A note on Oscar Chisini mean value definition

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Abstract. Mainly on the basis of some notable physical examples reported in a 1929 Oscar Chisini paper, in this brief note it is exposed further possible historic-critical remarks on the definition of statistical mean which lead us towards the realm of Integral Geometry, via the Felix Klein *Erlanger Programm.*

1. Introduction

If one identifies, from the mathematical viewpoint, the concept of *statistical variable* (of Statistic) with that of *random variable* (of Probability Theory) according to what established in (Dall'Aglio, 1987, Capitolo IV, § IV.2), then the notion of *mean value* may be included in the most general one of *expectation value* of a random variable[1], in turn included in the wider class of the *moments* of a random variable.

Following (Piccolo, 1998, Capitolo 4), that of mean is a primitive concept for the human being, so that it is perceived with immediacy, even if its measure is arbitrary since depending on the synthesis criterion adopted; through this last, then, it will be possible to state a formal definition of it. The first notion of mean was due to[2] A.L. Cauchy in 1821 who defined it as an intermediate value between the maximum and minimum values of a given statistical variable; such a definition, nowadays is considered as a simple *internality Cauchy condition*. Instead, a great attention had a formal definition of mean value due to Oscar Chisini in 1929, according to whom the mean $\mathcal{M}$ of given statistical variable *X*, is that value which, respects to another given synthetic function *f* defined on the frequency distribution of *X*, leaves invariant the values of the latter, that is to say[3]

$$(1) \qquad f(x_1, \dots, x_n) = f(\mathcal{M}, \dots, \mathcal{M}) \qquad \forall (x_1, \dots, x_n) \in dom\, f.$$

Following (Girone & Salvemini, 2000, Capitolo 6, § 6.1) and (Ferrauto, 1996, Capitolo 4), this mean value $\mathcal{M}$ warrants to leave unchanged a predetermined quantity assumed to be invariant, and formally expressed by the function *f*. This Chisini's theoretical criterion defining a mean, is made operative by specifying the function *f* in dependence on the formal properties (like additivity, multiplicativity or invertibility) of the random variable *X*, so reaching to various possible types of means on the basis of the *f* chosen (Piccolo, 1998, Capitolo 4, § 4.2); such a choice is strictly dependent on the context of the involved problem.

Other possible definitions of mean have also been proposed, like that according to O. Wald (1950) and that according to M. Nagumo, A.N. Kolmogorov and B. De Finetti (Piccolo, 1998, Capitolo 4, § 4.2), which substantially make use of methods analogous to the functional one of Chisini whose essential idea is the following: through the function *f*, it is possible to consider the *transferability* of the initial statistical variable *X* amongst the unities of the statistical population in which it is defined.

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[1] In this regards, see also (De Finetti, 1930) and what will be said in Section 2.

[2] For some related historic-bibliographical notes, see (Berzolari, 1972, Articolo LV, Capitolo II).

[3] For instance, to get the usual arithmetic mean, we choose the following weighted invariant function $f(x_1,\dots,x_n) = \sum_{i=1}^{n} p_i\, x_i$ and we impose that be $\sum_{i=1}^{n} p_i\, x_i = \sum_{i=1}^{n} \mathcal{M}\, x_i$, whence $\mathcal{M} = \sum_{i=1}^{n} p_i\, x_i / \sum_{i=1}^{n} p_i$ which is the *weighted arithmetic mean* of the variables x_i with weights p_i. Instead, the invariant function giving rise the simple *geometrical mean* is as follows $f(x_1,\dots,x_n) = \prod_{i=1}^{n} x_i$, from which, applying (1), it follows $\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \mathcal{M} = \mathcal{M}^n$, whence $\mathcal{M} = \sqrt[n]{\prod_{i=1}^{n} x_i}$. Finally, for the weighted harmonic mean, it is $f(x_1,\dots,x_n) = \sum_{i=1}^{n} p_i / x_i$, hence $\sum_{i=1}^{n} p_i / x_i = \sum_{i=1}^{n} p_i / \mathcal{M}$ whence $\mathcal{M} = (\sum_{i=1}^{n} p_i)/(\sum_{i=1}^{n} p_i / x_i)$ which is the *weighted harmonic mean* with weights p_i. For further information in this regards, see (Girone & Salvemini, 2000, Capitolo 6).

In this brief note, we want above all dwell with the notion of mean value according to Chisini, on which Bruno De Finetti has mainly based his important paper (De Finetti, 1930).

2. On Chisini's mean definition

In the general framework of a critical discussion of the notion on mean value in Statistics, De Finetti centers his paper on a review of the notion of statistical mean according to Oscar Chisini with its possible features and applications, first stating as through it an extension of the concept of mean to an arbitrary random variable is also possible.

Oscar Chisini, whose main research field was in Algebraic Geometry, in 1929 incidentally was led to consider some statistical questions from which derived his brief but meaningful note on the general notion of a mean value. In it, he first of all criticizes the old 1821 Cauchy definition of mean simply conceived as a certain value comprised between the minimum and maximum values of the set of values of a given variable, because it does not provide neither any synthetic information giving a global vision of the phenomenon described by this variable nor puts into evidence the typical relative character that a mean must have. According to Chisini, these last requirements might be accomplished, for instance, by means of the choice of a certain function depending on the observed quantities of this phenomenon, to this purpose referring to some meaningful kinematical[4] and geometrical[5] problems as practical examples of this his basic point of view: for instance, to point out the relative character of the mean, that is to say, its dependence on the circumstances of the involved problematic situation, he argues, among others, on a physical problem concerning the determination of the mean resistance of three conductors, whose result clearly depend on the geometry of the this physical problem which is related to parallel or sequential disposition of these conductors. At last, he also considers the determination of this statistical parameter – a mean value – as regard very interesting physical problems concerning the oscillations of certain physical systems (like a pendulum), in which it is also involved some not negligible geometrical considerations connected, for example, to mass distribution problems whose inertial momenta are nothing more that second order statistical momenta (see (De Finetti, 1970, Volume I, Capitolo II, §§ 8, 9, 10)).

Therefore, Chisini provides a general definition of mean of an arbitrary distribution of a quantity given in certain circumstances and situations[6], as that unique value of it which may be substituted without any change in the above contextual problematic framework. To our purposes, we stress on this last peculiarity, that is to say, the just mentioned requirement of general invariance about the circumstantial and situational setting of the given distribution. In the general case of an arbitrary random variable ξ with distribution given by the partition function $\Phi(\xi)$, then we should consider a functional of the type $\mathcal{F}[\Phi(\xi)] = \int \psi(\xi) d\Phi(\xi)$ instead of $f(x_1, \dots, x_n)$, and request to be valid the condition $\mathcal{F}[\Phi(\xi)] = F_\xi(x)$ if x is the required mean for such a random variable, with $F_\xi(x) = \delta(\xi - x)$ distribution function of the random variable ξ centered at x. Therefore, under the hypothesis of invertibility of F, we have $x = F_\xi^{-1}(\mathcal{F}[\Phi(\xi)])$.

All these above considerations have been drew from the papers (Chisini, 1929) and (De Finetti, 1930). In this regards, see also (De Finetti, 1970).

[4] In this regards, it is classical examples those related to the computation of the mean velocity of certain kinematical problems, those same usually reported by the common treatises and textbooks on Statistics and Probability Theory; see, for instance, besides (De Finetti, 1930), also (Girone & Salvemini, 2000, Capitolo 6, § 6.12) and (Dall'Aglio, 1987, Capitolo IV, § 2, Esempio IV.2.1).

[5] Above all, the examples reported at points 4. and 6. of the paper (Chisini, 1929), are very meaningful to show the dependence of some types of means by the geometrical aspects of the problem in which they are involved. In particular, the first example reported at point 6. might be extended considering in the formula (12), of the Chisini's paper, a path integral along the distribution line of the values given by $x = x(t)$ instead of a scalar integral which, besides, depends too by the geometry of the problem since it is the area underlying this line of equation $x = x(t)$; it is likewise interesting the other following examples of the same point 6., from which it result to be always non-negligible the geometrical aspects of the considered problem. Finally, the argumentations carried out at the final point 7. clearly show what significant effects has a change of independent variables of the function f of (1), leading us toward the more general group theory considerations which will be given in the next Section 4. However, for a more in-depth discussion of these type of argumentations, see (De Finetti, 1970, Volume I, Capitolo II, §§ 8, 9, 10).

[6] About the choice of a given mean, De Finetti, in (De Finetti, Volume I, Capitolo II, § 9), speaks of the *relative* and *functional* meaning that it must be identified for answering to the purpose of the given problem; according to the author, this problem's purpose may be summarized in the German term *zweckmässig* where *zweck* means ''purpose'' whereas *mässig* means ''suitable'', that is to say, the aim of the problem must be ''suitable to the purpose'' (zweckmässig).

3. A particular case related to non-commutativity.

One of the main formal properties of a statistical mean is that of commutativity, or else its invariance under the action of the permutation group. Indeed, following the very important paper[7] (Stevens, 1946), the first measurement approach to statistical variables both qualitative and quantitative, consists in their classification according to one of the four main measure levels stated by S.S. Stevens, precisely the *nominal*, *ordinal*, *interval* and *ratio scales*, of which we herein reports what the same Stevens says in (Stevens, 1946, p. 677)

> *«Paraphrasing N. R. Campbell (Final Report, p. 340), we may say that measurement, in the broadest sense, is defined as the assignment of numerals to objects or events according to rules. The fact that numerals can be assigned under different rules leads to different kinds of scales and different kinds of measurement. The problem then becomes that of making explicit (a) the various rules for the assignment of numerals, (b) the mathematical properties (or group structure) of the resulting scales, and (c) the statistical operations applicable to measurements made with each type of scale».*

Subsequently, at page 678 of (Stevens, 1946), about the description of the third column of the basic Table I (see later), he states that

> *«In the column which records the group structure of each scale are listed the mathematical transformations which leave the scale-form invariant. Thus, any numeral, x, on a scale can be replaced by another numeral, x', where x' is the function of x listed in this column. Each mathematical group in the column is contained in the group immediately above it. The last column presents examples of the type of statistical operations appropriate to each scale. This column is cumulative in that all statistics listed are admissible for data scaled against a ratio scale. The criterion for the appropriateness of a statistic is invariance under the transformations in column 3».*

We herein report the Table I of (Stevens, 1946) with the additions and corrections given in (Stevens, 1958)

Table I

Measurement Scale	Basic Empirical Operations	Mathematical Group Structure	Permissible Statistics (Invariantive)	Typical examples
NOMINAL	Determination of equalities	*Permutation group* $x' = f(x)$ with f bijective correspondence	Number of cases, Mode, Contingency correlation, Information measure	Numerations
ORDINAL	Determination of greater or less	*Isotonic group* $x' = f(x)$ with f injective map	Median, Percentiles, Ordinary correlations	Intelligence test coarse scorings, Mineral hardness
INTERVAL	Determination of equality of intervals or differences	*General linear group* $x' = ax+b$	Mean, Standard deviation, Rank-order correlation, Product-moment correlation	^{o}F and ^{o}C temperatures, Line position, Intelligence test standard scorings
RATIO	Determination of equality or ratio	*Similarity group* $x' = ax$	Coefficient of variation, Geometrical and Harmonic means	Densities, ^{o}K temperatures, Sones, Brils

[7] See also (Ferrauto, 1996, Capitolo 1) and (Piccolo, 1998, Capitolo 2, § 2.3).

According to what said by the same Stevens in (Lerner, 1977, Capitolo 3), the central characteristic on which is based this scale classification is that of invariance through which it is possible to get an objective scientific information from a given set of data if and only if they are invariant respect to a certain group of transformations, the invariance group of the given scale.

The four measurement levels are cumulative and therefore, in particular, the mathematical properties of one level are included into those of the higher levels[8] (see (Ferrauto, 1996, Capitolo 1)), and hence the commutativity, formally given by the invariance respect to the permutation group of the first measurement level, is one of the main formal properties owned by the various statistical tools therein mentioned. From this last conclusion, it is also possible to argue what follows.

Following (Bernardini, 1968, Capitolo XV), (Kittel et al., 1970, Capitolo 2, § 2.6) and (Tonzig, 1981, 3), the finite angular displacements and velocities are directional quantities which yet are not of vectorial nature because they does not verify the commutative law for the sum, so that it is not possible to consider an any their mean in the above sense[9]. On the other hand, the non-commutativity of finite rotations[10] is due to the non-commutativity of the rotation differential operators (*generators*) L_x, L_y and L_z of the group SO(3), which, amongst other, lead to mathematics of the addition of quantum angular momenta and related selection rules. These last quantum observables cannot be summed among them with the ordinary rules of a commutative algebra but according to the irreducible representation methods of SO(3) (see (Onofri & Destri, 1996, Capitolo 8, § 8.3); in particular, it is not possible to consider, for them, the usual statistical means.

The observations made so far, above all those related to the basic above mentioned work of Stevens, clearly lead us towards a major consideration of the relationships elapsing between Group Theory and Statistic, hence between Geometry and Statistic if one takes into account the well-known 1872 Felix Klein *Erlanger Programm*, whose principle of the method sets that the main formal properties of geometrical entities are those invariant respect to the action of well-determined groups. Hence, following this Klein's idea, central concepts and tools of Geometry will be those of group invariance and symmetry, this program having had notable and fruitful features both in pure and applied mathematics, as well as in Physics: one of these, concerns that branch of Mathematics known as *Integral Geometry*, which is closely connected to the notion of *geometric probability* and related arguments.

4. Towards the Integral Geometry

Following[11] (Stoka, 1982, Capitolo III), if G_m is an m parameter Lie group of transformations of $\mathbb{R}^n$ of the type

$$y_i = \psi_i(x_1, \dots, x_n;\ a_1, \dots, a_m) = \psi_i(x; a) \qquad i = 1, \dots, n \tag{2}$$

depending on m parameters a_j $j = 1, \dots, m$, then a function $\Phi(x_1, \dots, x_n)$ is said to be an *integral invariant* of the group G_m if

$$\int \dots \int_S \Phi(x_1, \dots, x_n)\ dx_1 \dots dx_n = \int \dots \int_S \Phi(y_1, \dots, y_n)\ dy_1 \dots dy_n \tag{3}$$

for every $S \subseteq \mathbb{R}^n$ for which there exist the given integrals. On the other hand, if

$$J(x; y) \stackrel{\text{def}}{=} \frac{D(y_1, \dots, y_n)}{D(x_1, \dots, x_n)}$$

is the Jacobian determinat related to the variable change $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$ given by (2), then, from (3), it follows that

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[8] As already said by the same Stevens, when he says that «[…] *each mathematical group in the column 3 is contained in the group immediately above it*».

[9] Analogously, the usual mean values, in general, cannot be applied to computations involving the so-called *intensive* physical quantities, like the temperatures notwithstanding these last commute among them.

[10] But not of the infinitesimal ones.

[11] For a more complete reference, see (Stoka, 1968).

(4) $$\Phi(x_1,\dots,x_n)=J(x;y)\,\Phi(y_1,\dots,y_n)=J(x;y)\,\Phi\big(\psi_1(x;a),\dots,\psi_n\,(x;a)\big).$$

Now, the relation (1), written for Φ instead of f, is of the type (4) when $\psi_1(x;a)=\cdots=\psi_n\,(x;a)=M$ and $J(x;y)=1$ (or a nonzero constant), so that the (1) is a particular case of the more general relation (4).

If $\xi_j(x_1,\dots,x_n)$ $j=1,\dots,m$ are the infinitesimal generators of G_m, then a *theorem of R. Deltheil* (see (Stoka, 1982, Capitolo III, § 3.1)) states that for $\Phi(x_1,\dots,x_n)$ be an integral invariant of G_m it is necessary and sufficient that Φ be solution of the following system of first order partial differential equations

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(\xi_{ij}(x)\Phi(x)\big)=0,\qquad j=1,\dots,m,$$

whence it follows a close relationship between the group structure of G_m and its integral invariant functions Φ. The group G_m is said to be *measurable* if it admits an unique integral invariant function Φ, at most up to a multiplicative constant.

Let $\mathfrak{F}_p$ be a family of p (≥ 1) dimensional and q parametric manifolds V_p of $\mathbb{R}^n$ each of which is given by the system of (parametric) equations

$$F^j\big(x_1,\dots,x_n;\;\alpha_1,\dots,\alpha_q\big)=0\qquad j=1,\dots,n-p$$

with any F^j analytic and $\alpha_1,\dots,\alpha_q$ arbitrary parameters, the variability of this family being given only by the variability of these parameters α_r and not by the functions F^j. Let G be a group acting on $\mathfrak{F}_p$, that is to say, such that T: $\mathfrak{F}_p\to\mathfrak{F}_p$ for every $T\in G$, and let $\mathfrak{I}_G=\bigoplus_{V_p\in\mathfrak{F}_p}\mathfrak{I}_{V_p}$ be the internal direct product of the isotropy groups $\mathfrak{I}_{V_p}=\{T;\;T\in G,\;\;T(V_p)=V_p\}$, each of which is a normal subgroup of G. Hence, let $\mathfrak{G}_G=G/\mathfrak{I}_G$ be the related quotient group which has the property of leaving globally invariant the family $\mathfrak{F}_p$ without containing any transformation (different from the identity) which leaves invariant every manifold V_p of $\mathfrak{F}_p$; such a group will be said the *maximal invariance group* of $\mathfrak{F}_p$.

If $\mathfrak{G}_G$ is a Lie group of transformations of $\mathbb{R}^n$ of the type (2), said $\alpha_1,\dots,\alpha_q$ the parameters of a manifold V_p, then the parameters $\beta_1,\dots,\beta_q$ of the manifold $V_p'=T(V_p)$ will be such that

$$F^j\big(x_1,\dots,x_n;\beta_1,\dots,\beta_q\big)=F^j\big(\psi_1(x;a),\dots,\psi_n(x;a);\;\alpha_1,\dots,\alpha_q\big)\qquad j=1,\dots,n-p$$

where

(5) $$\beta_k=\vartheta_k\big(\alpha_1,\dots,\alpha_q;\;a_1,\dots,a_r\big)\qquad k=1,\dots,q$$

for certain functions ϑ_k. Therefore, if $\mathfrak{D}_q\subseteq\mathbb{R}^q$ is the space of the parameters $\alpha_1,\dots,\alpha_q$ of the family $\mathfrak{F}_p$, then to the maximal invariance group $\mathfrak{G}_G$, whose elements are of the type (2), it is possible to associate, relatively to the space $\mathfrak{D}_q$, the family of transformations (5) which form a group isomorphic to $\mathfrak{G}_G$ and that will be denoted by $\mathfrak{H}_r(\alpha)$; hence $\mathfrak{H}_r(\alpha)\cong\mathfrak{G}_G$, the first group being also said *associated* to $\mathfrak{G}_G$ respect to the family $\mathfrak{F}_p$. Thus, if $\mathfrak{H}_r(\alpha)$ is a measurable group with invariant integral function $\Phi\big(\alpha_1,\dots,\alpha_q\big)$, then we can define a measure on $\mathfrak{F}_p$ as follows: said $\mathcal{A}$ a subset of $\mathfrak{F}_p$, we put

(6) $$\mu_{\mathfrak{G}_G}(\mathcal{A})\overset{\text{def}}{=}\int\dots\int_{\mathcal{A}_\alpha}\big|\Phi\big(\alpha_1,\dots,\alpha_q\big)\big|\;d\alpha_1\dots d\alpha_q$$

where $\mathcal{A}_\alpha$ is the bounded set of the parameter space $\mathfrak{D}_q$, corresponding to $\mathcal{A}$ through the (5); evidently, such a definition depends on the basic isomorphism $\mathfrak{H}_r(\alpha)\cong\mathfrak{G}_G$. Thus, we can now define a *geometric probability* as follows: if $\tilde{\mathcal{A}}\subseteq\mathcal{A}$, then the (geometric) probability for a manifold $V_p\in\mathcal{A}$ belongs to $\tilde{\mathcal{A}}$, is given by

$$P_{\mathfrak{G}_G}(\tilde{\mathcal{A}}) \stackrel{\text{def}}{=} \frac{\mu_{\mathfrak{G}_G}(\tilde{\mathcal{A}})}{\mu_{\mathfrak{G}_G}(\mathcal{A})}.$$

Moreover, if ξ is an arbitrary random variable associated to the set $\mathcal{A} \subseteq \mathfrak{F}_p$, then the *h-th geometric moment* of ξ is defined by

$$\mu^h_{\mathfrak{G}_G}(\xi) \stackrel{\text{def}}{=} \frac{\int \dots \int_{\mathcal{A}_\alpha} \xi^h(\alpha_1, \dots, \alpha_q)|\Phi(\alpha_1, \dots, \alpha_q)| \; d\alpha_1 \dots d\alpha_q}{\int \dots \int_{\mathcal{A}_\alpha} |\Phi(\alpha_1, \dots, \alpha_q)| \; d\alpha_1 \dots d\alpha_q}$$

which, as known[12], generalize the various notions of means (like the arithmetic, harmonic and geometric ones) of the discrete case. From here, it is possible to make out a certain geometric background in Statistic.

6. Conclusions

As seen, the various notions so far introduced strictly depend on the Lie group of transformation $\mathfrak{G}_G$ of the type (2), of which we have considered a possible isomorphic image, namely $\mathfrak{H}_r(\alpha)$. Furthermore, in these discussions it has also been possible to verify as the basic Chisini invariant relation (1) may be included as a particular case of the more general invariant relation (4), upon which have been centred the various argumentations that followed. In turn, the latter are all closely related to the action of the given Lie group of transformations $\mathfrak{G}_G$ and its invariants (like (4)), so that the more properly geometric framework might make its appearance via the general philosophy of the already cited Felix Klein *Erlanger Programm*, as regards the geometric probability theory as a particular chapter of the Integral Geometry[13].

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[12] See, for instance, the notion of *power mean value* of *index h* for the discrete case in (Girone & Salvemini, 2000, Capitolo 6, § 6.11) which, amongst others, contain, as particular cases, the notions of arithmetic, harmonic and geometric mean. In turn, this power mean is a particular case (related to the discrete one) of the more general notion of *h-th moment* of an arbitrary random variable (see (Dall'Aglio, 1987, Capitolo IV, § IV.3)).

[13] For brief historical outlines of this fundamental mathematical branch with related possible applications, see, for instance, (Stoka, 1982).

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