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Tools for Thought: The Case of Mathematics

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ABSTRACT

The objective of this article is to take into account the functioning of *representational cognitive tools*, and in particular of notations and visualizations in mathematics. In order to explain their functioning, formulas in algebra and logic and diagrams in topology will be presented as case studies and the notion of *manipulative imagination* as proposed in previous work will be discussed. To better characterize the analysis, the notions of *material anchor* and *representational affordance* will be introduced.

Introduction: (Representational) Cognitive Tools

The distinction between image and word is commonplace in the human sciences. Both images and words are representations; however, the philosophical literature on depiction generally distinguishes between images, which resemble the objects they depict, and words, which have meaning thanks to conventions. Much of the philosophical discussion either takes this distinction for granted or argues in its favor. One of the aims of the present article will be to go beyond this distinction and to introduce the category of *representational cognitive tools*.

The notion of representational cognitive tools refers to specific cases where external and material representations are used as *inferential* tools, that is, as tools for thinking about what they represent. To quote Sybille Krämer, there exists "a sizable class of representational tools" such as writing, tables, graphs, diagrams, or maps that arises at the conjunction of word and image; she defines this class "the diagrammatic."[1] Following her suggestion, in the remainder of the paper I will focus on a number of representations that belong to the diagrammatic. As Krämer claims, an important feature characterizing these objects is that they make "showing" and "saying" work together so as to create an "operative iconicity": as representations, they have an iconic nature; however, they are meant to be changed, transformed, put into use to the aim of learning something new about what they represent.

To give an example, a map is of course assumed to represent some environment—let us say the Stanford University campus—but it is also intended to be oriented by its users in the appropriate way depending both on their position in physical space and on their purpose, for instance if they want to figure out how to go from the entrance of the campus to the library. People must rely on several different capacities in order to use the map: they have to perceive and select its relevant visual features, make the appropriate connections between what is sketched on the map and what is physically in their vicinity, locate themselves in the map right where they are physically standing, read the labels, correctly interpret the colors in the map, and so on. As a consequence, the map *simultaneously* shows them the campus and, in metaphorical terms that need to be accounted for, "tells" them where to go if they want to reach the library. In Krämer's words, the external and material representations belonging to the diagrammatic "are not only a medium for the representation of the objects of knowledge, but also at the same time an instrument through which those very objects can be *generated* and *explored*."[2] I will assume this as the main characterizing feature of representational cognitive tools: they are intended both as representations *and* as instruments to think about what they represent; their dual nature transcends the distinction between image and word.

Representational cognitive tools are a subcategory of cognitive tools, that is, of external and material objects that humans have manufactured and produced in their cultural evolution. Think, for example, of the abacus: despite not being in any way straightforwardly representational, it has been introduced as an instrument to perform arithmetic calculations. The manufacture and the production of such tools may have had an influence on human thinking. Jack Goody's work on *writing* is revealing in this regard: according to the anthropologist, the externalization of thought that

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[1] Sybille Krämer, "Trace, Writing, Diagram: Reflections on Spatiality, Intuition, Graphical Practices and Thinking," in *The Power of the Image. Emotion, Expression, Explanation*, ed. András Benedek and Kristóf Nyiri, 3–22 (Frankfurt am Main: Peter Lang, 2014), 3.

[2] Krämer, "Trace, Writing, Diagram" (ref. 1), 3.

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0160-9327/

has happened with writing has changed forever the very structure of our thinking.[3] Goody argues that it is wrong to think of oral cultures as totally analogous to written cultures *minus* writing, because the two are qualitatively distinct cultures, where memory and narrative forms are *structurally* different. In his view, the literate activity transforms human life, for example by allowing the accumulation of knowledge about the surrounding world. Writing is intended here in a wide sense. For example, a mathematical table is for Goody a product of writing despite the fact that, thanks to the spatial arrangement of its entries, people who can neither read nor write can easily use it. However, this is of course conditional on the kind of information contained in the table: for example, a minimal literacy level would be required to recognize numerical entries. The use of a street map presupposes writing but may also imply literacy: someone who is familiar with the arbitrary order of letters of the Latin alphabet will have no problem in finding the address of a friend in an unknown town. The important point here is that for Goody these external and material objects provide us with “a special cognitive tool, a *technology for the intellect*.”[4]

The third point to stress about cognitive tools concerns the conditions for their use. I have proposed elsewhere that humans possess some kind of ability that I call “diagramming.”[5] Thanks to diagramming, humans were able to create and employ cognitive tools as a medium in which an external connection is obtained between several different systems already available in other contexts, in view of a totally novel *cognitive* task: enhancing inference and reasoning. In other words, cognitive tools are multi-recruiting systems: they constitute an “interface” with which to integrate information coming from perception or action, already functioning in pragmatic contexts, and other more cognitive resources such as conceptual knowledge. The cognitive tool is perceived, changed, manipulated in view of getting some new information: the simultaneous activation of all these systems becomes relevant for a particular epistemic purpose. To go back to the map example, it is thanks to the map and, more importantly, *on* the map that users combine pieces of information coming from their visuo-spatial system with others obtained from their conceptual system and to some extent from their motor system, for example by reorienting the map in space; the result is an inference about the direction to take to reach the desired destination. It is important to note that such an inference would *not* have been accessible to the visuo-spatial, the conceptual, or the motor system alone.

To sum up, I characterize cognitive tools as follows: (i) they are external and material objects that constitute a *technology* for the intellect, since they are the product of our cultural evolution and have structured our thought; (ii) their use triggers a form of *diagramming*; that is, they are a medium to recruit multiple systems in view of a novel cognitive objective. Moreover, some cognitive tools are also *representational* and correspond to Krämer’s class of the diagrammatic, transcending the distinction between image and word. Compared to other cognitive tools, they present an additional feature: (iii) they are representations and *at the same time* instruments to study what they represent. In the case of representational cognitive tools, the triggering of diagramming corresponds to what Krämer labels “operative iconicity.” In the remainder of the article, I will give some examples of the use of representational cognitive tools in mathematics and analyze them in relation to the characterization just given.

[3] Jack Goody, *The Power of the Written Tradition* (Washington, DC: Smithsonian Institution Press, 2000).

[4] Goody, *Written Tradition* (ref. 3), 148. Emphasis added.

[5] Valeria Giardino, “Diagramming: Connecting Cognitive Systems to Improve Reasoning,” in Benedek and Nyiri, eds., *Power of the Image* (ref. 1), 23–34.

Two Case Studies from Mathematics

Ten years ago, Paolo Mancosu, in his introduction to a pioneering collection of essays on the philosophy of mathematical practice, claimed that attention to mathematical *practice* was a necessary condition for a renewal of the philosophy of mathematics.[6] One of the main features of the practice of mathematics is the use of many heterogeneous cognitive tools, ranging from formal languages to figures and illustrations, to the aim of experimenting, discovering new results, and explaining already established ones. As José Ferreirós argues in a recent book, mathematical practices typically involve writing, in particular complex semiotic systems of written symbols.[7] My proposal is then to look at the practice of mathematics by focusing on some of the representational cognitive tools it involves. First, I will consider the role of notation in algebra and logic; second, I will present part of my work in collaboration with Silvia De Toffoli on the practice of topology.[8]

Formulas in Algebra and Propositional Logic

As discussed above, the notion of representational cognitive tool as characterized in the introduction allows going beyond the opposition of image and word. For example, algebraic formulas and geometric figures do not belong to different categories when considered as representational cognitive tools: first, they are both the product of cultural evolution and they both structure our thought; second, they are both representations and at the same time instruments for thought. However, a third feature characterizing representational cognitive tools is that they trigger diagramming; that is, they happen to be multi-recruiting systems. In this section, the question will be the following: is it also the case for formulas? Geometric figures will be taken into account later.

In an interesting study, David Landy and Robert Goldstone considered simple equations and their physical layout to the aim of evaluating whether it may affect their segmentation.[9] As everyone with minimal competence in algebra knows, when reading a notational form, one has to segment it into its formal components; for example, in the simplest cases, a formula displaying additions and multiplications has to be parsed in such a way that multiplication is performed before addition (see Figure 1, fourth line, for an example). Hence, one may conclude that mathematics concerns only abstract reasoning and is totally disconnected from its (visual) written form: segmentation would be cognitively executed through the mere application of formal rules to individual notational symbols. However, this would bring to the “appealing and tempting assumption” that it is trivial for the cognitive agent—the parser—to extract abstract symbol sequences from physical notations.[10] To challenge this claim, the two researchers asked the participants in one of their experiments to judge the validity of equations containing both multiplication and addition, to which they added visual cues such as spacing,

[6] Paolo Mancosu, ed. *The Philosophy of Mathematical Practice* (Oxford: Oxford University Press, 2008).

[7] José Ferreirós, *Mathematical Knowledge and the Interplay of Practices* (Princeton: Princeton University Press, 2015).

[8] Silvia De Toffoli and Valeria Giardino, “Forms and Roles of Diagrams in Knot Theory,” *Erkenntnis* 79, no. 4 (2014): 829–42; “An Inquiry into the Practice of Proving in Low-Dimensional Topology,” *Boston Studies in the Philosophy and History of Science* 308 (2015): 315–36.

[9] David Landy and Robert L. Goldstone, “How Abstract is Symbolic Thought?,” *Journal of Experimental Psychology: Learning, Memory, and Cognition* 33, no. 4 (2007): 720–33.

[10] Landy and Goldstone, “Symbolic Thought” (ref. 9), 721.

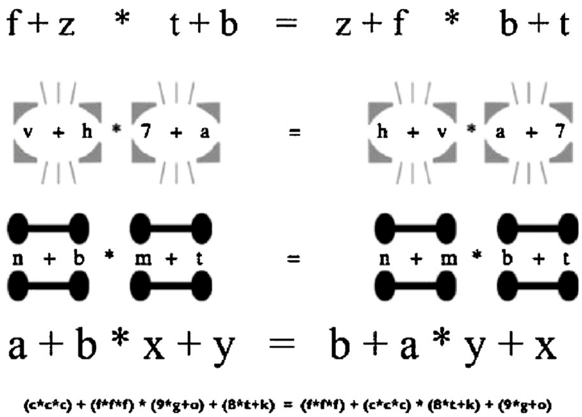

Figure 1. Some of the formats employed by Landy and Goldstone (2007). In the formula on the first line, spaces are added between the two factors to multiply, so as to encourage a visual grouping that is not in line with the correct order of the operators. The figure is taken from David Landy, Colin Allen, and Carlos Zednik, "A Perceptual Account of Symbolic Reasoning," *Frontiers in Psychology* 5, no. 275 (2014).

lines, or circles. The hypothesis was that such cues would have an influence on the application of perceptual grouping mechanisms and consequently on the participants' capacity for symbolic reasoning (see Figure 1). The results of the experiment show that validity judgments are more likely to be correct if visual groupings are *in line with* valid operator precedence (multiplication comes before addition); the physical layout of the equation seems thus to have an influence on performance.

Of course, there is a trivial sense in which the physical layout of a formula clearly has an influence on performance: for example, people would have difficulties in segmenting a formula that is written in very small type or in which the addition signs are tilted in such a way that they look like multiplication signs. However, the case at stake here is more interesting because the experimenters' manipulations, despite the fact that the formula is perfectly readable, still interfere with perception by activating groupings that are here not relevant for the segmentation task. If this is true, then it shows that formulas trigger diagramming and the activation of perceptual systems, such as groupings mechanisms, and this may have an influence on cognitive performance.

Based on these and similar results, in a more recent paper, David Landy, Colin Allen, and Carlos Zednik put forward the *Perceptual Manipulations Theory*, according to which most of symbolic reasoning emerges from the ways in which notational formalisms are perceived and manipulated; in their view, notations serve as *targets* for powerful perceptual and sensorimotor systems.[11] It should be noted that there is a sense in which the fact that these systems are triggered by the use of these cognitive tools might have its advantages in facilitating inference, for example by *liberating* the user from semantic processing in symbol manipulations. For example, in the task of dividing 3/4 by 7/8, thinking semantically would be particularly unhelpful, since our normal understanding of division makes divisions by a fraction difficult to grasp. However, the material transformation that allows us to invert the divisor and then multiply rather than divide is very easy to apply: $3/4 \div 7/8 = 3/4 \times 8/7 = 6/7$. Once the users learn how to correctly group the numbers—that is, to switch numerator and denominator of the second fraction and change the sign from division to multiplication—they can easily proceed thanks to "de-semantification" (another term that is introduced by Krämer to define "operative writing").[12]

Other scholars have underlined the importance of perception and action in the use of mathematical notations. For example, Philip Kellman, Christine Massey, and Ji Son, explicitly write about *perceptual learning* in mathematics, that is, the improvements that are produced by the familiarity with a cognitive tool.[13] In their reconstruction, except for occasionally mentioning pattern recognition, the literature on education dismisses perceptual learning as irrelevant. For example, studies involving sensory discriminations among a small set of fixed stimuli are considered to have little connection to learning tasks in the real world, in particular when it comes to high-level, explicit and symbolic domains such as mathematics. On the contrary, according to the authors, improvements in information extraction are motivated by perceptual learning as a result of practice and are shown to be very relevant for acquiring the appropriate expertise in almost any domain. There is no reason to exclude mathematics from these dynamics. This would pave the way toward new strategies for teaching mathematics: students should be trained in recognizing symbolic expressions by using standard perceptual learning techniques and this might lead to lasting gains both in equation reading and comprehension and in algebraic problem-solving. An important point would be to focus on the selection of relevant information and on the fluent extraction of structure.

Of course, considering notation as a cognitive tool does not amount to claiming that mathematical reasoning only involves perception, or that the possible actions on the notation are driven by perceptual groupings alone. On the contrary, cognitive tools trigger diagramming; that is, they recruit multiple systems, one of which can certainly be the conceptual system. The important point is that the influence of perception and action on mathematical reasoning is a further aspect of mathematical practice that has been commonly neglected and adds up to more complex cognitive processes.

In line with these observations, and from within the philosophy of mathematical practice, a trend that we might call philosophy of notation has emerged with the objective of defining the *design principles* that can be found behind the introduction of specific notations with a particular intended aim, and the *trade-offs* that these principles imply. In a very recent study, Dirk Schlimm analyzed the physical features of the notation that Gottlob Frege developed in his *Begriffsschrift* by focusing in particular on propositional logic.[14] Schlimm's conclusion is that Frege's two-dimensional notation was intended to be effective, in particular to encourage groupings according to some specific perceptual chunks and thus to make the statements more easily readable so as to facilitate further transformations.

In Frege's notation, each line corresponds to an individual proposition that is logically linked to the ones above and underneath (see Figure 2). As Frege himself explains, "the *Begriffsschrift* makes the most of the two-dimensionality of the writing surface by allowing the assertible contents to follow one *below* the other while each of these *extends* [*separately*] *from left to right*. Thus, the *separate* contents are clearly *separated* from each

[11] David Landy, Colin Allen, and Carlos Zednik, "A Perceptual Account of Symbolic Reasoning," *Frontiers in Psychology* 5, no. 275 (2014): 1–10.

[12] Sybille Krämer, "Writing, Notational Iconicity, Calculus: On Writing as a Cultural Technique," *Modern Languages Notes* 118, no. 3 (2003): 518–37.

[13] Philip J. Kellman, Christine M. Massey, and Ji Y. Son, "Perceptual Learning Modules in Mathematics: Enhancing Students' Pattern Recognition, Structure Extraction, and Fluency," *Topics in Cognitive Science* 2, no. 2 (2010): 285–305.

[14] Dirk Schlimm, "On Frege's Begriffsschrift Notation for Propositional Logic: Design Principles and Trade-Offs," *History and Philosophy of Logic*, 39, no. 1 (2018): 53–79.

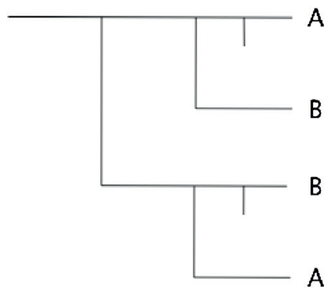

Figure 2. An example of Frege's two-dimensional notation: the segment on the left is used to express implication between the implication underneath and the one above. The small vertical segment is used to express negation.

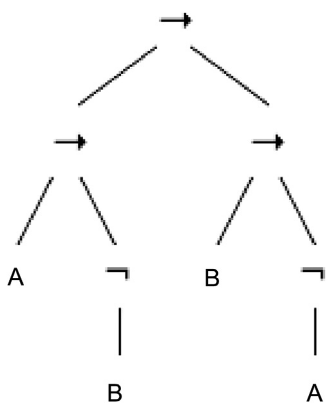

Figure 3. An example of a syntax tree, starting from the main connective and constructing other trees as branches.

other, and yet their logical relations are *easily visible at a glance*."[15] He thus designed his notation by following these principles: he wanted "contents" to be to be organized along both the vertical and horizontal axes, to be clearly *separated* from each other, and their "logical relations" to be *easy visible at a glance*.

To give an example, compare different notations that express the same logical proposition.[16] The statement $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$, as displayed by the standard notation, becomes $A \supset \sim B. \supset. B \supset \sim A$ in Russell and Whitehead's notation, where points are used instead of parentheses, or CCANBCBNA in the Polish notation, or the schema in Figure 2 in Frege's notation. Alternatively, another possible notation is based on syntax trees and is shown in Figure 3.

Schlimm's point is that syntax trees should be considered the canonical notation for the reason that, thanks to their physical features, they make all relevant structural relations explicit. In a syntax tree, it is very easy to find the main connective: it suffices to look at the top node, regardless of the complexity of the entire formula. Moreover, subformulas are very easy to individuate because they are simply subtrees to the left and to the right of the top node. This task becomes cumbersome in the case of very long propositions as expressed by one of the linear notations, in which parentheses and points are required.

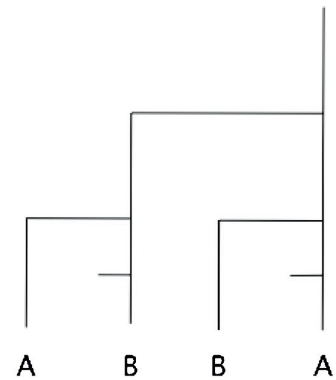

Figure 4. A rotation of 90° degrees of Frege's notation makes it "look like" a syntax tree.

Surprisingly, Frege's notation happens to be very close to syntax trees (which were introduced in linguistics long after Frege). In fact, it suffices to rotate Frege's notation 90° to the left as displayed in Figure 4, and then to insert the appropriate label nodes, to obtain the corresponding syntax tree in Figure 3.

Details about this sort of transformation can be found in Schlimm, where the correspondences between the geometrical features of Frege's *Begriffsschrift* and the logical operators are thoroughly presented.[17] However, part of my aim in this section is precisely to point out at the fact that without giving these correspondences explicitly, the reader should be able to extract them from the visual features of the two-dimensional notations, supposedly with no particular cognitive effort. A different but related point is that it is an empirical question to decide which notation would be more suitable for which set of further transformations and as a consequence for a particular epistemic use, on the basis of the way we spontaneously perceive and spontaneously act on them.

An explanation is needed for the reasons why Frege's notation, despite its alleged cognitive advantages, was in the end unsuccessful. In Schlimm's reconstruction, anybody familiar with logical systems in the nineteenth century (when no syntax trees were available) would have recognized Frege's notation as quite odd. Frege himself seemed to be aware of that, since he mentions in the preface of his work that readers might be "frightened off by the first impression of unfamiliarity."[18] The main reason for the failure of Frege's attempt might then be found in the profound unfamiliarity of his two-dimensional notation for Frege's contemporaries, and not in some intrinsic cognitive inconvenience. This case suggests that the designers of new innovative cognitive tools should worry about them to be "familiar enough," if they want them to be adopted by a particular community of practitioners.

I will come back to formulas later in the discussion.

Diagrams in Topology

My second example of representational cognitive tools in mathematics relates to some figures that are used in the practice of topology, which were the target of some previous work with De Toffoli.[19] Topology is a branch of geometry that deals with those

[15] Gottlob Frege, "Über den Zweck der Begriffsschrift," *Jenaische Zeitschrift für Naturwissenschaft* 16, Neue Folge 9, suppl. (1882–1883): 1–10, on 7–8. English translation: "On the Aim of the 'Conceptual Notation,'" in Gottlob Frege, *Conceptual Notation and Related Articles*, 90–100 (Oxford: Oxford University Press, 1972). Emphasis added.

[16] Of course, more complex and mathematically relevant examples might be given, which would be closer to the challenge Frege wanted to respond to. However, for the sake of clarity, I will refer to a very simple one.

[17] Schlimm, "On Frege's Begriffsschrift Notation" (ref. 14).

[18] Gottlob Frege, *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Halle an der Salle: Louis Nebert, 1879), 7. English translation: "Begriffsschrift, A Formula Language, Modeled upon that for Arithmetic," in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, in Jean van Heijenoort, ed., 1–82 (Cambridge, MA: Harvard University Press, 1967).

[19] De Toffoli and Giardino, "Forms and Roles of Diagrams"; "Inquiry into the Practice" (ref. 8).

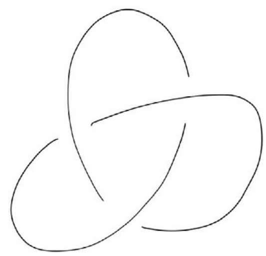

Figure 5. A knot diagram of the trefoil knot.

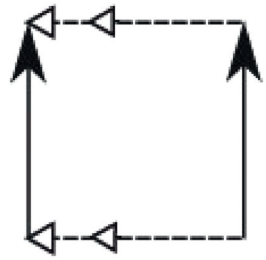

Figure 6. A representation of the *torus* as constructed from a square with its sides identified. Reproduced with permission from Silvia De Toffoli.

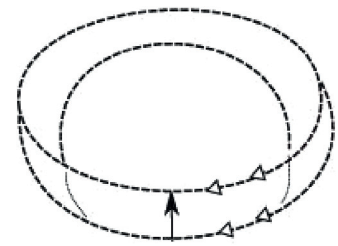

Figure 7. A representation of a cylinder as obtained by identifying two sides of a square in the same direction. Reproduced with permission from Silvia De Toffoli.

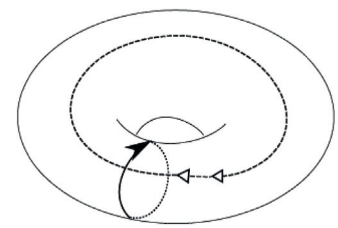

Figure 8. A representation of the *torus*; the two marked curves indicate where the gluings (the identifications) were made. Reproduced with permission from Silvia De Toffoli.

properties of geometric configurations that remain unchanged under continuous deformations such as stretching or twisting, that is, up to homeomorphisms; in other words, one can think of topology as "rubber sheet" geometry.

In a first article devoted to knot theory, which studies topological knots, we focused on the use of *knot diagrams*, like the one depicted in Figure 5. We argued that knot diagrams are not only visual projections of a knot but also *dynamic* tools: when looking at a knot diagram, experts envisage possible moves on it, in order to infer new information about the properties of the corresponding knot. Knot diagrams are representational cognitive tools because they are external and material representations produced by the practice of topology on which an expert can carefully arrange the pattern of inferences. Moreover, they trigger a specific form of diagramming that is *manipulative* imagination: experts know how to perform or how to imagine performing some inferential actions on the diagrams. Finally, they are representational cognitive tools because they are at the same time representations of knots and tools supporting mathematical operations on them.

To give an example of how manipulative imagination works, consider the construction of the *torus* as a square with its sides identified (see Figure 6). This example is taken from some further work on the practice of low-dimensional topology.[20]

Given a square with boundary, that is, a surface homeomorphic to a disk D^2, it is possible to identify or, in manipulative terms, "to glue," two of its opposite sides in order to obtain another surface. If the two sides are glued in the same direction, as indicated by the black arrows, then a *cylinder* is obtained (see Figure 7). Then, the other two sides of the square in the direction of the white double arrows are glued,[21] thus obtaining the *torus* (see Figure 8).

The square diagram in Figure 6 is a cognitive tool because: (i) it is an external and material object that can be shared among practitioners and structure their thoughts; (ii) it triggers manipulative imagination and therefore it is a multi-recruiting system: one has to imagine bending and gluing back the sides of the square; finally, it is representational because (iii) it represents the torus but it also displays the information on how to manipulate the square in order to construct it.

A first important point to make here is that not all the conceivable transformations that might be applied to this diagram are legitimate; that is, only some of them are recognized as *meaningful* in relation to the specific mathematical context. These actions are also *epistemic*: they allow obtaining new mathematical results. To clarify, the term "epistemic action" is taken from the experimental study presented years ago by David Kirsh and Paul Maglio, where they distinguish between *pragmatic* actions, which aim to bring the agent closer to a physical goal, and *epistemic* actions, which use the external world to improve cognition.[22] In their definition, epistemic actions are "physical actions that make mental computation easier" and are performed *outside* the mind on the physical objects that are available, with the result of enhancing inference and reasoning.[23] De Cruz and De Smedt claim that this might be the case also for the actions that are applied to notations in mathematics, since mathematical symbols "enable us to perform mathematical operations that we would not be able to do in the mind alone, they are *epistemic actions.*"[24]

[20] De Toffoli and Giardino, "Inquiry into the Practice" (ref. 8).

[21] The particular appearance of the two couples of arrows—here, black and white double arrows—is arbitrary; they simply have to be different so as to *visually* allow distinguishing *easily* the couple to glue and the direction in which to glue.

[22] David Kirsh and Paul Maglio, "On Distinguishing Epistemic from Pragmatic Action," *Cognitive Science* 18 (1994): 513–49. Kirsh and Maglio's study is on the performance of *Tetris* players. Tetris is a popular videogame from the 1980s, where some pieces that are shaped as geometric figures and composed of four squared blocks fall one after the other down onto the playing field. The objective of the game is to move the pieces down so as to create a horizontal line below, composed of ten blocks presenting no gaps. Once a whole line is created, it disappears, and any block that is above the deleted line falls down. The game is over if the stack of pieces reaches the top of the playing field; however, if a certain number of lines is cleared, then the game enters a new level (in which the pieces will go down faster). In order to create the line below to the aim of having it cleared, the player can move the pieces sideways and rotate them by 90-degree units. In their observations, the authors noticed that players act on the pieces of the game not only to directly achieve the final goal but also to *understand* what the best move to create a line down onto the playing field is.

[23] Kirsh and Maglio, "On Distinguishing" (ref. 22), 514.

[24] Helen De Cruz and Johan De Smedt, "Mathematical Symbols as Epistemic Actions," *Synthese* 190, no. 1 (2013): 3–19, on 4.

A second important point is made by Mark Colyvan, who mentions as well the case of square diagrams in topology and claim that they are "both *notation* and *a kind of blueprint for construction* of the objects in question," and for this reason they simultaneously present features belonging to algebra and to geometry.[25] This is in line with our characterization of representational cognitive tools: his suggestion is that "whichever way you look at it, we have a powerful piece of notation here that does *some genuine mathematical work for us*."[26] Later, I will consider again this claim.

Good Design and Mathematical Practices

In the previous sections, we saw that formulas in algebra and logic and knot diagrams and square diagrams in topology are cognitive tools because they are (i) external material objects that are the product of the culture of mathematics and that structure our thinking and (ii) they are subject to conceptual but also to sensorimotor considerations: they are targets for perception and action to be redeployed for more cognitive and abstract uses. Moreover, they are representational because (iii) they serve as representations and at the same time as instruments to study what they represent.

Cognitive tools are the product of cultural evolution and they are intended to fulfill some specific cognitive or conceptual aim. For this reason, ideally cognitive tools should be designed and selected by the community of practitioners as having the appropriate physical features so as to trigger diagramming and thus enhance reasoning. Consider Frege's notation or square diagrams again. Thanks to their physical features, their content gets structured in a way that makes it easily graspable and allows for applying particular physical transformations that correspond to meaningful operations, in line with the designer's intentions. However, Frege's notation was not successful because it was too unfamiliar at the time, whereas square diagrams are commonly adopted today for teaching topology.[27] In some cases, cognitive tools are configured very carefully to respect our organizational perceptual skills. For example, it is easy to interpret two interrupted lines in a knot diagram as a single thread going below another, which aligns with the Gestalt law of continuity; thanks to manipulative imagination, we can even imagine grasping one of these threads, pulling it off and inferring the consequences of our move.

By acknowledging all these elements, the practice of mathematics does not seem to correspond entirely to the disembodied process of obtaining more and more theorems by carrying out long chains of deductions based on a limited number of mathematical elementary propositions—the axioms. Evidence is given in favor of the claim that in the everyday activity of doing mathematics, the physical features of the tools that are used—and their interpretation in line with the transformations allowed by the practice and depending on the level of expertise of the agent—may have an influence on mathematical reasoning.

In the next and final section, I will try to better characterize the functioning of representational cognitive tools, by focusing on their material structure and on the epistemic actions that they make accessible.

[25] Mark Colyvan, *An Introduction to the Philosophy of Mathematics* (Cambridge: Cambridge University Press, 2012), 139. Emphasis added.

[26] Colyvan, *Philosophy of Mathematics* (ref. 25), 139.

[27] The introduction of these diagrams from an historical point of view is partly reconstructed in Christophe Eckes and Valeria Giardino, "The Classificatory Function of Diagrams: Two Examples from Mathematics," *Lectures Notes on Computer Science*, 10871, 120–136.

The Functioning of Representational Cognitive Tools

Leaving metaphors aside, how could it be that a map "tells" its user where the library is, or that square diagrams "do some genuine mathematical work" for the topologist? In order to clarify these claims, I will draw on the notion of *material anchor* as defined in cognitive anthropology by Edwin Hutchins and I will add some elements coming from psychology, in particular by referring to James Gibson's notion of *affordance*.[28]

Representational Cognitive Tools as Material Anchors

As Hutchins explains, one of the principal findings of studies of situated cognition is that humans make "opportunistic use of structure."[29] To give an example, in the *method of loci* the orator who has to memorize a speech associates elements of speech with architectural features of the place where the speech is delivered. Designed physical objects can also serve as structures to enhance or support cognition. Of course, cognitive artifacts are always embedded in the larger cultural system of practices in which they are used. According to Hutchins, in most cases, cognitive processes are *distributed* in several ways: across the members of a social group, through time, and, more importantly for the present purpose, by the coordination between internal and external and material structure; in his words, "distributed cognition looks for a broader class of cognitive events and does not expect all such events to be encompassed by the skin or skull of an individual."[30]

In particular, Hutchins introduces the notion of "material anchor" by presenting a continuum of cases ranging from those in which external and material objects provide little or no structure to a conceptual model to those in which virtually all aspects of the conceptual model are *embodied* in the structure of the external and material objects. In all examples, "the cognitive operations performed on the model are implemented as physical manipulation of material structure."[31]

Hutchins' idea is that we refer to some particular conceptual model to build the constraints of the task *into* the structure of the cognitive tool. For example, a visualization technique used by Japanese students allow them to compute the day of the week of any day in the year by looking at a set of regions on the first three fingers of their left hand. Another example is the ancient schema of the wind rose, which allows making correspondences between direction and time; as Hutchins explains, "by blending the conceptual structure of time with the material structure of the compass rose, the navigator can experience direction as an expression of time."[32]

Both memory and processing load can be reduced if the constraints of the task can be built into the physical structure of a material device. This is a result of a cultural practice: a physical structure is not a material anchor because of some intrinsic quality, but because it is used as such.

Representational cognitive tools, as characterized in the introduction, can be considered to be material anchors for conceptual blends, enhancing and supporting reasoning by

[28] Edwin Hutchins, "Material Anchors for Conceptual Blends," *Journal of Pragmatics* 37, no. 10 (2005): 1555–77; James J. Gibson, *The Ecological Approach to Visual Perception* (Boston: Houghton Mifflin, 1979).

[29] Edwin Hutchins, "Cognitive Artifacts," in *The MIT Encyclopedia of the Cognitive Sciences*, ed. Frank C. Keil and Robert A. Wilson, 126–128 (Cambridge, MA: MIT Press, 1999), 126.

[30] Edwin Hutchins, "Cognition, Distributed," in *The International Encyclopedia of the Social and Behavioral Sciences*, ed. Neil J. Smelser and Paul B. Baltes, 2068–2072 (Oxford: Elsevier, 2001), 2068.

[31] Hutchins, "Material Anchors" (ref. 28), 1574.

[32] Hutchins, "Material Anchors" (ref. 28), 1569.

relieving cognitive load and structuring thought. A great part of thinking depends on the many physical structures that are available for us as the result of the cultural process of “crystallization” of conceptual models into external and material objects. These objects can be manipulated meaningfully relative to some conceptual context and they serve as material anchors for cognitive tasks. In our particular case, their physical manipulations correspond to mathematical operations and reduce memory and processing loads. Moreover, the structure of the tool embodies the structure of the represented mathematical content in such a way that in some cases it may have an influence on understanding or conceptualization; questions about what constitute good and bad design in mathematics arise. In the continuum of cases discussed by Hutchins, representational cognitive tools are in the vicinity of those in which virtually all aspects of the conceptual model get embodied in their structure.

However, if thinking processes involve complex manipulations of conceptual structure in some cases, and cognitive operations performed on the model are implemented as physical manipulations of material structure, how are these transformations defined? How do the users know at each time which actions to perform on the cognitive tool in order to enhance or support their reasoning? How do perception and action get involved? In the case of representational cognitive tools in mathematics, these actions depend both on the physical features of the tools and on the properties of the mathematical content that gets represented. To address these questions, I will refer in particular on an extension of the notion of *affordance*, as originally introduced by Gibson.[33]

Representational Affordances

In his proposal to revise experimental psychology, Gibson pointed out that the real research target in the empirical research on vision should not be the physical world, which is the object of other scientific disciplines—physics for example—but rather the *environment* humans with vision live and move in. Once the environment is considered, it is possible to recognize *affordances* in it.[34]

An “affordance,” a term derived by the verb “to afford” but invented by Gibson, refers at the same time to the environment and to the subject, since it relates to both, and thus implies the complementarity between the two. In Gibson’s words, “the *affordances* of the environment are what it *offers* the animal, what it provides or furnishes, either for good or ill.”[35] To give an example, take a surface with some physical properties: it is horizontal, flat, extended, and rigid. However, what is relevant *to us* is not that it possesses the aforementioned physical properties, but that thanks to them it *affords* support.

Affordances are not only important for the natural environment, but they are also crucial for designed objects. Consider a hammer, which is conceived with a particular function coupling with some particular affordance; for example, a good hammer affords good grasping. However, there may be cases in which the designed affordances do not align with the function of the designed object. A simple example is the following: a door, with a handle, which has to be pushed to open, is badly designed: it affords pulling but it requires the opposite action to fulfill its function.

Natural things afford specific actions in the environment, and designed objects are conceived to afford some particular action in view of some intended pragmatic aim. What about representational cognitive tools? In this case, the objects in question are external and material but serve as representations and at the same time as instruments for some epistemic aim.

My proposal is that representational cognitive tools also afford some actions on them, and some of them may correspond to *epistemic* actions. Since they are material objects but also representational objects, I will call these affordances *representational*. Further distinctions have to be made. Analogously to other concrete objects in the environment, representational cognitive tools present a first layer of affordances that simply depend on our perceptual system. For example, the formulas described in the previous sections triggered perceptual grouping mechanisms; tables or graphs may afford other kinds of transformations depending on the symmetries they display, to which we attend very spontaneously. At this first level, the nature of the cognitive tool as external and material object brings about competences originating in our relationship with concrete objects in the environment.

However, another important feature of representational affordances is that some of them are highly *context dependent*: they are conditional on interpretation and on the function of the cognitive tool. The same notation or diagram might afford different actions and transformations when having different functions or across different disciplines. For example, a triangle in Euclidean geometry affords among other things translation but only an expert knows that it cannot afford, for example, stretch; however, this affordance may well be available in other geometrical practices. It is at this level that the conceptual system gives more clearly its contribution to diagramming by recognizing the constraints imposed to the material anchor by the conceptual model.

As a consequence, I will distinguish between *perceptual* representational affordances and *context-dependent* representational affordances. On the one hand, cognitive tools spontaneously afford some actions because they are part of our material environment: it is thanks to good continuation, which applies to other percept as well, that the threads in a knot diagram are seen as uninterrupted lines despite the fact that they are interrupted in the diagram (see Fig. 1). On the other hand, cognitive tools also afford actions that are only “for the expert”: this second kind of affordances determines at each time what is correct and what is not, depending on the mathematical context.

If this is the case, then two general consequences can be drawn. First, a design principle can be defined, according to which it is commendable to prevent perceptual representational affordances from being sources of errors. This is the case of the manipulations applied to the equations by Landy and Goldstone: adding spaces or lines triggers features of our cognitive systems, such as perceptual groupings, and this may lead to hindering effects of sensorimotor considerations on mathematical reasoning. On the contrary, it is convenient to exploit perceptual representational affordances to facilitate the relevant and correct transformations and make them correspond to legitimate actions that are also meaningful in the mathematical context. This is the case of flipping upside-down fractions when changing a division into a multiplication, or of displaying a long logical proposition by a syntax tree to identify the main connector, or of using a square diagram to define the path of a point on a torus. Second, the context dependence of some representational affordances has to be recognized and seriously taken into account: different contexts or functions can make different context-dependent representational affordances emerge, but these affordances will still be conditional on the physical features of the material anchor and will keep interacting with the perceptual representational ones. One might also envisage cases where a perceptual representational affordance provided by a particular tool is so forceful that it contributes to the definition of new meaningful and epistemic actions.

[33] Gibson, *Ecological Approach* (ref. 28).
[34] Gibson, *Ecological Approach* (ref. 28).
[35] Gibson, *Ecological Approach* (ref. 28), 127.

To sum up, my proposal is that representational cognitive tools in mathematics are material anchors that are well designed when they spontaneously afford the meaningful transformations that lead experts to inferential and epistemic actions. The hypothesis is that a cognitive tool is well designed when context-dependent representational affordances are in line with perceptual representational affordances; in other words, a representational cognitive tool serves as a good material anchor when diagramming is at its best, that is, when our perception and our action are spontaneously driven towards conceptually meaningful manipulations, thus facilitating inference.

Conclusions: Representational Cognitive Tools in Mathematics

I have shown the value of considering representations such as formulas or diagrams in mathematics as representational cognitive tools, thus bypassing the standard dichotomy between image and word. Cognitive tools are external material objects that are the product of our culture and structure our thinking and are shared and used in view of some inferential aim. They can be inspected and manipulated thanks to diagramming: as multi-recruiting systems, they are aimed at enhancing reasoning. Moreover, *representational* cognitive tools are at the same time representations and instruments to study the objects they represent. The introduction and the use of cognitive tools in mathematics are crucial for the development and the advancement of mathematical thinking.

Moreover, following Hutchins, I hypothesized that these tools are material anchors to support and enhance reasoning about some conceptual model, and they allow users to act on them by performing epistemic and inferential actions. Such actions correspond to the representational affordances provided by the tool, which are based both on our cognitive system and on expertise. The affordances that are recognized by the experts are context dependent and correspond to meaningful manipulations by respecting the constraints that are determined by the larger mathematical context. Thanks to diagramming, different systems already available for other more pragmatic tasks such as the visuo-spatial system and the motor system, work together with the conceptual system on the material anchor in view of some inferential objective.

This framework might be relevant for possible applications. First, educators might consider whether the present strategies for teaching mathematics take into account the dynamic use of representational cognitive tools; for example, it might be important for students to familiarize with the functioning of cognitive tools by learning how to recognize their affordances and to perform appropriate epistemic actions on them. Second, developers might be interested in exploring a further extension of affordances, for example when cognitive tools are presented on touch screens that allow users to transform and change them on line even more dynamically. What happens to representational affordances when they meet these new technologies? Do the constraints on the structure depending on the epistemic context get mixed up with the affordances of the new screens? The answer to these questions is matter of future research.

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