $s \Rightarrow s'$ and $c \Rightarrow c'$, then $\mathbf{nrec} r s \mu\alpha.c \Rightarrow \mu\alpha.c'[\alpha := \alpha (\mathbf{nrec} r' s' \Box)]$.
- (t7) If $t \Rightarrow t'$ and $\alpha \notin \text{FCV}(t)$, then $\mu\alpha.[\alpha]t \Rightarrow t'$.
- (c1) If $t \Rightarrow t'$, then $[\alpha]t \Rightarrow [\alpha]t'$.
- (c2) If $c \Rightarrow c'$, then $[\alpha]\mu\beta.c \Rightarrow c'[\beta := \alpha\Box]$.

As has been observed in [Fuj99], Parigot's original parallel reduction relation is not confluent. Similarly, the parallel reduction as defined above for $\lambda\mu^{\mathbf{T}}$ is not confluent. Let us (as in [BHF01]) consider the term $(\mu\alpha.[\alpha]\mu\gamma.[\alpha]x)y$, this term contains both a (t6.2) and a (c2)-redex. However, after contracting the

(t6.2)-redex, we obtain the term $\mu\alpha.[\alpha](\mu\gamma.[\alpha]xy)y$, in which the (c2)-redex is blocked.



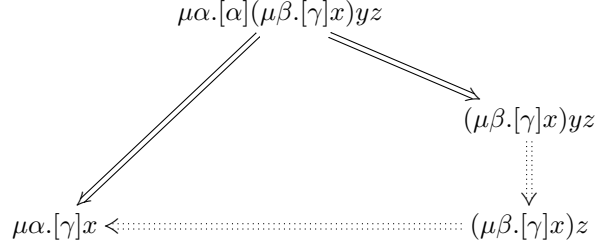
Although it is possible to prove that this relation is weakly confluent, weak confluence is not quite satisfactory. Of course, since $\lambda\mu^{\mathbf{T}}$ is strongly normalizing (Theorem 6.34), it would give confluence by Newman’s lemma. However, an untyped version of $\lambda\mu^{\mathbf{T}}$ is of course not strongly normalizing, hence we do not obtain confluence for raw terms this way.

Baba, Hirokawa and Fujita [BHF01] noticed that this problem could be repaired by allowing a $\mu\beta$ to “jump over a whole context” to its corresponding $[\alpha]$. Their version of the (c2)-rule is as follows.

(c2) If $c \Rightarrow c'$ and $E \Rightarrow E'$, then $[\alpha]E[\mu\beta.c] \Rightarrow c'[\beta := \alpha E']$.

Here E and E' are contexts and parallel reduction on contexts is defined by reducing all its components in parallel. This (c2)-rule performs “deep” structural substitutions and renaming in one step and thus covers and extends the original rules (t6.1-4) and (c2)

Baba *et al.* [BHF01] have shown that their relation \Rightarrow is confluent for $\lambda\mu$ without the (t7) rule. It is not confluent if the (t7) rule is included. Let us (as in [BHF01]) consider the term $\mu\alpha.[\alpha](\mu\beta.[\gamma]x)yz$.



In the conclusion of their work they suggest that this problem can be repaired by considering a series of structural substitutions (t6.1-4) as one step. This approach has been carried out successfully by Nakazawa for a call-by-value variant of $\lambda\mu$ [Nak03]. However, Nakazawa did not use the notion of complete development. We will follow the approach suggested by Baba *et al.* for $\lambda\mu^{\mathbf{T}}$ and use the notion of complete development.

Definition 5.1. Parallel reduction $t \Rightarrow t'$ on terms is mutually inductively defined with parallel reduction $c \Rightarrow c'$ on commands and parallel reduction $E \Rightarrow E'$ on contexts as follows.

- (t1) $x \Rightarrow x$
- (t2) $0 \Rightarrow 0$
- (t3) If $t \Rightarrow t'$, then $\lambda x.t \Rightarrow \lambda x.t'$.
- (t4) If $t \Rightarrow t'$ and $E^s \Rightarrow E'$, then $E^s[t] \Rightarrow E'[t']$.
- (t5) If $t \Rightarrow t'$ and $r \Rightarrow r'$, then $(\lambda x.t)r \Rightarrow t'[x := r']$.
- (t6) If $c \Rightarrow c'$ and $E \Rightarrow E'$, then $E[\mu\alpha.c] \Rightarrow \mu\alpha.c'[\alpha := \alpha E']$.
- (t7) If $t \Rightarrow t'$ and $\alpha \notin \text{FCV}(t)$, then $\mu\alpha.[\alpha]t \Rightarrow t'$.
- (t8) If $r \Rightarrow r'$, then $\mathbf{nrec} \ r \ s \ 0 \Rightarrow r'$.
- (t9) If $r \Rightarrow r'$ and $s \Rightarrow s'$, then $\mathbf{nrec} \ r \ s \ (\mathbf{S}\underline{n}) \Rightarrow s' \ \underline{n} \ (\mathbf{nrec} \ r' \ s' \ \underline{n})$.
- (c1) If $t \Rightarrow t'$, then $[\alpha]t \Rightarrow [\alpha]t'$.
- (c2) If $c \Rightarrow c'$ and $E \Rightarrow E'$, then $[\alpha]E[\mu\beta.c] \Rightarrow c'[\beta := \alpha E']$.
- (E1) $\square \Rightarrow \square$
- (E2) If $E \Rightarrow E'$ and $t \Rightarrow t'$, then $Et \Rightarrow E't'$.
- (E3) If $E \Rightarrow E'$, then $\mathbf{S}E \Rightarrow \mathbf{S}E'$.
- (E4) If $E \Rightarrow E'$, $r \Rightarrow r'$ and $s \Rightarrow s'$, then $\mathbf{nrec} \ r \ s \ E \Rightarrow \mathbf{nrec} \ r' \ s' \ E'$.

Furthermore, \Rightarrow^* denotes the transitive closure of \Rightarrow .

For conciseness of presentation, we specify most of the forthcoming lemmas just for terms. Yet they can always be mutually stated and mutually inductively proven for commands and contexts.

Lemma 5.2. *Parallel reduction is reflexive. That is, $t \Rightarrow t$ for all terms t .*

Proof. By induction on t . We use the rules (t1-4), (t6), (c1) and (E1-4). □

Lemma 5.3. *If $E \Rightarrow E'$ and $t \Rightarrow t'$, then $E[t] \Rightarrow E'[t']$.*

Proof. By induction on the derivation of $E \Rightarrow E'$. □

Lemma 5.4. *If E^s is singular and $E^s \Rightarrow E'$, then E' is singular.*

Proof. By a case analysis on the derivation of $E^s \Rightarrow E'$. □

Lemma 5.5. *If $t \Rightarrow t'$, then $\text{FV}(t') \subseteq \text{FV}(t)$ and $\text{FCV}(t') \subseteq \text{FCV}(t)$.*

Proof. By induction on the derivation of $t \Rightarrow t'$. □

Lemma 5.6. *Parallel reduction is preserved under (structural) substitution.*

1. If $t \Rightarrow t'$ and $s \Rightarrow s'$, then $t[x := s] \Rightarrow t'[x := s']$.

2. If $t \Rightarrow t'$ and $E \Rightarrow E'$, then $t[\alpha := \beta E] \Rightarrow t'[\alpha := \beta E']$.

Proof. By induction on the derivation of $t \Rightarrow t'$. We treat some cases.

(t6) Let $F[\mu\gamma.c] \Rightarrow \mu\gamma.c'[\gamma := \gamma F']$ with $c \Rightarrow c'$ and $F \Rightarrow F'$. Now we have $c[\alpha := \beta E] \Rightarrow c'[\alpha := \beta E']$ and $F[\alpha := \beta E] \Rightarrow F'[\alpha := \beta E']$ by the induction hypothesis. Therefore we have the following.

$$\begin{aligned} (F[\mu\gamma.c])[\alpha := \beta E] &\equiv (F[\alpha := \beta E])[\mu\gamma.c[\alpha := \beta E]] \\ &\Rightarrow \mu\gamma.c'[\alpha := \beta E'][\gamma := \gamma(F'[\alpha := \beta E'])] \\ &\equiv \mu\gamma.c'[\gamma := \gamma F'][\alpha := \beta E'] \\ &\equiv (\mu\gamma.c'[\gamma := \gamma F'])[\alpha := \beta E'] \end{aligned}$$

In the before last step we use a substitution lemma. This is possible because $\gamma \notin \text{FCV}(E)$ by the Barendregt convention and thus $\gamma \notin \text{FCV}(E')$ by Lemma 5.5.

(c2) Let $[\alpha]F[\mu\gamma.c] \Rightarrow c'[\gamma := \alpha F']$ with $c \Rightarrow c'$ and $F \Rightarrow F'$. Now we have $c[\alpha := \beta E] \Rightarrow c'[\alpha := \beta E']$ and $F[\alpha := \beta E] \Rightarrow F'[\alpha := \beta E']$ by the induction hypothesis. Therefore we have the following.

$$\begin{aligned} ([\alpha]F[\mu\gamma.c])[\alpha := \beta E] &\equiv [\beta]E(F[\alpha := \beta E])[\mu\gamma.c[\alpha := \beta E]] \\ &\Rightarrow c'[\alpha := \beta E'][\gamma := \beta E'(F'[\alpha := \beta E'])] \\ &\equiv c'[\gamma := \alpha F'][\alpha := \beta E'] \\ &\equiv (c'[\gamma := \alpha F'])[\alpha := \beta E'] \end{aligned}$$

In the before last step we use a substitution lemma. This is possible because $\gamma \notin \text{FCV}(E)$ by the Barendregt convention and thus $\gamma \notin \text{FCV}(E')$ by Lemma 5.5. \square

A crucial property of a parallel reduction is that a one step reduction is an instance of a parallel reduction and that a parallel reduction is an instance of a multi-step reduction.

Lemma 5.7. *Parallel reduction enjoys the intended behavior. That is:*

1. If $t \rightarrow t'$, then $t \Rightarrow t'$.
2. If $t \Rightarrow t'$, then $t \twoheadrightarrow t'$.

Proof. The first property is proven by induction on the derivation of $t \rightarrow t'$ using that parallel reduction is reflexive (Lemma 5.2). The second by induction on the derivation of $t \Rightarrow t'$ using an obvious substitution lemma for \twoheadrightarrow . \square

To define the complete development of a term t , we need to decide which redexes to contract. This job is non-trivial because \Rightarrow is very strong: In one step it is able to move a subterm that is located very deeply in the term to the outside. For example, consider the command e :

$$e \equiv E_n[\mu\alpha_n.[\alpha_n] \dots E_1[\mu\alpha_1.[\alpha_1]E_0[\mu\alpha_0.c]] \dots] \quad (1)$$

where all the $\mu\alpha_i.[\alpha_i]E_{i-1}$ are $\mu\eta$ -redexes. That is, $\alpha_i \notin \text{FCV}(E_j)$ for all $0 \leq j < i \leq n$ and $\alpha_i \notin \text{FCV}(c)$ for all $0 \leq i \leq n$. Intuitively one would be urged to contract the (t7)-redexes immediately. That yields:

$$E'_n[\dots E'_1[E'_0[\mu\alpha_0.c']]]$$

given complete developments E'_i of E_i and c' of c . However, this is not the complete development of e . We have $[\alpha_{i+1}]E_i[\mu\alpha_i.d] \Rightarrow d$ for each i such that $0 \leq i < n$, hence the whole command e reduces to c' . As this example indicates, it is impossible to determine whether a (t7)-redex should be contracted without looking more deeply into the term. In order to define the complete development we introduce a special kind of context consisting of a series of nested (t7)-redexes, as in (1). Furthermore, we define a case distinction on terms.

Definition 5.8. A $\lambda\mu^{\mathbf{T}}$ η -context (or simply: an η -context) is defined as follows.

$$H ::= \square \mid E[\mu\alpha.[\alpha]H] \quad \text{provided that } \alpha \notin \text{FCV}(H)$$

The operation of substitution of a term for the hole in an η -context is defined in the usual way. However, since these contexts contain μ -binders it is important that this operation is *capture avoiding* for μ -variables. Note also that—in general—an η -context is not a context in the sense of Definition 3.5.

Lemma 5.9. Each term t is of exactly one of the following shapes.

- | | |
|----------|--|
| variable | 1. x |
| value | 2. \underline{n} |
| | 3. $\lambda x.s$ |
| redex | 4. $(\lambda x.s)r$ |
| | 5. $\mathbf{nrec} \ r \ s \ \underline{n}$ |
| | 6. $H[r]$ with $H \not\equiv \square$ and $r \equiv E[\lambda x.s]$, $r \equiv E[0]$ or $r \equiv E[x]$ |
| | 7. $H[E[\mu\beta.c]]$ with $c \equiv [\gamma]s$ and $\gamma \neq \beta$, or $c \equiv [\beta]s$ and $\beta \in \text{FCV}(s)$ |
| other | 8. sr with $s \not\equiv E[\mu\beta.c]$ and $s \not\equiv \lambda x.t$ |
| | 9. $\mathbf{nrec} \ r \ s \ u$ with $u \not\equiv E[\mu\beta.c]$ and $u \not\equiv \underline{n}$ |
| | 10. Su with $u \not\equiv E[\mu\beta.c]$ and $u \not\equiv \underline{n}$ |

Proof. We prove that t is always of one of the given shapes by induction on the structure of t . Furthermore, because these shapes are non-overlapping it is immediate that t is always of exactly one of the given shapes. \square

Definition 5.10. The complete development t^\diamond of a term t is defined (using the case distinction established in Lemma 5.9) as:

1. $x^\diamond := x$

2. $\underline{n}^\diamond := \underline{n}$
3. $(\lambda x.s)^\diamond := \lambda x.s^\diamond$
4. $((\lambda x.s)r)^\diamond := s^\diamond[x := r^\diamond]$
5. $(\mathbf{nrec} \ r \ s \ 0)^\diamond := r^\diamond$
6. $(\mathbf{nrec} \ r \ s \ (\mathbf{S}\underline{n}))^\diamond := s^\diamond \ \underline{n} \ (\mathbf{nrec} \ r^\diamond \ s^\diamond \ \underline{n})$
7. $(H[r])^\diamond := H^\diamond[r^\diamond]$
provided that $H \not\equiv \square$ and $r \equiv E[\lambda x.s]$, $r \equiv E[0]$ or $r \equiv E[x]$.
8. $(H[E[\mu\beta.c]])^\diamond := \mu\beta.c^\diamond[\beta := \beta H^\diamond E^\diamond]$
provided that $c \equiv [\gamma]s$ and $\gamma \neq \beta$, or $c \equiv [\beta]s$ and $\beta \in \text{FCV}(s)$.
9. $(sr)^\diamond := s^\diamond r^\diamond$
provided that $s \not\equiv E[\mu\beta.c]$ and $s \not\equiv \lambda x.t$
10. $(\mathbf{nrec} \ r \ s \ u)^\diamond := \mathbf{nrec} \ r^\diamond \ s^\diamond \ u^\diamond$
provided that $u \not\equiv E[\mu\beta.c]$ and $u \not\equiv \underline{n}$
11. $(\mathbf{S}u)^\diamond := \mathbf{S}u^\diamond$
provided that $u \not\equiv E[\mu\beta.c]$ and $u \not\equiv \underline{n}$

with the complete development c^\diamond of a command c defined as:

1. $([\alpha]E[\mu\beta.c])^\diamond := c^\diamond[\beta := \alpha E^\diamond]$
2. $([\alpha]t)^\diamond := [\alpha]t^\diamond$
provided that $t \not\equiv E[\mu\beta.c]$

the complete development E^\diamond of a context E defined as:

1. $\square^\diamond := \square$
2. $(Et)^\diamond := E^\diamond t^\diamond$
3. $(SE)^\diamond := SE^\diamond$
4. $(\mathbf{nrec} \ r \ s \ E)^\diamond := \mathbf{nrec} \ r^\diamond \ s^\diamond \ E^\diamond$

and the complete development H^\diamond of an η -context H defined as:

1. $\square^\diamond := \square$
2. $(E[\mu\alpha.[\alpha]H])^\diamond := E^\diamond H^\diamond$

Towards a proof of confluence, we now want to prove the following property: if $t \Rightarrow t'$, then $t' \Rightarrow t^\diamond$. This is proven by induction on the structure of t ; the most interesting cases are when $t \equiv H[r]$ (case 7 of Definition 5.10) or $t \equiv H[E[\mu\beta.c]]$ (case 8 of Definition 5.10). For these cases we need some special lemmas.

Lemma 5.11. *Let r be a term such that $r \equiv E[\lambda x.s]$, $r \equiv E[0]$ or $r \equiv E[x]$, and H an η -context. If $[\alpha]H[r] \Rightarrow c$ with $\alpha \notin \text{FCV}(H[r])$, then $c \equiv [\alpha]s$ with $H[r] \Rightarrow s$ and $\alpha \notin \text{FCV}(s)$.*

Proof. By induction on the structure of H . \square

Lemma 5.12. *Let r be a term such that $r \equiv E[\lambda x.s]$, $r \equiv E[0]$ or $r \equiv E[x]$, and H an η -context such that $H \not\equiv \square$. If $H[r] \Rightarrow t$ and for every strict subexpression e of $H[r]$ we have $e \Rightarrow e'$ implies $e' \Rightarrow e^\circ$, then $t \Rightarrow H^\circ[r^\circ]$.*

Proof. We have to consider three cases for the reduction $H[r] \Rightarrow t$.

- (t4) Let $H[r] \equiv E_s[E_r[\mu\beta.[\beta]H_1[r]]] \Rightarrow E'_s[s]$ with E_s a singular context such that $E_s \Rightarrow E'_s$, and $E_r[\mu\beta.[\beta]H_1[r]] \Rightarrow s$. By assumption we have $E'_s \Rightarrow E_s^\circ$ and $s \Rightarrow (E_r[\mu\beta.[\beta]H_1[r]])^\circ \equiv E_r^\circ[H_1^\circ[r^\circ]]$. Therefore, by Lemma 5.3, we obtain that $E'_s[s] \Rightarrow E_s^\circ[E_r^\circ[H_1^\circ[r^\circ]]] \equiv (H[r])^\circ$.
- (t6) Let $H[r] \equiv E[\mu\beta.[\beta]H_1[r]] \Rightarrow \mu\beta.c[\beta := \beta E']$ with $E \Rightarrow E'$ and moreover $[\beta]H_1[r] \Rightarrow c$. By Lemma 5.11, we know that $c \equiv [\beta]s$ with $H_1[r] \Rightarrow s$ and $\beta \notin \text{FCV}(s)$. So we are in the situation

$$H[r] \equiv E[\mu\beta.[\beta]H_1[r]] \Rightarrow \mu\beta.[\beta]E'[s]$$

with $E \Rightarrow E'$ and $H_1[r] \Rightarrow s$. Now $E' \Rightarrow E^\circ$ and $s \Rightarrow (H_1[r])^\circ \equiv H_1^\circ[r^\circ]$ by assumption. Therefore $\mu\beta.[\beta]E'[s] \Rightarrow E^\circ[H_1^\circ[r^\circ]] \equiv (H[r])^\circ$ by Lemma 5.3 and rule (t7).

- (t7) Let $H[r] \equiv \mu\beta.[\beta]H_1[r] \Rightarrow s$ with $H_1[r] \Rightarrow s$. By assumption we have $s \Rightarrow (H_1[r])^\circ \equiv H_1^\circ[r^\circ]$. Therefore $s \Rightarrow H_1^\circ[r^\circ] \equiv (H[r])^\circ$. \square

Lemma 5.13. *Let E be a context, H an η -context, γ a μ -variable, and let d be a command such that $d \equiv [\beta]s$ with $\beta \neq \gamma$ or $d \equiv [\gamma]s$ with $\gamma \in \text{FCV}(s)$. If $H[E[\mu\gamma.d]] \Rightarrow t$ and for every strict subexpression e of $H[E[\mu\gamma.d]]$ we have $e \Rightarrow e'$ implies $e' \Rightarrow e^\circ$, then $t \Rightarrow \mu\alpha.d^\circ[\gamma := \alpha H^\circ E^\circ]$.*

Proof. We prove this result by simultaneously proving the following three properties by induction on the length of H .

1. If $H[E[\mu\gamma.d]] \Rightarrow t$, then $E_2[t] \Rightarrow \mu\alpha.d^\circ[\gamma := \alpha E_2^\circ H^\circ E^\circ]$.
2. If $H[E[\mu\gamma.d]] \Rightarrow t$, then $[\alpha]E_2[t] \Rightarrow d^\circ[\gamma := \alpha E_2^\circ H^\circ E^\circ]$.
3. If $[\alpha]H[E[\mu\gamma.d]] \Rightarrow c$, then $c \Rightarrow d^\circ[\gamma := \alpha H^\circ E^\circ]$.

The base case is where $H \equiv \square$. We only treat a number of instances for the step case, so let $H \equiv E_1[\mu\beta.[\beta]H_1]$.

1. Let $E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow t$. Analyzing the possible steps we prove that for every context E_2 we have:

$$E_2[t] \Rightarrow \mu\alpha.d^\circ[\gamma := \alpha E_2^\circ E_1^\circ H_1^\circ E^\circ].$$

- (t4) Let $E_1 \equiv E_s E_r$ where E_s is a singular context and let $t \equiv E'_s[s]$ with $E_s \Rightarrow E'_s$ and $E_r[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow s$. We can apply the induction hypothesis for property (1) to $E_r[\mu\beta.[\beta]H_1]$. Now we find that for every context E_2 we have:

$$E_2[E'_s[s]] \Rightarrow \mu\alpha.d^\circ[\gamma := \alpha E_2^\circ E_s^\circ E_r^\circ H_1^\circ E^\circ].$$

- (t6) Let $E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow \mu\beta.c[\beta := \beta E'_1]$ with $E_1 \Rightarrow E'_1$ and $[\beta]H_1[E[\mu\gamma.d]] \Rightarrow c$. The induction hypothesis for property (3) yields $c \Rightarrow d^\circ[\gamma := \beta H_1^\circ E^\circ]$. Using the substitution Lemma 5.6 and the rule (t6), we conclude that for any context E_2 we have:

$$E_2[\mu\beta.c[\beta := \beta E'_1]] \Rightarrow \mu\beta.d^\circ[\gamma := \beta E_2^\circ E_1^\circ H_1^\circ E^\circ].$$

- (t7) Let $E_1 \equiv \square$ and $\mu\beta.[\beta]H_1[E[\mu\gamma.d]] \Rightarrow s$ with $H_1[\mu\gamma.d] \Rightarrow s$. The induction hypothesis for property (1) applied to $H_1[E[\mu\gamma.d]]$ tells us that for any context E_2 we have:

$$E_2[s] \Rightarrow \mu\gamma.d^\circ[\gamma := \gamma E_2^\circ H_1^\circ E^\circ].$$

2. A similar argument to the one used for (1) also proves (2).
 3. Let $[\alpha]E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow c$. Analyzing the possible steps we prove that we have:

$$c \Rightarrow d^\circ[\gamma := \alpha E_1^\circ H_1^\circ E^\circ].$$

- (c1) Let $[\alpha]E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow [\alpha]s$ with $E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow s$. To close this case, we have to make a finer case analysis of the possible steps that have led to s . This is similar to what we have done for property (1) above. To close the case we also need the induction hypothesis for property (1) and property (2).
 (c2) Let $[\alpha]E_1[\mu\beta.[\beta]H_1[E[\mu\gamma.d]]] \Rightarrow c[\beta := \alpha E'_1]$ with $E_1 \Rightarrow E'_1$ and $[\beta]H_1[E[\mu\gamma.d]] \Rightarrow c$. We apply the induction hypothesis for property (3) to conclude that $c \Rightarrow d^\circ[\gamma := \beta H_1^\circ E^\circ]$. Therefore we have $c[\beta := \alpha E'_1] \Rightarrow d^\circ[\gamma := \alpha E_1^\circ H_1^\circ E^\circ]$ by the substitution Lemma 5.6 and we are done. \square

Theorem 5.14. *If $t_1 \Rightarrow t_2$, then $t_2 \Rightarrow t_1^\circ$.*

Proof. We prove this result by mutual induction on the structure of terms, commands and contexts. We use the case distinction made in Lemma 5.9. We consider some interesting cases.

1. Let $t_1 \equiv x$. In this case just reduction (t1) is possible, so $x \Rightarrow x^\circ \equiv x$.
2. Let $t_1 \equiv (\lambda x.s_1)r_1$. In this case the following reductions are possible.
 - (t4) $(\lambda x.s_1)r_1 \Rightarrow (\lambda x.s_2)r_2$ with $s_1 \Rightarrow s_2$ and $r_1 \Rightarrow r_2$. Now we have $s_2 \Rightarrow s_1^\circ$ and $r_2 \Rightarrow r_1^\circ$ by the induction hypothesis. Therefore we have $(\lambda x.s_2)r_2 \Rightarrow ((\lambda x.s_1)r_1)^\circ \equiv s_1^\circ[x := r_1^\circ]$.

- (t5) $(\lambda x.s_1)r_1 \Rightarrow s_2[x := r_2]$ with $s_1 \Rightarrow s_2$ and $r_1 \Rightarrow r_2$. Now we have $s_2 \Rightarrow s_1^\diamond$ and $r_2 \Rightarrow r_1^\diamond$ by the induction hypothesis. Therefore $s_2[x := r_2] \Rightarrow ((\lambda x.s_1)r_1)^\diamond \equiv s_1^\diamond[x := r_1^\diamond]$ by Lemma 5.6.
3. Let $t_1 \equiv H_1[r_1]$ with $H_1 \neq \square$ and $r_1 \equiv E[\lambda x.s]$, $r_1 \equiv E[0]$ or $r_1 \equiv E[x]$. Suppose $t_1 \Rightarrow t_2$. Then $t_2 \Rightarrow H_1^\diamond[r_1^\diamond] \equiv t_1^\diamond$ by Lemma 5.12.
 4. Let $t_1 \equiv H_1[E_1[\mu\beta.c_1]]$ with $c_1 \equiv [\gamma]s$ and $\gamma \neq \beta$, or $c_1 \equiv [\beta]s$ and $\beta \in \text{FCV}(s)$. Suppose $t_1 \Rightarrow t_2$, then $t_2 \Rightarrow \mu\alpha.c_1^\diamond[\beta := \alpha H_1^\diamond E_1^\diamond] \equiv t_1^\diamond$ by Lemma 5.13.
 5. Let $t_1 \equiv s_1 r_1$ with $s_1 \neq E[\mu\alpha.c]$ and $s_1 \neq \lambda x.s$. In this case just reduction (t4) is possible, so $s_1 r_1 \Rightarrow s_2 r_2$ with $s_1 \Rightarrow s_2$ and $r_1 \Rightarrow r_2$. Now $s_1 \Rightarrow s_2^\diamond$ and $r_2 \Rightarrow r_1^\diamond$ by the induction hypothesis, so $s_2 r_2 \Rightarrow (s_1 r_1)^\diamond \equiv s_1^\diamond r_1^\diamond$. \square

Corollary 5.15. *Parallel reduction satisfies the diamond property. That is, if $t_1 \Rightarrow t_2$ and $t_1 \Rightarrow t_3$, then there exists a term t_4 such that $t_2 \Rightarrow t_4$ and $t_3 \Rightarrow t_4$.*

Proof. Let $t_4 = t_1^\diamond$. Now we have $t_2 \Rightarrow t_1^\diamond$ and $t_3 \Rightarrow t_1^\diamond$ by Theorem 5.14. \square

Theorem 5.16. *Reduction on $\lambda\mu^{\mathbf{T}}$ is confluent. That is, if $t_1 \rightarrow t_2$ and $t_1 \rightarrow t_3$, then there exists a term t_4 such that $t_2 \rightarrow t_4$ and $t_3 \rightarrow t_4$.*

Proof. By Corollary 5.15 and the fact that $t \Rightarrow^* t'$ if and only if $t \rightarrow t'$, which follows immediately from Lemma 5.7. \square

6 Strong normalization of $\lambda\mu^{\mathbf{T}}$

In this section we prove that the $\lambda\mu^{\mathbf{T}}$ -calculus is strongly normalizing. Unfortunately we cannot use the CPS-translation as defined in Section 4 to prove this result. Our CPS-translation merely preserves typing and convertibility whereas it does not preserve reduction. Defining a CPS-translation that is strictly reduction preserving (each reduction step corresponds to one or more reduction steps under the translation) is already non-trivial for the $\lambda\mu$ -calculus, as Ikeda and Nakazawa [IN06] have shown. We failed to extend their approach to $\lambda\mu^{\mathbf{T}}$ due to difficulties translating the `nrec` construct.

Instead we prove strong normalization by defining two reductions \rightarrow_A and \rightarrow_B such that $\rightarrow = \rightarrow_{AB} := \rightarrow_A \cup \rightarrow_B$. In Section 6.1 we prove, using the reducibility method, that \rightarrow_A is strongly normalizing. In Section 6.2 we prove that \rightarrow_B is strongly normalizing and that both reductions commute in a way that we can obtain strong normalization for \rightarrow_{AB} .

To prove strong normalization of the second order call-by-value $\lambda\mu$ -calculus, Nakazawa [Nak03] characterizes reductions whose strictness is preserved by a modified CPS-translation. Nakazawa also uses a postponement argument, but the proof is very different from ours.

Definition 6.1. *Let \rightarrow_A denote the compatible closure of the reduction rules \rightarrow_β , $\rightarrow_{\mu\mathbf{S}}$, $\rightarrow_{\mu\mathbf{R}}$, \rightarrow_0 , $\rightarrow_{\mathbf{S}}$ and $\rightarrow_{\mu\mathbf{N}}$. Let \rightarrow_B denote the compatible closure of the reduction rules $\rightarrow_{\mu\eta}$ and $\rightarrow_{\mu i}$.*

Definition 6.2. Given a notion of reduction \rightarrow_X (e.g. \rightarrow_A or \rightarrow_B), the set of strongly normalizing terms, notation SN_X , is inductively defined as follows.

1. If for all terms t' with $t \rightarrow_X t'$ we have $t' \in \text{SN}_X$, then $t \in \text{SN}_X$.

Fact 6.3. If t is in \rightarrow_X -normal form, then $t \in \text{SN}_X$.

Fact 6.4. If $t \in \text{SN}_X$ and $t \rightarrow_X t'$, then $t' \in \text{SN}_X$.

6.1 Strong normalization of \rightarrow_A

In this subsection we prove that \rightarrow_A -reduction is strongly normalizing using the reducibility method. Our proof is inspired by Parigot's proof of strong normalization for the $\lambda\mu$ -calculus [Par97].

Since we only consider \rightarrow_A -reduction we will omit subscripts from all notations. Moreover, for conciseness of notation we specify most of the forthcoming lemmas only for terms and not for commands.

The reducibility method is originally due to Tait [Tai67], who proposed the following interpretation for \rightarrow -types.

$$\begin{aligned} \llbracket \alpha \rrbracket &:= \text{SN} \\ \llbracket \sigma \rightarrow \tau \rrbracket &:= \{t \mid \forall s \in \llbracket \sigma \rrbracket . ts \in \llbracket \tau \rrbracket\} \end{aligned}$$

This interpretation makes it possible to prove strong normalization of $\lambda\rightarrow$ in a very short and elegant way [Geu08, for example]. Instead of proving that a term t of type ρ is strongly normalizing one proves a slight generalization, namely $t \in \llbracket \rho \rrbracket$. This method also extends to $\lambda^{\mathbf{T}}$ [GTL89, for example].

Unfortunately, for $\lambda\mu$ it becomes more complicated. If a term of the shape $\lambda x.r$ consumes an argument, the λ -abstraction vanishes. However, if a term of the shape $\mu\alpha.c$ consumes an argument the μ -abstraction remains, hence it is not possible to predict how many arguments $\mu\alpha.c$ will consume. To repair this issue Parigot has proposed a way to switch between a term that is a member of a certain *reducibility candidate* and one that is strongly normalizing when applied to a certain set of sequences of arguments.

In $\lambda\mu^{\mathbf{T}}$ a term of the shape $\mu\alpha.c$ is not only able to consume arguments on its right hand side, but is also able to consume an unknown number of \mathbf{S} 's and \mathbf{nrec} 's. Therefore we generalize Parigot's idea to contexts so that we are able to switch between a term that is a member of a certain reducibility candidate and one that is strongly normalizing in a certain set of contexts.

Before going into the details of the proof we state some facts.

Fact 6.5. If $t \in \text{SN}$, then we have that the length of each \rightarrow_A -reduction sequence starting at t is bounded. We use the notation $\nu(t)$ to denote this bound.

Proof. The result holds because \rightarrow_A -reduction is finitely branching. □

Fact 6.6. If $t \in \text{SN}$ and $t \rightarrow t'$, then $\nu(t') < \nu(t)$.

Fact 6.7. \rightarrow_A -reduction is preserved under (structural) substitution.

1. If $t \rightarrow t'$, then $t[x := s] \rightarrow t'[x := s]$.
2. If $s \rightarrow s'$, then $t[x := s] \rightarrow t[x := s']$.
3. If $t \rightarrow t'$, then $E[t] \rightarrow E[t']$ and $t[\alpha := \beta E] \rightarrow t'[\alpha := \beta E]$.
4. If $E \rightarrow E'$, then $E[t] \rightarrow E'[t]$ and $t[\alpha := \beta E] \rightarrow t[\alpha := \beta E']$.

We now extend the notion of strongly normalizing terms to strongly normalizing contexts. Informally a context is strongly normalizing if all its sub-terms are strongly normalizing.

Definition 6.8. *The set of strongly normalizing contexts, notation SN^\square , is inductively defined as follows.*

1. $\square \in \text{SN}^\square$
2. If $E \in \text{SN}^\square$ and $t \in \text{SN}$, then $Et \in \text{SN}^\square$.
3. If $E \in \text{SN}^\square$, then $SE \in \text{SN}^\square$.
4. If $E \in \text{SN}^\square$, $r \in \text{SN}$ and $s \in \text{SN}$, then $\text{nrec } r s E \in \text{SN}^\square$.

Parigot's approach has another advantage; for the expansion lemmas we do not need to worry about the interpretation of types. We merely need the notion of being strongly normalizing (with respect to some context).

Lemma 6.9. *Let E be a context and r a term such that $r \equiv x$, $r \equiv (\lambda x.r)t$, $r \equiv \text{nrec } r s \underline{n}$ or $r \equiv E^s[\mu\alpha.c]$. If $E[r] \rightarrow t$, then we have:*

1. $t \equiv E[r']$ with $r \rightarrow r'$, or,
2. $t \equiv E'[r]$ with $E \rightarrow E'$.

Proof. We prove the result by induction on the structure of E . We consider only the case $E \equiv Ft$. Here we use the assumption about the shape of r to derive that $F[r]$ cannot be of the shape $\lambda x.s$ or $\mu\beta.c$. This guarantees that $F[r]t$ is not a redex, by which the result follows immediately. \square

Lemma 6.10. *If $r \in \text{SN}$ and $E[t[x := r]] \in \text{SN}$, then $E[(\lambda x.t)r] \in \text{SN}$.*

Proof. We use Fact 6.5 to prove this result by well-founded induction on $\nu(r) + \nu(E[t[x := r]])$. By Definition 6.2 we have to show that for each term w with $E[(\lambda x.t)r] \rightarrow w$ we have $w \in \text{SN}$.

1. Let $w \equiv E[t[x := r]]$. Now $E[t[x := r]] \in \text{SN}$ by assumption.
2. Let $w \equiv E[(\lambda x.t')r]$ and $t \rightarrow t'$. Now $E[t[x := r]] \rightarrow E[t'[x := r]]$ by Fact 6.7, hence $E[t'[x := r]] \in \text{SN}$. By the induction hypothesis we have $E[(\lambda x.t')r] \in \text{SN}$ since $\nu(E[t'[x := r]]) < \nu(E[t[x := r]])$.
3. Let $w \equiv E[(\lambda x.t)r']$ and $r \rightarrow r'$. Now $E[t[x := r]] \rightarrow E[t[x := r']]$ by Fact 6.7 and therefore $E[t[x := r']] \in \text{SN}$. By the induction hypothesis we have $E[(\lambda x.t)r'] \in \text{SN}$ since $\nu(r') < \nu(r)$.

4. Let $w \equiv E[(\lambda x.t)r]$ and $E \rightarrow E'$. Now $E[t[x := r]] \rightarrow E'[t[x := r]]$ by Fact 6.7, hence $E'[t[x := r]] \in \text{SN}$. By the induction hypothesis we have $E'[(\lambda x.t)r] \in \text{SN}$ since $\nu(E'[t[x := r]]) < \nu(E[t[x := r]])$.

Lemma 6.9 guarantees that we have considered all possible shapes of w . \square

Lemma 6.11. *If $F^s \in \text{SN}^\square$ and $E[\mu\alpha.c[\alpha := \alpha F^s]] \in \text{SN}$, then $E[F^s[\mu\alpha.c]] \in \text{SN}$.*

Proof. The proof is similar to the proof of Lemma 6.10. \square

Corollary 6.12. *If $F \in \text{SN}^\square$ and $E[\mu\alpha.c[\alpha := \alpha F]] \in \text{SN}$, then $E[F[\mu\alpha.c]] \in \text{SN}$.*

Proof. By induction on the structure of F .

1. Let $F \equiv \square$. We have $E[\mu\alpha.c] \equiv E[\mu\alpha.c[\alpha := \alpha \square]]$ for each context E and command c , so by assumption we are done.
2. Let $F \equiv G^s H$. By an obvious substitution lemma and assumption we have $E[\mu\alpha.c[\alpha := \alpha H][\alpha := \alpha G^s]] \equiv E[\mu\alpha.c[\alpha := \alpha F]] \in \text{SN}$. Therefore we have $E[G^s[\mu\alpha.c[\alpha := \alpha H]]] \in \text{SN}$ by Lemma 6.11. Hence $E[G^s[H[\mu\alpha.c]]] \in \text{SN}$ by the induction hypothesis. \square

Lemma 6.13. *For each context E we have the following.*

1. *If $E[r] \in \text{SN}$ and $s \in \text{SN}$, then $E[\text{nrec } r \ s \ 0] \in \text{SN}$.*
2. *If $E[s \ \underline{n} \ (\text{nrec } r \ s \ \underline{n})] \in \text{SN}$, then $E[\text{nrec } r \ s \ (\text{SN})] \in \text{SN}$.*

Proof. We use Fact 6.5 and prove (1) by induction on $\nu(E[r]) + \nu(s)$ and (2) by induction on $\nu(E[s \ \underline{n} \ (\text{nrec } r \ s \ \underline{n})])$. Similar to the proof of Lemma 6.10 we distinguish various cases. \square

Parigot extends the well-known *functional construction* of two sets of terms S and T ($S \rightarrow T := \{t \mid \forall u \in S . tu \in T\}$) to a set \mathcal{S} of sequences of terms and a set T of terms as follows.

$$\mathcal{S} \rightarrow T := \{t \mid \forall \vec{u} \in \mathcal{S} . t\vec{u} \in T\}$$

Moreover, he defines the notion of reducibility candidates in such way that each reducibility candidate R can be expressed as $\mathcal{S} \rightarrow \text{SN}$ for a certain set of sequences of terms \mathcal{S} . Therefore he is able to switch between the proposition $t \in R$ and the proposition $t\vec{u} \in \text{SN}$ for all $\vec{u} \in \mathcal{S}$. We extend Parigot's notion of functional construction to contexts in the obvious way.

Definition 6.14. *Given a set of contexts \mathcal{E} and a set of terms T , the functional construction $\mathcal{E} \rightarrow T$ is defined as follows.*

$$\mathcal{E} \rightarrow T := \{t \mid \forall E \in \mathcal{E} . E[t] \in T\}$$

Given two sets of terms S and T , then $S \rightarrow T$ is defined as follows.

$$S \rightarrow T := \{\square u \mid u \in S\} \rightarrow T$$

Remark that, for sets of terms S and T , our definition of the functional construction $S \rightarrow T$ is equivalent to the ordinary definition.

$$S \rightarrow T = \{\square u \mid u \in S\} \rightarrow T = \{t \mid \forall u \in S. tu \in T\}$$

Keeping in mind that we wish to express each reducibility candidate R as $\mathcal{E} \rightarrow \mathbf{SN}$ for some \mathcal{E} , one might try to define the collection of reducibility candidates as the smallest set that contains \mathbf{SN} and is closed under functional construction and arbitrary intersection. But then $\{\mathbf{nrec} \ \Omega \ \Omega \ \square\} \rightarrow \mathbf{SN} = \emptyset$ is a valid candidate too. To avoid this we should be a bit more careful.

Definition 6.15. *We define the collection of reducibility candidates, \mathcal{R} , inductively as follows.*

- (sn) $\mathbf{SN} \in \mathcal{R}$
- (\bigcap) If $\emptyset \subset \mathbf{R} \subseteq \mathcal{R}$, then $\bigcap \mathbf{R} \in \mathcal{R}$.
- (app) If $S, T \in \mathcal{R}$, then $S \rightarrow T \in \mathcal{R}$.
- (suc) If $T \in \mathcal{R}$, then $\{\mathbf{S}\square\} \rightarrow T \in \mathcal{R}$.
- (nrec) If $S, T \in \mathcal{R}$, then $\{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T \in \mathcal{R}$.

Lemma 6.16. *For each $R \in \mathcal{R}$ we have the following.*

1. $R \subseteq \mathbf{SN}$
2. $E[x] \in R$ for each x and $E \in \mathbf{SN}^\square$.

Proof. We prove these results simultaneously by induction on the generation of R . We consider some interesting cases.

- (sn) Let $R = \mathbf{SN}$. We certainly have $R \subseteq \mathbf{SN}$. Also, $E[x] \in \mathbf{SN}$ by Lemma 6.9.
- (\bigcap) Let $R = \bigcap \mathbf{R}$. By the induction hypothesis we have $T \subseteq \mathbf{SN}$ for each $T \in \mathbf{R}$. Therefore we have $\bigcap \mathbf{R} \subseteq \mathbf{SN}$, so the first property holds.
By the induction hypothesis we also have $E[x] \in T$ for each $T \in \mathbf{R}$ and $E \in \mathbf{SN}^\square$. Therefore we have $E[x] \in \mathbf{R}$ for each $E \in \mathbf{SN}^\square$, so the second property holds as well.
- (suc) Let $R = \{\mathbf{S}\square\} \rightarrow T$. To prove the first property, we suppose that $t \in R$. This means that $\mathbf{S}t \in T$. Therefore $\mathbf{S}t \in \mathbf{SN}$ because $T \subseteq \mathbf{SN}$ by the induction hypothesis. Now certainly $t \in \mathbf{SN}$, so the first property holds.
To prove the second property we have to show that $E[x] \in R$. By the induction hypothesis we have $E[x] \in T$ for each $E \in \mathbf{SN}^\square$. In particular we have $\mathbf{S}E[x] \in T$. This means that $E[x] \in R$, so the second property holds as well.

(nrec) Let $R = \{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T$. To prove the first property, we suppose that $t \in R$. This means that $\mathbf{nrec} \ r \ s \ t \in T$ for each $r \in T$ and $s \in S \rightarrow T \rightarrow T$. By the induction hypothesis we have an $x \in T$ and $y \in S \rightarrow T \rightarrow T$, hence $\mathbf{nrec} \ x \ y \ t \in T$. Thus $t \in \mathbf{SN}$ because $T \subseteq \mathbf{SN}$ by the induction hypothesis, so the first property holds.

To prove the second property we have to show that $E[x] \in R$. By the induction hypothesis we have $E[x] \in T$ for each $E \in \mathbf{SN}^\square$. In particular we have $\mathbf{nrec} \ r \ s \ E[x] \in T$. This means that $E[x] \in R$, so the second property holds as well. \square

As we have remarked before, we wish to express each reducibility candidate R as $\mathcal{E} \rightarrow \mathbf{SN}$ for some set of contexts \mathcal{E} . Now we will make that idea precise.

Definition 6.17. *Given an $R \in \mathcal{R}$, a set of contexts R^\perp is inductively defined on the generation of R as follows.*

$$\begin{aligned} \mathbf{SN}^\perp &:= \{\square\} \\ (\bigcap \mathbf{R})^\perp &:= \bigcup \{T^\perp \mid T \in \mathbf{R}\} \\ (S \rightarrow T)^\perp &:= \{\square\} \cup \{E(\square u) \mid u \in S, E \in T^\perp\} \\ (\{\mathbf{S}\square\} \rightarrow T)^\perp &:= \{\square\} \cup \{E(\mathbf{S}\square) \mid E \in T^\perp\} \\ (\{\mathbf{nrec} \ r \ s \ \square\} \rightarrow T)^\perp &:= \{\square\} \cup \{E(\mathbf{nrec} \ r \ s \ \square) \mid r \in T, s \in S \rightarrow T \rightarrow T, E \in T^\perp\} \end{aligned}$$

Fact 6.18. *For each $R \in \mathcal{R}$ we have $\square \in R^\perp$.*

Lemma 6.19. *For each $R \in \mathcal{R}$ we have $R = R^\perp \rightarrow \mathbf{SN}$.*

Proof. By induction on the generation of R . We consider some interesting cases.

(sn) Let $R = \mathbf{SN}$. We have $R = \{\square\} \rightarrow \mathbf{SN}$, so we are done.

(\bigcap) Let $R = \bigcap \mathbf{R}$. By the induction hypothesis we have $T = T^\perp \rightarrow \mathbf{SN}$ for each $T \in \mathbf{R}$. Therefore we have the following.

$$\begin{aligned} R &= \bigcap \{T \mid T \in \mathbf{R}\} \\ &= \bigcap \{T^\perp \rightarrow \mathbf{SN} \mid T \in \mathbf{R}\} \\ &= \bigcap \{\{t \mid \forall E \in T^\perp . E[t] \in \mathbf{SN}\} \mid T \in \mathbf{R}\} \\ &= \{t \mid \forall T \in \mathbf{R}, E \in T^\perp . E[t] \in \mathbf{SN}\} \\ &= \{t \mid \forall E \in \bigcup \{T^\perp \mid T \in \mathbf{R}\} . E[t] \in \mathbf{SN}\} \\ &= \bigcup \{T^\perp \mid T \in \mathbf{R}\} \rightarrow \mathbf{SN} \end{aligned}$$

(nrec) Let $R = \{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T$. By the induction hypothesis we have $T = T^\perp \rightarrow \mathbf{SN}$. Therefore we have the following.

$$\begin{aligned}
R &= \{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T \\
&= \{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T^\perp \rightarrow \mathbf{SN} \\
&= \{t \mid \forall r \in T, s \in S \rightarrow T \rightarrow T. \mathbf{nrec} \ r \ s \ t \in T^\perp \rightarrow \mathbf{SN}\} \\
&= \{t \mid \forall E \in T^\perp, r \in T, s \in S \rightarrow T \rightarrow T. E[\mathbf{nrec} \ r \ s \ t] \in \mathbf{SN}\} \\
&= \{t \mid t \in \mathbf{SN} \wedge \forall E \in T^\perp, r \in T, s \in S \rightarrow T \rightarrow T. E[\mathbf{nrec} \ r \ s \ t] \in \mathbf{SN}\} \\
&= (\{\square\} \cup \{E(\mathbf{nrec} \ r \ s \ \square) \mid r \in T, s \in S \rightarrow T \rightarrow T, E \in T^\perp\}) \rightarrow \mathbf{SN}
\end{aligned}$$

The before last step holds because for all terms t , if $E[\mathbf{nrec} \ r \ s \ t] \in \mathbf{SN}$ for all $E \in T^\perp$, $r \in T$, $s \in S \rightarrow T \rightarrow T$, then also $t \in \mathbf{SN}$. This is because T^\perp , T and $S \rightarrow T \rightarrow T$ are non-empty by Fact 6.18 and Lemma 6.16. \square

Lemma 6.20. *For each $R \in \mathcal{R}$ we have $t \in R$ iff $E[t] \in \mathbf{SN}$ for all $E \in R^\perp$.*

Proof. We have $t \in R$ iff $t \in R^\perp \rightarrow \mathbf{SN}$ by Lemma 6.19, and $t \in R^\perp \rightarrow \mathbf{SN}$ iff $E[t] \in \mathbf{SN}$ for all $E \in R^\perp$ by Definition 6.14. \square

Now, to prove strong normalization of \rightarrow_A , it remains to give an interpretation $\llbracket \rho \rrbracket \in \mathcal{R}$ for each type ρ . As a first attempt, we could adapt the definition for $\lambda \rightarrow$, which we have given in the introduction of this section.

$$\begin{aligned}
\llbracket \mathbf{N} \rrbracket &:= \mathbf{SN} \\
\llbracket \sigma \rightarrow \tau \rrbracket &:= \llbracket \sigma \rrbracket \rightarrow \llbracket \tau \rrbracket
\end{aligned}$$

Unfortunately, the interpretation of \mathbf{N} does not contain enough structure to prove the following properties.

1. If $t \in \mathbf{SN}$, then $\mathbf{S}t \in \mathbf{SN}$.
2. If $t \in \mathbf{SN}$, $r \in S$ and $s \in \mathbf{SN} \rightarrow S \rightarrow S$, then $\mathbf{nrec} \ r \ s \ t \in S$.

Here, the term t could reduce to a term of the shape $\mu\alpha.c$ and is thereby able to consume the surrounding \mathbf{S} or \mathbf{nrec} . To define an interpretation of \mathbf{N} that contains more structure we introduce the following definition.

Definition 6.21. *We define the collection \mathcal{N} inductively as follows.*

- (sn) $\mathbf{SN} \in \mathcal{N}$
- (suc) If $S \in \mathcal{N}$, then $\{\mathbf{S}\square\} \rightarrow S \in \mathcal{N}$.
- (nrec) If $S \in \mathcal{N}$ and $T \in \mathcal{R}$, then $\{\mathbf{nrec} \ r \ s \ \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T \in \mathcal{N}$.

Fact 6.22. $\mathcal{N} \subseteq \mathcal{R}$

Definition 6.23. The interpretation $\llbracket \rho \rrbracket$ of a type ρ is defined as follows.

$$\begin{aligned}\llbracket \mathbb{N} \rrbracket &:= \bigcap \mathcal{N} \\ \llbracket \sigma \rightarrow \tau \rrbracket &:= \llbracket \sigma \rrbracket \rightarrow \llbracket \tau \rrbracket\end{aligned}$$

Fact 6.24. For each type ρ we have $\llbracket \rho \rrbracket \in \mathcal{R}$.

Lemma 6.25. For each $n \in \mathbb{N}$ we have $\underline{n} \in \llbracket \mathbb{N} \rrbracket$.

Proof. In order to prove this result we have to show that $\underline{n} \in R$ for all $R \in \mathcal{N}$ and $n \in \mathbb{N}$. We proceed by induction on the generation of R .

- (var) Let $R = \text{SN}$. Now we have to show that $\underline{n} \in \text{SN}$ for all $n \in \mathbb{N}$. However, \underline{n} is in normal form, so we certainly have $\underline{n} \in \text{SN}$.
- (suc) Let $R = \{\text{S}\square\} \rightarrow S$. Now we have $\underline{n} \in S$ for all $n \in \mathbb{N}$ by the induction hypothesis. It remains to show that $\text{S}\underline{n} \in S$ for all $n \in \mathbb{N}$. However, $\text{S}\underline{n} \equiv \underline{n+1}$, so the required result follows from the induction hypothesis.
- (nrec) Let $R = \{\text{nrec } r \text{ s } \square \mid r \in T, s \in S \rightarrow T \rightarrow T\} \rightarrow T$. Now we have $\underline{n} \in S$ for all $n \in \mathbb{N}$ by the induction hypothesis. It remains to show that $\text{nrec } r \text{ s } \underline{n} \in T$ for all $S \in \mathcal{N}, T \in \mathcal{R}, r \in T, s \in S \rightarrow T \rightarrow T$ and $n \in \mathbb{N}$. We proceed by induction on n .
 - (a) Let $n = 0$. We have $E[r] \in \text{SN}$ for all $E \in T^\perp$ by Lemma 6.20 and $s \in \text{SN}$ by Lemma 6.16. Hence $E[\text{nrec } r \text{ s } 0] \in \text{SN}$ by Lemma 6.13 and therefore $\text{nrec } r \text{ s } 0 \in T$ by Lemma 6.20.
 - (b) Let $n > 0$. We have $\text{nrec } r \text{ s } \underline{n-1} \in T$ by the induction hypothesis. Furthermore, because $s \in S \rightarrow T \rightarrow T$ and $\underline{n-1} \in S$, we have $s \underline{n-1} (\text{nrec } r \text{ s } \underline{n-1}) \in T$, so $E[s \underline{n-1} (\text{nrec } r \text{ s } \underline{n-1})] \in \text{SN}$ for all $E \in T^\perp$ by Lemma 6.20. Therefore $E[\text{nrec } r \text{ s } (\text{S}\underline{n-1})] \in \text{SN}$ by Lemma 6.13, so $\text{nrec } r \text{ s } \underline{n} \in T$ by Lemma 6.20. \square

Lemma 6.26. If $t \in \llbracket \mathbb{N} \rrbracket$, then $\text{St} \in \llbracket \mathbb{N} \rrbracket$.

Proof. Assume that $t \in \llbracket \mathbb{N} \rrbracket$. This means, $t \in R$ for all $R \in \mathcal{N}$. Now we have to prove that $\text{St} \in R$ for all $R \in \mathcal{N}$. But for all $R \in \mathcal{N}$ we have $\{\text{S}\square\} \rightarrow R \in \mathcal{N}$, hence $t \in \{\text{S}\square\} \rightarrow R$ by assumption and therefore $\text{St} \in R$. \square

Lemma 6.27. If $r \in \llbracket \rho \rrbracket$, $s \in \llbracket \mathbb{N} \rightarrow \rho \rightarrow \rho \rrbracket$ and $t \in \llbracket \mathbb{N} \rrbracket$, then $\text{nrec } r \text{ s } t \in \llbracket \rho \rrbracket$.

Proof. We have $\llbracket \mathbb{N} \rrbracket \in \mathcal{N}$ by Definition 6.23, so if $t \in \llbracket \mathbb{N} \rrbracket$, then $\text{nrec } r \text{ s } t \in T$ for all $T \in \mathcal{R}, r \in T$ and $s \in \llbracket \mathbb{N} \rrbracket \rightarrow T \rightarrow T$ by Definition 6.21. Also $\llbracket \rho \rrbracket \in \mathcal{R}$ by Fact 6.24 and $\llbracket \mathbb{N} \rightarrow \rho \rightarrow \rho \rrbracket = \llbracket \mathbb{N} \rrbracket \rightarrow \llbracket \rho \rrbracket \rightarrow \llbracket \rho \rrbracket$, hence $\text{nrec } r \text{ s } t \in \llbracket \rho \rrbracket$. \square

Theorem 6.28. Let $x_1 : \rho_1, \dots, x_n : \rho_n; \alpha_1 : \sigma_1, \dots, \alpha_m : \sigma_m \vdash t : \tau$ such that $r_i \in \llbracket \rho_i \rrbracket$ for all $1 \leq i \leq n$ and $E_j \in \llbracket \sigma_j \rrbracket^\perp$ for all $1 \leq j \leq m$, then:

$$t[x_1 := r_1, \dots, x_n := r_n, \alpha_1 := \alpha_1 E_1, \dots, \alpha_m := \alpha_m E_m] \in \llbracket \tau \rrbracket.$$

Proof. Abbreviate $\Gamma = x_1 : \rho_1, \dots, x_n : \rho_n$, $\Delta = \alpha_1 : \sigma_1, \dots, \alpha_m : \sigma_m$, with $t' \equiv t[x_1 := r_1, \dots, x_n := r_n, \alpha_1 := \alpha_1 E_1, \dots, \alpha_m := \alpha_m E_m]$, and $c' \equiv c[x_1 := r_1, \dots, x_n := r_n, \alpha_1 := \alpha_1 E_1, \dots, \alpha_m := \alpha_m E_m]$. Now by mutual induction we prove that $\Gamma; \Delta \vdash t : \tau$ implies $t' \in \llbracket \tau \rrbracket$ and that $\Gamma; \Delta \vdash c : \perp$ implies $c' \in \text{SN}$.

- (var) Let $\Gamma; \Delta \vdash x : \sigma$ with $x : \sigma \in \Gamma$. Now we have $x' \in \llbracket \sigma \rrbracket$ by assumption.
- (λ) Let $\Gamma; \Delta \vdash \lambda x : \sigma. t : \sigma \rightarrow \tau$ with $\Gamma, x : \sigma; \Delta \vdash t : \tau$. Moreover let $u \in \llbracket \rho \rrbracket$ and $E \in \llbracket \tau \rrbracket^\perp$. Now we have $t'[x := u] \in \llbracket \tau \rrbracket$ by the induction hypothesis and so $E[t'[x := u]] \in \text{SN}$ by Lemma 6.20. Therefore $E[(\lambda x.t')u] \in \text{SN}$ by Lemma 6.10 and hence $(\lambda x.t')u \in \llbracket \tau \rrbracket$ by Lemma 6.20, so $\lambda x.t' \in \llbracket \sigma \rightarrow \tau \rrbracket$ by Definition 6.14.
- (app) Let $\Gamma; \Delta \vdash ts : \tau$ with $\Gamma; \Delta \vdash t : \sigma \rightarrow \tau$ and $\Gamma; \Delta \vdash s : \sigma$. Now we have $t' \in \llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \rightarrow \llbracket \tau \rrbracket$ and $s' \in \llbracket \sigma \rrbracket$ by the induction hypothesis, hence $t's' \in \llbracket \tau \rrbracket$ by Definition 6.14.
- (zero) Let $\Gamma; \Delta \vdash 0 : \mathbb{N}$. Now we have $0 \in \llbracket \mathbb{N} \rrbracket$ by Lemma 6.25.
- (suc) Let $\Gamma; \Delta \vdash St : \mathbb{N}$ with $\Gamma; \Delta \vdash t : \mathbb{N}$. Now we have $t' \in \llbracket \mathbb{N} \rrbracket$ by the induction hypothesis and therefore $St' \in \llbracket \mathbb{N} \rrbracket$ by Lemma 6.26.
- (nrec) Let $\Gamma; \Delta \vdash \mathbf{nrec} r s t : \rho$ with $\Gamma; \Delta \vdash r : \rho$, $\Gamma; \Delta \vdash s : \mathbb{N} \rightarrow \rho \rightarrow \rho$ and $\Gamma; \Delta \vdash t : \mathbb{N}$. Now we have $r' \in \llbracket \rho \rrbracket$, $s' \in \llbracket \mathbb{N} \rightarrow \rho \rightarrow \rho \rrbracket$ and $t' \in \llbracket \mathbb{N} \rrbracket$ by the induction hypothesis. Therefore $\mathbf{nrec} r' s' t' \in \llbracket \rho \rrbracket$ by Lemma 6.27.
- (act) Let $\Gamma; \Delta \vdash \mu\alpha : \rho. c : \rho$ with $\Gamma; \Delta. \alpha : \rho \vdash c : \perp$. Moreover let $E \in \llbracket \rho \rrbracket^\perp$. Now we have $c'[\alpha := \alpha E] \in \text{SN}$ by the induction hypothesis. Hence $\mu\alpha. c'[\alpha := \alpha E] \in \text{SN}$ and therefore $E[\mu\alpha. c'] \in \text{SN}$ by Corollary 6.12, so $\mu\alpha. c' \in \llbracket \rho \rrbracket$ by Lemma 6.20.
- (pas) Let $\Gamma; \Delta \vdash [\alpha]t : \perp$ with $\alpha : \sigma \in \Delta$ and $\Gamma; \Delta \vdash t : \sigma$. Now we have $t' \in \llbracket \sigma \rrbracket$ by the induction hypothesis. Also, we have a context $E \in \llbracket \sigma \rrbracket^\perp$ by assumption. Therefore $E[t'] \in \text{SN}$ by Lemma 6.20 and so $[\alpha]E[t'] \in \text{SN}$ because $([\alpha]t)' = [\alpha]E[t']$. \square

Corollary 6.29. *If $\Gamma; \Delta \vdash t : \rho$, then $t \in \text{SN}_A$.*

Proof. We have $x_i \in \llbracket \rho_i \rrbracket$ for each $x_i : \rho_i \in \Gamma$ by Lemma 6.16 and $\square \in \llbracket \sigma_j \rrbracket^\perp$ for each $\alpha_j : \sigma_j \in \Delta$ by Fact 6.18. Therefore $t \in \llbracket \rho \rrbracket$ by Theorem 6.28 and hence $t \in \text{SN}_A$ by Fact 6.24 and Lemma 6.16. \square

6.2 Strong normalization of \rightarrow_{AB}

In this section we prove that \rightarrow_B is strongly normalizing and that \rightarrow_A -steps can be advanced. Together with strong normalization of \rightarrow_A this is sufficient to prove strong normalization of \rightarrow_{AB} . Proving strong normalization of \rightarrow_{AB}

from \rightarrow_A and \rightarrow_B is not specific to $\lambda\mu^T$. Krebbers² provides a proof of this result based on abstract relations in the Coq proof assistant.

Lemma 6.30. *For each term t we have $t \in \text{SN}_B$.*

Proof. By performing a $\rightarrow_{\mu\eta}$ or $\rightarrow_{\mu i}$ -reduction step on t , the term t reduces strictly in its size and therefore \rightarrow_B -reduction is strongly normalizing. \square

Lemma 6.31. *A single \rightarrow_A -reduction step can be advanced. That means, if $t_1 \rightarrow_B t_2 \rightarrow_A t_3$, then there is a t_4 such that the following diagram commutes.*

$$\begin{array}{ccc} t_1 & \xrightarrow{B} & t_2 \\ A \downarrow & & \downarrow A \\ t_4 & \xrightarrow{AB} & t_3 \end{array}$$

Proof. We prove this lemma by distinguishing cases on $t_1 \rightarrow_B t_2$ and $t_2 \rightarrow_A t_3$, we treat some interesting cases.

1. Let $(\lambda x.t)r \rightarrow_B (\lambda x.t)r' \rightarrow_A \lambda x.t[x := r']$ with $r \rightarrow_B r'$. Now by an obvious substitution lemma we have $t[x := r] \rightarrow_{AB} t[x := r']$, hence the following diagram commutes.

$$\begin{array}{ccc} (\lambda x.t)r & \xrightarrow{B} & (\lambda x.t)r' \\ A \downarrow & & \downarrow A \\ t[x := r] & \xrightarrow{AB} & t[x := r'] \end{array}$$

2. Let $E^s[\mu\alpha.[\alpha]\mu\beta.c] \rightarrow_B E^s[\mu\alpha.c[\beta := \alpha \square]] \rightarrow_A \mu\alpha.c[\beta := \alpha \square][\alpha := \alpha E^s]$. Now the following diagram commutes by an obvious substitution lemma.

$$\begin{array}{ccc} E^s[\mu\alpha.[\alpha]\mu\beta.c] & \xrightarrow{B} & E^s[\mu\alpha.c[\beta := \alpha \square]] \\ A \downarrow & & \downarrow A \\ \mu\alpha.[\alpha]E^s[\mu\beta.c[\alpha := \alpha E^s]] & & \mu\alpha.c[\beta := \alpha \square][\alpha := \alpha E^s] \\ & \searrow A \quad \nearrow B & \\ & \mu\alpha.[\alpha]\mu\beta.c[\alpha := \alpha E^s][\beta := \beta E^s] & \end{array} \quad \square$$

Corollary 6.32. *A single \rightarrow_A -reduction step after multiple \rightarrow_B -reduction steps can be advanced. That means, if $t_1 \rightarrow_B t_2 \rightarrow_A t_3$, then there is a t_4 such that the following diagram commutes.*

$$\begin{array}{ccc} t_1 & \xrightarrow{B} & t_2 \\ A \downarrow & & \downarrow A \\ t_4 & \xrightarrow{AB} & t_3 \end{array}$$

²The Coq proof is available at http://robertkrebbers.nl/misc/sn_commute.{v,html}.

Proof. The result holds by repeatedly applying Lemma 6.31 starting from right to left as the diagram indicates. \square

$$\begin{array}{ccccccc}
 t_1 & \xrightarrow{B} & t_2 & \xrightarrow{B} & t_{n-1} & \xrightarrow{B} & t_n \\
 \downarrow A & & \downarrow A & & \downarrow A & & \downarrow A \\
 t'_1 & \xrightarrow{AB} & t'_2 & \xrightarrow{AB} & t'_{n-1} & \xrightarrow{AB} & t'_n
 \end{array}$$

Lemma 6.33. *If $t \in \text{SN}_A$, then $t \in \text{SN}_{AB}$.*

Proof. We prove this result by induction on the derivation of $t \in \text{SN}_A$, so by the induction hypothesis we obtain that for each term t' with $t \rightarrow_A t'$ we have $t' \in \text{SN}_{AB}$. By Lemma 6.30 we have $t \in \text{SN}_B$, hence it suffices to prove that for all reduction sequences $t \rightarrow_B t_2 \rightarrow_A t_3$ we have $t_3 \in \text{SN}_{AB}$. Now by Corollary 6.32 we obtain a t_4 such that the following diagram commutes.

$$\begin{array}{ccc}
 t & \xrightarrow{B} & t_2 \\
 \downarrow A & & \downarrow A \\
 t_4 & \xrightarrow{AB} & t_3
 \end{array}$$

By the induction hypothesis we have $t_4 \in \text{SN}_{AB}$. Therefore, since $t_4 \rightarrow_{AB} t_3$, we have $t_3 \in \text{SN}_{AB}$ by Fact 6.4, so we are done. \square

Theorem 6.34. *If t is well-typed, then $t \in \text{SN}_{AB}$.*

Proof. This result follows directly from Theorem 6.33 and Corollary 6.29. \square

7 Conclusions and further work

In this paper we have introduced the $\lambda\mu^{\mathbf{T}}$ -calculus, an extension of Parigot's $\lambda\mu$ -calculus to include a type of natural numbers \mathbb{N} with primitive recursion `nrec`, à la Gödel's \mathbf{T} . We have proven the main meta-theoretical properties and have shown that exactly the provably recursive functions in first-order arithmetic can be represented.

In order to maintain confluence and a normal form theorem the $\lambda\mu^{\mathbf{T}}$ -calculus is not a straightforward combination of the $\lambda\mu$ -calculus and Gödel's \mathbf{T} . Both these systems are originally call-by-name, whereas $\lambda\mu^{\mathbf{T}}$ is a call-by-name system with strict evaluation on datatypes.

In our treatment of the reduction rules in $\lambda\mu^{\mathbf{T}}$, we have observed a tension between the call-by-name features taken directly from Parigot's original calculus, and the need to restrict the rules for the datatypes to be call-by-value. We plan to investigate a fully-fledged call-by-value version of $\lambda\mu^{\mathbf{T}}$ (see for example [OS97, Py98] for definitions of a call-by-value variant of $\lambda\mu$). We expect that, apart from our proof of strong normalization, most of our results will extend

to such a system. For a proof of strong normalization we will likely experience problems related to those discussed in [DN05]. The key issue is our Lemma 6.10, which states that if $r \in \mathbf{SN}$ and $E[t[x := r]] \in \mathbf{SN}$, then $E[(\lambda x.t)r] \in \mathbf{SN}$. In a call-by-value variant the reduction rule $v(\mu\alpha.c) \rightarrow \mu\alpha.c[\alpha := \alpha (v\Box)]$ will complicate this because $(\lambda x.t)r$ is not solely a β -redex anymore.

Instead of the $\lambda\mu$ -calculus it would be interesting to consider a system with the control operators `catch` and `throw` as primitive (see Figure 4 for the typing rules). Such a system is described by Crolard [Cro99], who proves a correspondence with $\lambda\mu$. Herbelin [Her10] also considers a variant of such a system to define an intuitionistic logic that proves a variant of Markov’s principle.

$$\frac{\Gamma; \Delta, \alpha : \rho \vdash t : \rho}{\Gamma; \Delta \vdash \mathbf{catch}_\alpha t : \rho} \quad \frac{\Gamma; \Delta \vdash t : \rho \quad \alpha : \rho \in \Delta}{\Gamma; \Delta \vdash \mathbf{throw}_\alpha t : \tau}$$

(a) catch (b) throw

Figure 4: The typing rules for the primitives `catch` and `throw`.

The further reaching goal would be to define a dependently typed λ -calculus with datatypes and control operators that allows program extraction from classical proofs. In such a calculus one can write specifications of programs, which can then be proven using classical logic. The extraction mechanism would then extract a program from such a proof, where the classical reasoning steps are extracted to control operators. This would yield programs-with-control that are *correct by construction* because they are extracted from a proof of the specification. This would extend the well-known extraction method for constructive proofs, see [PM89] for example, to classical proofs.

This goal is particularly useful to obtain provably correct algorithms where the use of control operators would really pay off (for example if a lot of backtracking is involved). See [CGU00] for applications to classical search algorithms. The work of Makarov [Mak06] may also be useful here, as it gives ways to optimize program extraction to make it feasible for practical programming.

Acknowledgments We are grateful to the anonymous referees who spotted some mistakes in earlier versions of this paper and provided several helpful suggestions.

References

- [Acz78] Peter Aczel. A general Church-Rosser theorem. Technical report, University of Manchester, 1978.
- [AH03] Zena M. Ariola and Hugo Herbelin. Minimal Classical Logic and Control Operators. In Jos C. M. Baeten, Jan Karel Lenstra, Joachim Parrow, and Gerhard J. Woeginger, editors, *ICALP*, volume 2719 of *LNCS*, pages 871–885. Springer, 2003.

- [BB96] Franco Barbanera and Stefano Berardi. A symmetric lambda calculus for classical program extraction. *Information and Computation*, 125(2):103–117, 1996.
- [BBS00] Ulrich Berger, Wilfried Buchholz, and Helmut Schwichtenberg. Refined Program Extraction from Classical Proofs. In *Annals of Pure and Applied Logic*, pages 77–97. Springer Verlag, 2000.
- [BHF01] Kensuke Baba, Sachio Hirokawa, and Ken-etsu Fujita. Parallel Reduction in Type Free $\lambda\mu$ -calculus. *ENTCS*, 42:52–66, 2001.
- [BS95] Ulrich Berger and Helmut Schwichtenberg. Program development by proof transformation. In Helmut Schwichtenberg, editor, *Proof and Computation, NATO Advanced Study Institute, International Summer School held in Marktoberdorf, Germany, 1993*, volume 139 of *Series F: Computer and Systems Sciences*, pages 1–45. Springer-Verlag, 1995.
- [BU02] Gilles Barthe and Tarmo Uustalu. CPS Translating Inductive and Coinductive Types. In Peter Thiemann, editor, *PEPM*, pages 131–142. ACM, 2002.
- [CF98] Loïc Colson and Daniel Fredholm. System T, call-by-value and the minimum problem. *Theoretical Computer Science*, 206(1-2):301–315, 1998.
- [CGU00] James L. Caldwell, Ian P. Gent, and Judith Underwood. Search Algorithms in Type Theory. *Theoretical Computer Science*, 232(1-2):55–90, 2000.
- [Coq96] Thierry Coquand. Computational content of classical logic. In *Semantics and Logics of Computation*, pages 470–517. Cambridge University Press, 1996.
- [CP11] Tristan Crolard and Emmanuel Polonowski. A program logic for higher-order procedural variables and non-local jumps, 2011. Technical report TR-LACL-2011-4. <http://arxiv.org/abs/1112.1554>.
- [Cro99] Tristan Crolard. A confluent lambda-calculus with a catch/throw mechanism. *Journal of Functional Programming*, 9(6):625–647, 1999.
- [dG94] Philippe de Groote. A CPS-translation of the $\lambda\mu$ -calculus. In Sophie Tison, editor, *CAAP*, volume 787 of *LNCS*, pages 85–99. Springer, 1994.
- [DN05] René David and Karim Nour. Why the usual candidates of reducibility do not work for the symmetric $\lambda\mu$ -calculus. *ENTCS*, 140:101–111, 2005.
- [FH92] Matthias Felleisen and Robert Hieb. The Revised Report on the Syntactic Theories of Sequential Control and State. *Theoretical Computer Science*, 103(2):235–271, 1992.
- [Fuj97] Ken-etsu Fujita. Calculus of Classical Proofs I. In R. K. Shyamasundar and Kazunori Ueda, editors, *ASIAN*, volume 1345 of *LNCS*, pages 321–335. Springer, 1997.
- [Fuj99] Ken-etsu Fujita. Explicitly Typed $\lambda\mu$ -calculus for Polymorphism and Call-by-Value. In Jean-Yves Girard, editor, *TLCA*, volume 1581 of *LNCS*, pages 162–176. Springer, 1999.
- [Geu08] Herman Geuvers. Introduction to type theory. In Ana Bove, Luís Soares Barbosa, Alberto Pardo, and Jorge Sousa Pinto, editors, *LerNet ALFA Summer School*, volume 5520 of *LNCS*, pages 1–56. Springer, 2008.

- [Gri90] Timothy G. Griffin. A Formulae-as-Types Notion of Control. In *POPL*, pages 47–58. ACM, 1990.
- [GTL89] Jean Y. Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Cambridge University Press, 1989.
- [Her10] Hugo Herbelin. An Intuitionistic Logic that Proves Markov’s Principle. In *LICS*, pages 50–56. IEEE Computer Society, 2010.
- [IN06] Satoshi Ikeda and Koji Nakazawa. Strong normalization proofs by CPS-translations. *Information Processing Letters*, 99(4):163–170, 2006.
- [Mak06] Yevgeni Makarov. Practical program extraction from classical proofs. In *MFPS*, volume 155 of *ENTCS*, pages 521 – 542, 2006.
- [Mur90] Chetan Murthy. *Extracting Constructive Content from Classical Proofs*. PhD thesis, Cornell University, 1990.
- [Nak03] Koji Nakazawa. Confluency and Strong Normalizability of Call-by-Value $\lambda\mu$ -calculus. *Theoretical Computer Science*, 290(1):429–463, 2003.
- [OS97] C.-H. Luke Ong and Charles A. Stewart. A Curry-Howard foundation for functional computation with control. In *POPL*, pages 215–227, 1997.
- [Par92] Michel Parigot. $\lambda\mu$ -calculus: An Algorithmic Interpretation of Classical Natural Deduction. In Andrei Voronkov, editor, *LPAR*, volume 624 of *LNCS*, pages 190–201. Springer, 1992.
- [Par93] Michel Parigot. Classical Proofs as Programs. In Georg Gottlob, Alexander Leitsch, and Daniele Mundici, editors, *Kurt Gödel Colloquium*, volume 713 of *LNCS*, pages 263–276. Springer, 1993.
- [Par97] Michel Parigot. Proofs of Strong Normalisation for Second Order Classical Natural Deduction. *Journal of Symbolic Logic*, 62(4):1461–1479, 1997.
- [PM89] Christine Paulin-Mohring. Extracting F_ω ’s programs from proofs in the Calculus of Constructions. In *POPL*. ACM, 1989.
- [Py98] Walter Py. *Confluence en $\lambda\mu$ -calcul (in French)*. PhD thesis, Université de Savoie, 1998.
- [RS94] Jakob Rehof and Morten Heine Sørensen. The λ_Δ -calculus. In Masami Hagiya and John C. Mitchell, editors, *TACS*, volume 789 of *LNCS*, pages 516–542. Springer, 1994.
- [Ste72] Sören Stenlund. *Combinators, λ -terms and proof theory*. D. Reidel Dordrecht, 1972.
- [SU06] Morten Heine Sørensen and Pawel Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, volume 149 of *Studies in Logic and The Foundations of Mathematics*. Elsevier Science, 2006.
- [Tai67] William W. Tait. Intensional Interpretations of Functionals of Finite Type I. *Journal of Symbolic Logic*, 32(2):198–212, 1967.
- [Tak95] Masako Takahashi. Parallel Reductions in λ -Calculus. *Information and Computation*, 118(1):120–127, 1995.
- [vBLL05] Steffen van Bakel, Stéphane Lengrand, and Pierre Lescanne. The Language χ : Circuits, Computations and Classical Logic. In Mario Coppo, Elena Lodi, and G. Michele Pinna, editors, *ICTCS*, volume 3701 of *LNCS*, pages 81–96. Springer, 2005.