Electron Spin
or
"Classically Non-Describable Two-Valuedness"

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Abstract

In December 1924 Wolfgang Pauli proposed the idea of an inner degree of freedom of the electron, which he insisted should be thought of as genuinely quantum mechanical in nature. Shortly thereafter Ralph Kronig and, independently, Samuel Goudsmit and George Uhlenbeck took up a less radical stance by suggesting that this degree of freedom somehow corresponded to an inner rotational motion, though it was unclear from the very beginning how literal one was actually supposed to take this picture, since it was immediately recognised (already by Goudsmit and Uhlenbeck) that it would very likely lead to serious problems with Special Relativity if the model were to reproduce the electron's values for mass, charge, angular momentum, and magnetic moment. However, probably due to the then overwhelming impression that classical concepts were generally insufficient for the proper description of microscopic phenomena, a more detailed reasoning was never given. In this contribution I shall investigate in some detail what the restrictions on the physical quantities just mentioned are, if they are to be reproduced by rather simple classical models of the electron within the framework of Special Relativity. It turns out that surface stresses play a decisive role and that the question of whether a classical model for the electron does indeed contradict Special Relativity can only be answered on the basis of an *exact* solution, which has hitherto not been given.

Contents

1 Introduction

The discovery of electron spin is one of the most interesting stories in the history of Quantum Mechanics; told e.g. in van der Waerden's contribution to the Pauli Memorial Volume ([10], pp. 199-244), in Tomonaga's book [38], and also in various first-hand reports [39][16] [24]. This story also bears fascinating relations to the history of understanding Special Relativity. One such relation is given by Thomas' discovery of what we now call "Thomas precession" [36][37], which explained for the first time the correct magnitude of spin-orbit coupling and hence the correct magnitude of the fine-structure split of spectral lines, and whose mathematical origin can be traced to precisely that point which marks the central difference between the Galilei and the Lorentz group (this is e.g. explained in detail in Sects. 4.3-4.6 of [14]). In the present paper I will dwell a little on another such connection to Special Relativity.

As is widely appreciated, Wolfgang Pauli is a central figure, perhaps *the* most central figure, in the story of spin . Being the inventor of the idea of an inner (quantum mechanical) degree of freedom of the electron, he was at the same time the strongest opponent to attempts to relate it to any kind of interpretation in terms of kinematical concepts that derive from the picture of an extended material object in a state of rotation. To my knowledge, Pauli's hypothesis of this new intrinsic feature of the electron, which he cautiously called "a classical non-describable two valuedness", was the first instance where a quantum-mechanical degree of freedom was claimed to exist without a corresponding classical one. This seems to be an early attempt to walk without the crutches of some 'correspondence principle'. Even though the ensuing developments seem to have re-installed – mentally at least – the more classical notion of a spinning electron through the ideas of Ralph Kronig (compare section 4 of van der Waerden's contribution to [10], pp. 209-216) and, independently, Samuel Goudsmit and George Uhlenbeck [17][18], Pauli was never convinced, despite the fact that he lost the battle against Thomas[1] and declared "total surrender" in a letter to Bohr written on March 12. 1926 ([22], Vol. I, Doc. 127, pp. 310). For Pauli the spin of the electron remained an abstract property which receives its ultimate and irreducible explanation in terms of group theory, as applied to the subgroup[2] of spatial rotations (or its double cover) within the full symmetry group of space-time, may it be the Galilei or the Lorentz group (or their double cover).[3] In this respect, Pauli's 1946 Nobel Lecture contains the following instructive passage (here and throughout this paper I enclose my annotations to quotes within square brackets):

> Although at first I strongly doubted the correctness of this idea [of the electron spin in the sense of Kronig, Goudsmit and Uhlenbeck] because of its classical-mechanical character, I was finally converted to it by Thomas'

[1] At this point Frenkel's remarkable contribution [11] should also be mentioned, which definitely improves on Thomas' presentation and which was motivated by Pauli sending Frenkel Thomas' manuscript, as Frenkel acknowledges in footnote 1 on p. 244 of [11]. A more modern account of Frenkel's work is given in [35].

[2] It is more correct to speak of the conjugacy class of subgroups of spatial rotations, since there is no (and cannot be) a single distinguished subgroup group of 'spatial' rotations in Special Relativity.

[3] Half-integer spin representations only arise either as proper ray-representations (sometimes called 'double-valued' representations) of spatial rotations $SO(3)$ or as faithful true representations (i.e. 'single-valued') of its double-cover group $SU(2)$, which are subgroups of the Galilei and Lorentz groups or their double-cover groups respectively.

calculations on the magnitude of doublet splitting. On the other hand, my earlier doubts as well as the cautions expression ≪classically non-describable two-valuedness≫ experienced a certain verification during later developments, since Bohr was able to show on the basis of wave mechanics that the electron spin cannot be measured by classically describable experiments (as, for instance, deflection of molecular beams in external electromagnetic fields) and must therefore be considered as an essential quantum-mechanical property of the electron.[4] ([25], p. 30)

This should clearly not be misunderstood as saying that under the impression of Thomas' calculations Pauli accepted spin in its 'classical-mechanical' interpretation. In fact, he kept on arguing fiercely against what in a letter to Sommerfeld from December 1924 he called "model prejudices" ([22], Vol. I, Doc. 72, p. 182) and did not refrain from ridiculing the upcoming idea of spin from the very first moment (cf. Fig. 1). What Pauli accepted was the idea of the electron possessing an intrinsic magnetic moment and angular momentum, the latter being interpreted *exclusively* in a formal fashion through its connection with the generators of the subgroup of rotations within the Lorentz group, much like we nowadays view it in modern relativistic field theory. To some extent it seems fair to say that, in this case, Pauli was a pioneer of the modern view according to which abstract concepts based on symmetry-principles are seen as primary, whereas their concrete interpretation in terms of localised material structures, to which e.g. the kinematical concept of 'rotation' in the proper sense applies, is secondary and sometimes even dispensable. But one should not forget that this process of emancipation was already going on in connection with the notion of classical fields, as Einstein used to emphasise, e.g., in his 1920 Leiden address "Ether and the Theory of Relativity"[5] ([34], Vol. 7, Doc. 38, pp. 308-320). We will come back to this point below.[6]

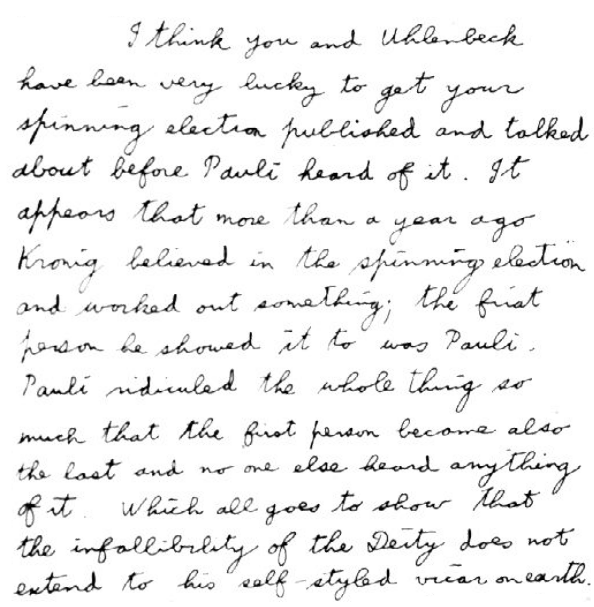
I think you and Uhlenbeck have been very lucky to get your spinning electron published and talked about before Pauli heard of it. It appears that more than a year ago Kronig believed in the spinning electron and worked out something; the first person he showed it to was Pauli. Pauli ridiculed the whole thing so much that the first person become also the last and no one else heard anything of it. Which all goes to show that the infallibility of the Deity does not extend to his self-styled vicar on earth.

Figure 1: Part of a letter by L.H. Thomas to S. Goudsmit dated March 25th 1926, taken from [15]

Besides being sceptical in general, Pauli once also made a *specific* remark as to the inadequateness of classical electron models; that was three years after Thomas' note, in a footnote in the addendum to his survey article "General Foundations of the Quantum

[4] At this point Pauli refers to the reports of the Sixth Physics Solvay Conference 1932.

[5] German original: Äther und Relativitätstheorie.

[6] The case of a classical electromagnetic field is of particular interesting insofar as the suggestive picture provided by Faraday's lines of force, which is undoubtedly helpful in many cases, also provokes to view these lines as objects in space, like ropes under tension, which can be attributed a variable state of motion. But this turns out to be a fatal misconception.

Theory of Atomic Structure"[7], that appeared 1929 as chapter 29 in 'Müller-Pouillets Lehrbuch der Physik'. There he said:

> Emphasising the kinematical aspects one also speaks of the 'rotating electron' (English 'spin-electron'). However, we do not regard the conception of a rotating material structure to be essential, and it does not even recommend itself for reasons of superluminal velocities one then has to accept. ([21], Vol. 1, pp. 721-722, footnote 2)

Interestingly, this is precisely the objection that, according to Goudsmit's recollections [15], Lorentz put forward when presented with Goudsmit's and Uhlenbeck's idea by Uhlenbeck, and which impressed Uhlenbeck so much that he asked Ehrenfest for help in withdrawing the already submitted paper [15]. He did not succeed, but the printed version contains at least a footnote pointing out that difficulty:

> The electron must now assume the property (a) [a g-factor of 2], which LANDÉ attributed to the atom's core, and which is hitherto not understood. The quantitative details may well depend on the choice of model for the electron. [...] Note that upon quantisation of that rotational motion [of the spherical hollow electron], the equatorial velocity will greatly exceed the velocity of light. ([17], p. 954)

This clearly says that a classical electron model cannot reproduce the observable quantities, mass, charge, angular momentum, and magnetic moment, without running into severe contradictions with Special Relativity.[8] The electron model they had in mind was that developed by Abraham in his 1903 classic paper on the "Principles of Electron Dynamics" [1] (cited in footnote 2 on p. 954 of [17]). Since then it has become standard textbook wisdom that classical electron models necessarily suffer from such defects (compare, e.g., [5], p. 155) and that, even in quantum mechanics, "the idea of the rotating electron is not be taken literally", as Max Born once put it ([5], p. 188). Modern references iterate this almost verbatim:

> The term 'electron spin' is not to be taken literally in the classical sense as a description of the origin of the magnetic moment described above. To be sure, a spinning sphere of charge can produce a magnetic moment, but the magnitude of the magnetic moment obtained above cannot be reasonably modelled by considering the electron as a spinning sphere.
> (Taken from ⟨ http://hyperphysics.phy-astr.gsu.edu/hbase/spin.html⟩)

In this contribution I wish to scrutinise the last statement. This is not done in an attempt to regain respect for classical electron models for modern physics, but rather to illuminate in some detail a specific and interesting case of the (well know) general fact that progress is often driven by a strange mixture of good and bad arguments, which hardly anybody cares to separate once progress is seen to advance in the 'right direction'. Also, the issues connected with an inner rotational motion of the electron

[7] German original: Allgemeine Grundlagen der Quantentheorie des Atombaues.

[8] The phrase "upon quantisation" in the above quotation is to be understood quantitatively, i.e. as "upon requiring the spin angular-momentum to be of magnitude $\hbar/2$ and the magnetic moment to be one magneton ($g = 2$)".

are hardly mentioned in the otherwise very detailed discussion of classical electron theories in the history-of-physics literature (compare [23][20]). Last but not least, the present investigation once more emphasises the importance of special-relativistic effects due to stresses, which are not necessarily connected with large velocities, at least in a phenomenological description of matter. But before giving a self-contained account, I wish to recall Pauli's classic paper of December 1924, where he introduced his famous "classically non-describable two-valuedness".

2 A classically non-describable two-valuedness

2.1 Preliminaries

We begin by recalling the notion of *gyromagnetic ratio*. Consider a (not necessarily continuous) distribution of mass and charge in the context of pre-Special-Relativistic physics, like, e.g., a charged fluid or a finite number of point particles. Let $\vec{v}(\vec{x})$ denote the corresponding velocity field with respect to an inertial frame and ρ_q and ρ_m the density distributions of electric charge and mass corresponding to the total charge q and mass m_0 respectively. The total angular momentum is given by ($\times$ denotes the antisymmetric vector product)

$$\vec{J} = \int d^3x\, \rho_m(\vec{x}) \big(\vec{x} \times \vec{v}(\vec{x})\big)\,. \tag{1}$$

The electric current distribution, $\vec{j}(\vec{x}) := \rho_q \vec{v}(\vec{x})$, is the source of a magnetic field which at large distances can be approximated by a sum of multipole components of increasingly rapid fall-off for large distances from the source. The lowest possible such component is the dipole. It has the slowest fall-off (namely $1/r^3$) and is therefore the dominant one at large distances. (A monopole contribution is absent due to the lack of magnetic charges.) The dipole field is given by[9]

$$\vec{B}_{\mathrm{dipole}}(\vec{x}) := \Big(\frac{\mu_0}{4\pi}\Big)\, \frac{3\vec{n}(\vec{n}\cdot\vec{M}) - \vec{M}}{r^3}\,, \tag{2}$$

where $r := \|\vec{x}\|$, $\vec{n} := \vec{x}/r$ and where $\vec{M}$ denotes the magnetic dipole moment of the current distribution, which is often (we shall follow this) just called *the* magnetic moment:

$$\vec{M} := \tfrac{1}{2}\int d^3x\, \rho_q(\vec{x})\, \big(\vec{x} \times \vec{v}(\vec{x})\big)\,. \tag{3}$$

Note the similarity in structure to (1), except for the additional factor of 1/2 in front of (3).

The *gyromagnetic ratio* of a stationary mass and charge current-distribution , R_g, is defined to be the ratio of the moduli of $\vec{M}$ and $\vec{J}$:

$$R_g := \frac{\|\vec{M}\|}{\|\vec{J}\|}\,. \tag{4}$$

[9] We use SI units throughout so that the electric and magnetic constants ε_0 and μ_0 will appear explicitly. Note that $\varepsilon_0\mu_0 = 1/c^2$ and that $\mu_0 = 4\pi\cdot 10^{-7}\ \mathrm{kg\cdot m\cdot C^{-2}}$ *exactly*, where C stands for 'Coulomb', the unit of charge.

We further define a dimensionless quantity g, called the *gyromagnetic factor*, by

$$R_g =: g\,\frac{q}{2m_0}\,. \tag{5}$$

These notions continue to make sense in non-stationary situations if $\vec{M}$ and $\vec{J}$ are slowly changing (compared to other timescales set by the given problem), or in (quasi) periodic situations if $\vec{M}$ and $\vec{J}$ are replaced by their time averages, or in mixtures of those cases where, e.g., $\vec{J}$ is slowly changing and $\vec{M}$ rapidly precesses around $\vec{J}$ (as in the case discussed below).

An important special case is given if charge and mass distributions are strictly proportional to each other, i.e., $\rho_q(\vec{x}) = \lambda\rho_m(\vec{x})$, where λ is independent of $\vec{x}$. Then we have

$$R_g = \frac{q}{2m_0} \Rightarrow g = 1\,. \tag{6}$$

In particular, this would be the case if charge and mass carriers were point particles of the same charge-to-mass ratio, like N particles of one sort, where

$$\rho_q(\vec{x}) = \frac{q}{N}\sum_{i=1}^{N}\delta^{(3)}(\vec{x}-\vec{x}_i) \qquad \text{and} \qquad \rho_m(\vec{x}) = \frac{m_0}{N}\sum_{i=1}^{N}\delta^{(3)}(\vec{x}-\vec{x}_i)\,. \tag{7}$$

After these preliminaries we now turn to Pauli's paper.

2.2 Pauli's paper of December 1924

On December 2nd 1924, Pauli submitted a paper entitled "On the influence of the velocity dependence of the electron mass upon the Zeeman effect"[10] ([21], Vol. 2, pp. 201-213) to the Zeitschrift für Physik. In that paper he starts with the general observation that for a point particle of rest-mass m_0 and charge q, moving in a bound state within a spherically symmetric potential, the velocity dependence of mass,

$$m = m_0/\sqrt{1-\beta^2}\,, \tag{8}$$

affects the gyromagnetic ratio. Here $\beta := v/c$, where $v := \|\vec{v}\|$. The application he aims for is the anomalous Zeeman effect for weak magnetic fields, a topic on which he had already written an earlier paper, entitled 'On the Rules of the anomalous Zeeman Effect'[11] ([21], Vol. 2, pp. 151-160), in which he pointed out certain connections between the weak-field case and the theoretically simpler case of a strong magnetic field. Note that "weak" and "strong" here refers to the standard set by the inner magnetic field caused by the electrons orbital motion, so that "weak" here means that the Zeeman split is small compared to the fine-structure.

Since the charge is performing a quasi periodic motion[12], its magnetic moment due to its orbital motion is given by the time average (I will denote the time average of a quantity X by $\langle X\rangle$)

$$\langle\vec{M}\rangle = q\,\langle\vec{x}\times\vec{v}\rangle/2\,. \tag{9}$$

[10] German original: Über den Einfluß der Geschwindigkeitsabhängigkeit der Elektronenmasse auf den Zeemaneffekt.

[11] German original: Über die Gesetzmäßigkeiten des anomalen Zeemaneffekts.

[12] Due to special-relativistic corrections, the bound orbits of a point charge in a Coulomb field are not closed. The leading order perturbation of the ellipse that one obtains in the Newtonian approximation is a prograde precession of its line of apsides.

On the other hand, its angular momentum is given by

$$\vec{J} = m\,(\vec{x} \times \vec{v}) = m_0\,(\vec{x} \times \vec{v})/\sqrt{1-\beta^2}\,. \tag{10}$$

It is constant if no external field is applied and slowly precessing around the magnetic field direction if such a field is sufficiently weak, thereby keeping a constant modulus. Hence we can write

$$\langle \vec{x} \times \vec{v} \rangle = \frac{\vec{J}}{m_0} \left\langle \sqrt{1-\beta^2} \right\rangle\,, \tag{11}$$

where the averaging period is taken to be long compared to the orbital period of the charge, but short compared to the precession period of $\vec{J}$ if an external magnetic field is applied. This gives

$$\frac{\|\langle \vec{M} \rangle\|}{\|\vec{J}\|} = \frac{|q|}{2m_0}\,\gamma\,, \tag{12}$$

where[13]

$$\gamma := \left\langle \sqrt{1-\beta^2} \right\rangle\,. \tag{13}$$

More specifically, Pauli applies this to the case on an electron in the Coulomb field of a nucleus. Hence m_0 from now on denotes the electron mass. Its charge is $q = -e$, and the charge of the nucleus is Ze. Using the virial theorem, he then gives a very simple derivation of

$$\gamma \;=\; 1 + W/m_0c^2\,, \tag{14}$$

where W denotes the electron's total energy (kinetic plus potential). For the quantised one-electron problem, an explicit expression for W in terns of the azimuthal quantum number k ($j+1$ in modern notation, where j is the quantum number of orbital angular-momentum) and the principal quantum number n ($n = n_r + k$, where n_r is the radial quantum number) was known since Sommerfeld's 1916 explanation of fine structure (see, e.g., [32], p. 53, formula (17)). Hence Pauli could further write:

$$\gamma \;=\; \left\{ 1 + \frac{\alpha^2 Z^2}{\left(n - k + \sqrt{k^2 - \alpha^2 Z^2} \right)^2} \right\}^{-1/2} \;\approx\; 1 - \frac{\alpha^2 Z^2}{2n^2}\,, \tag{15}$$

where the approximation refers to small values of $\alpha^2 Z^2$ and where $\alpha := e^2/4\pi\varepsilon_0\hbar c \approx 1/137$ is the fine-structure constant. For higher Z one obtains significant deviations from the classical value $\gamma = 1$. For example, $Z = 80$ gives $g = 0.812$.

The relativistic correction factor γ affects the angular frequency[14] with which the magnetic moment created by the electron's orbital motion will precess in a magnetic field of strength B. This angular frequency is now given by $\gamma\omega_0$, where ω_0 is the Larmor (angular) frequency:

$$\omega_0 = g_e \frac{eB}{2m_0}\,. \tag{16}$$

[13] This is Pauli's notation. Do not confuse this γ with the Lorentz factor $1/\sqrt{1-\beta^2}$, which nowadays is usually abbreviated by γ, though not in the present paper.

[14] We will translate all proper frequencies in Pauli's paper into angular frequencies. Hence there are differences in factors of 2π. This is also related to our usage of $\hbar := h/2\pi$ rather than h (Planck's constant).

Here we explicitly wrote down the gyromagnetic ratio, g_e, for of the electron's orbital motion even though $g_e = 1$, just to keep track of its appearance. The energy for the interaction of the electron with the magnetic field now likewise receives a factor of γ.

Pauli now applies all this to the "core model" for atoms with a single valence electron.[15] According to the simplest version of this model, the total angular momentum, $\vec{J}$, and the total magnetic moment, $\vec{M}$, are the vector sums of the angular and magnetic momenta of the core (indicated here by a subscript c) and the valence electron (indicated here by a subscript e):

$$\vec{J} = \vec{J}_c + \vec{J}_e\,, \tag{17a}$$

$$\vec{M} = \vec{M}_c + \vec{M}_e\,. \tag{17b}$$

The relations between the core's and electron's magnetic momenta on one side, and their angular momenta on the other, are of the form

$$\vec{M}_c = \frac{e g_c}{2m_0}\,\vec{J}_c\,, \tag{18a}$$

$$\vec{M}_e = \frac{e g_e}{2m_0}\,\vec{J}_e\,. \tag{18b}$$

The point is now that $\vec{M}$ is not a multiple of $\vec{J}$ if $g_e \neq g_c$. Assuming a constant $\vec{J}$ for the time being, this means that $\vec{M}$ will precess around $\vec{J}$. Hence $\vec{M}$ is the sum of a time independent part, $\vec{M}_\parallel$, parallel to $\vec{J}$ and a rotating part, $\vec{M}_\perp$, perpendicular to $\vec{J}$. The time average of $\vec{M}_\perp$ vanishes so that the effective magnetic moment is just given by $\vec{M}_\parallel$. Using (17) and (18), and resolving scalar products into sums and differences of squares,[16] we get

$$\begin{aligned}
\vec{M}_\parallel &= \frac{\vec{J}\cdot\vec{M}}{J^2}\,\vec{J} \\
&= \frac{e}{2m_0}\,\frac{g_e(\vec{J}\cdot\vec{J}_e) + g_c(\vec{J}\cdot\vec{J}_c)}{J^2}\,\vec{J} \\
&= \frac{e}{2m_0}\,\frac{g_e(J^2 + J_e^2 - J_c^2) + g_c(J^2 + J_c^2 - J_e^2)}{2J^2}\,\vec{J} \\
&= \frac{e}{2m_0}\,\left\{ g_e + \big(g_c - g_e\big)\frac{J^2 + J_c^2 - J_e^2}{2J^2} \right\}\,\vec{J}\,.
\end{aligned} \tag{19}$$

Setting again $g_e = 1$, the expression in curly brackets gives the gyromagnetic factor of the total system with respect to the effective magnetic moment. Its quantum analog is obtained by replacing $J^2 \to J(J+1)$ and correspondingly for J_c^2 and J_e^2, which is then called the *Landé factor* g_L. Hence

$$g_L := 1 + (g_c - 1)\,\frac{J(J+1) + J_c(J_c+1) - J_e(J_e+1)}{2J(J+1)}\,. \tag{20}$$

All this is still right to a good approximation if $\vec{J}$ is not constant, but if its frequency of precession around the direction of the (homogeneous) external field is much smaller

[15] Instead of the more modern expression "valence electron" Pauli speaks of "light electron" (German original: Lichtelektron). Sometimes the term "radiating electron" is also used (e.g., in [38]).

[16] Like, e.g., $\vec{J}\cdot\vec{J}_e = -\frac{1}{2}\big((\vec{J}-\vec{J}_e)^2 - J^2 - J_e^2\big) = -\frac{1}{2}\big(J_c^2 - J^2 - J_e^2\big)$.

than the precession frequency of $\vec{M}$ around $\vec{J}$, which is the case for sufficiently small external field strength .

Basically through the work of Landé it was known that $g_c = 2$ fitted the observed multiplets of alkalies and also earth alkalies quite well. This value clearly had to be considered anomalous, since the magnetic moment and angular momentum of the core were due to the orbital motions of the electrons inside the core, which inevitably would lead to $g_c = 1$, as explained in section 2.1. This was a great difficulty for the core model at the time, which was generally referred to as the "magneto-mechanical anomaly". Pauli pointed out that one could either say that the physical value of the core's gyromagnetic factor is twice the normal value, or, alternatively, that it is obtained by adding 1 to the normal value.

These two ways of looking at the anomaly suggested two different ways to account for the relativistic correction, which should only affect that part of the magnetic moment that is due to the orbital motion of the inner electrons, that is, the 'normal' part of g_c. Hence Pauli considered the following two possibilities for a relativistic correction of g_c, corresponding to the two views just outlined:

$$g_c = 2 \cdot 1 \;\rightarrow\; g_c = 2 \cdot \gamma \qquad \text{or} \qquad g_c = 1 + 1 \;\rightarrow\; g_c = 1 + \gamma\,. \tag{21}$$

Then comes his final observation, that neither of these corrections are compatible with experimental results on high-Z elements by Runge, Paschen and Back, which, like the low-Z experiments, resulted in compatibility with (20) only if $g_c = 2$. In a footnote Pauli thanked Landé and Back for reassuring him that the accuracy of these measurements where about one percent. Pauli summarises his findings as follows

> If one wishes to keep the hypothesis that the magneto-mechanical anomaly is also based in closed electron groups and, in particular, the K shell, then it is not sufficient to assume a doubling of the ratio of the group's magnetic moment to its angular momentum relative to its classical value. In addition, one also needs to assume a compensation of the relativistic correction. ([21], Vol. 2, p. 211)

After some further discussion, in which he stresses once more the strangeness[17] that lies in $g_c = 2$, he launches the following hypothesis, which forms the main result of his paper:

> The closed electron configurations shall not contribute to the magnetic moment and angular momentum of the atom. In particular, for the alkalies, the angular momenta of, and energy changes suffered by, the atom in an external magnetic field shall be viewed exclusively as an effect of the light-electron, which is also regarded as the location ["der Sitz"] of the magneto-mechanical anomaly. The doublet structure of the alkali spectra, as well as the violation of the Larmor theorem, is, according to this viewpoint, a result of a classically indescribable two-valuedness of the quantum-theoretic properties of the light-electron. ([21], Vol. 2, p. 212)

[17] For example: how can one understand the sudden doubling that the gyromagnetic factor of an outer electron must suffer when joining the core?

Note that this hypothesis replaces the atom's core as carrier of angular momentum by the valence electron. This means that (17), (18), and (20) are still valid, except that the subscript c (for "core") is now replaced by the subscript s (for "spin", anticipating its later interpretation), so that we now have a coupling of the electron's orbital angular momentum (subscript *e*) to its intrinsic angular momentum (subscript *s*). In (20), with g_c replaced by g_s, one needs to set $g_s = 2$ in order to fit the data. But now, as long as no attempt is made to relate the intrinsic angular momentum and magnetic moment of the electron to a common origin, there is no immediate urge left to regard this value as anomalous. Also, the problem in connection with the relativistic corrections (21) now simply disappeared, since it was based on the assumption that $\vec{J}_c$ and $\vec{M}_c$ were due to orbital motions of inner (and hence fast) electrons, whereas in the new interpretation only $\vec{J}_e$ and $\vec{M}_e$ are due to orbital motion of the outer (and hence slow) valence electron.

It is understandable that this hypothesis was nevertheless felt by some to lack precisely that kind of 'explanation' that Pauli deliberately stayed away from: a common dynamical origin of the electron's inner angular momentum and magnetic moment. From here the 'story of spin' takes its course, leading to the hypothesis of the rotating electron, first conceived by Kronig and a little later, and apparently independently, by Goudsmit and Uhlenbeck, and finally to its implementation into Quantum Mechanics by Pauli [26] ("Pauli Equation" for the non-relativistic case) and Dirac [8] (fully Lorentz invariant "Dirac Equation"). Since then many myths surrounding spin built up, like that the concept of spin, and in particular the value $g = 2$, was irreconcilable with classical (i.e. non-quantum) physics and that only the Dirac equation naturally predicted $g = 2$. As for the latter statement, it is well known that the principle of minimal coupling applied to the Pauli equation leads just as natural to $g = 2$ as in case of the Dirac equation (cf. [12] and [9], p. 37). Also, the very concept of spin has as natural a home in classical physics as in quantum physics if one starts from equally general and corresponding group-theoretic considerations.[18]

For the rest of this contribution I wish to concentrate on the particular side aspect already outlined in the introduction. Let me repeat the question: In what sense do the actual values of the electron parameters, mass, charge, intrinsic angular-momentum, and gyromagnetic factor, resist classical modelling in the framework of Special Relativity?

[18] The spaces of states in quantum and classical mechanics are Hilbert spaces and symplectic manifolds respectively. An *elementary system* is characterised in Quantum Mechanics by the requirement that the group of space-time symmetries act unitarily and irreducibly on its space of states. The corresponding requirement in Classical Mechanics is that the group action be symplectic and transitive [3]. The classification of homogeneous (with respect to the space-time symmetry group, be it the Galilei or Lorentz group) symplectic manifolds [2][19] leads then as natural to a classical concept of spin as the classification of unitary irreducible (ray-) representations leads to the quantum-mechanical spin concept. The mentioned classical structures are related to the quantum structures by various concepts of 'quantisation' like 'geometric quantisation'. Compare [41], in particular Chap. 6 on elementary systems.

3 Simple models of the electron

In this section we will give a self-contained summary of the basic features of simple electron models. The first model corresponds to that developed by Abraham [1], which was mentioned by Goudsmit and Uhlenbeck as already explained.[19] We will see that this model can only account for g factors in the interval between $3/2$ and $11/6$ if superluminal speeds along the equator are to be avoided. We also critically discuss the assumption made by Goudsmit and Uhlenbeck that this (i.e. Abraham's) model predicts $g = 2$. Since this model neglects the stresses that are necessary to prevent the charge distribution from exploding, we also discuss a second model in which such stresses (corresponding to a negative pressure in the electron's interior) are taken into account, at least in some slow-rotation approximation. This model, too, has been discussed in the literature before [7]. Here it is interesting to see that due to those stresses significantly higher values of g are possible, though not for small charges as we will also show.[20] Finally we discuss the restriction imposed by the condition of energy dominance, which basically says that the speed of sound of the stress-supporting material should not exceed the speed of light. This sets an upper bound on g given by $9/4$. Note that all these statements are made only in the realm where the slow-rotation approximation is valid. I do not know of any fully special-relativistic treatment on which generalisations of these statements could be based. In that sense, the general answer to our main question posed above is still lacking.

3.1 A purely electromagnetic electron

Consider a homogeneous charge distribution, ρ, of total charge Q on a sphere of radius R centred at the origin (again we write $r := \|\vec{x}\|$ and $\vec{n} := \vec{x}/r$):

$$\rho(\vec{x}) = \frac{Q}{4\pi R^2}\,\delta(r - R)\,. \tag{22}$$

For the moment we shall neglect the rest mass of the matter that sits at $r = R$ and also the stresses it must support in order to keep the charge distribution in place. The charge is the source of the scalar potential

$$\phi(\vec{x}) \;= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{\|\vec{x} - \vec{x}'\|}\,d^3x' = \frac{Q}{4\pi\varepsilon_0 R}\begin{cases} 1 & \text{for } r < R\,, \\ R/r & \text{for } r > R\,, \end{cases} \tag{23}$$

with corresponding electric field

$$\vec{E}(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 R^2}\begin{cases} \vec{0} & \text{for } r < R\,, \\ \vec{n} & \text{for } r > R\,. \end{cases} \tag{24}$$

[19] Since we are mainly concerned with the spin aspects, we will ignore the differences between Abraham's and, say, Lorentz' model (rigid versus deformable), which become important as soon as translational motions are considered. We mention Abraham not for any preference for his 'rigid' model, but for the reason that he considered rotational motion explicitly. Its interaction with the translational motion was further worked out in detail by Schwarzschild in [30], but this is not important here.

[20] This is another example of a special-relativistic effect which has nothing to do with large velocities.

Let now the charge distribution rotate rigidly with constant angular velocity $\vec{\omega}$. This gives rise to a current density

$$\vec{j}(\vec{x}) = (\vec{\omega} \times \vec{x})\, \rho(\vec{x}) = \frac{Q}{4\pi R^2}\, (\vec{\omega} \times \vec{x})\, \delta(r - R)\,, \tag{25}$$

which, in turn, is the source of a vector potential according to

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{x}')}{\|\vec{x} - \vec{x}'\|}\, d^3x' = \frac{\mu_0 Q}{12\pi R}\, \vec{\omega} \times \begin{cases} \vec{x} & \text{for } r < R\,, \\ \vec{x}\,(R/r)^3 & \text{for } r > R\,. \end{cases} \tag{26}$$

Hence, in the rotating case, there is an additional magnetic field in addition to the electric field (24):

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \begin{cases} 2\vec{M}/R^3 & \text{for } r < R\,, \\ \big(3\vec{n}(\vec{n} \cdot \vec{M}) - \vec{M}\big)/r^3 & \text{for } r > R\,, \end{cases} \tag{27}$$

where

$$\vec{M} := \tfrac{1}{3} Q R^2\, \vec{\omega}\,. \tag{28}$$

For $r < R$ this is a constant field in $\vec{\omega}$ direction. For $r > R$ it is a pure dipole field (i.e. all higher multipole components vanish) with moment (28).

3.1.1 Energy

The general expression for the energy of the electromagnetic field is[21]

$$\mathcal{E} = \int_{\mathbb{R}^3} \tfrac{1}{2} \Big(\varepsilon_0 E^2(\vec{x}) + \tfrac{1}{\mu_0} B^2(\vec{x}) \Big)\, d^3x\,. \tag{29}$$

For the case at hand, the electric and magnetic contributions to the energy are respectively given by

$$\mathcal{E}_e \;=\; \frac{Q^2}{8\pi\varepsilon_0 R} \begin{cases} 0 & \text{from } r < R \\ 1 & \text{from } r > R \end{cases} \tag{30a}$$

$$\mathcal{E}_m \;=\; \frac{\mu_0}{4\pi} M^2/R^3 \begin{cases} 2/3 & \text{from } r < R \\ 1/3 & \text{from } r > R\,. \end{cases} \tag{30b}$$

The total magnetic contribution can be written as

$$\mathcal{E}_m = \frac{\mu_0}{4\pi} M^2/R^3 = \tfrac{1}{2}\, I\, \omega^2\,, \tag{31}$$

where

$$I := \frac{\mu_0}{18\pi}\, Q^2 R \tag{32}$$

may be called the *electromagnetic moment of inertia* [1]. It has no mechanical interpretation in terms of a rigid rotation of the electrostatic energy distribution (see below)!

[21] From now on we shall denote the modulus of a vector simply by its core symbol, i.e., $\|\vec{E}\| = E$ etc.

The total electromagnetic energy can now be written as

$$\mathcal{E} = \mathcal{E}_e + \mathcal{E}_m = \frac{Q^2}{8\pi\varepsilon_0 R}\left\{1 + \tfrac{2}{9}\beta^2\right\}, \tag{33}$$

where we used $\varepsilon_0\mu_0 = 1/c^2$ and set $\beta := R\omega/c$. The ratio of magnetic ('kinetic') to total energy is then given by

$$\frac{\mathcal{E}_m}{\mathcal{E}} = \frac{\beta^2}{9/2 + \beta^2}, \tag{34}$$

which is a strictly monotonic function of β bounded above by 1 (as it should be). However, if we require $\beta < 1$, the upper bound is $2/11$.

3.1.2 Angular momentum

The momentum density of the electromagnetic field vanishes for $r < R$ and is given by

$$\vec{p}(\vec{x}) = \frac{\mu_0}{16\pi^2}\, Q\, (\vec{M} \times \vec{n})/r^5 \tag{35}$$

for $r > R$ ($1/c^2$ times 'Poynting vector'). The angular-momentum density also vanishes for $r < R$. For $r > R$ it is given by

$$\vec{\ell}(\vec{x}) = \vec{x} \times \vec{p}(\vec{x}) = \frac{\mu_0}{16\pi^2}\, Q\, \frac{\vec{M} - \vec{n}(\vec{n}\cdot\vec{M})}{r^4}. \tag{36}$$

Hence the total linear momentum vanishes, whereas the total angular momentum is given by

$$\vec{J} := \int_{r>R} \vec{\ell}(\vec{x})\, d^3x = I\vec{\omega} \tag{37}$$

with the *same* I (moment of inertia) as in (32).

3.1.3 The gyromagnetic factor

The gyromagnetic ratio now follows from expressions (28) for $\vec{M}$ and (37) for $\vec{J}$:

$$\frac{M}{J} = \frac{6\pi R}{\mu_0 Q} =: g\,\frac{Q}{2m}, \tag{38}$$

where m denotes the total mass, which is here given by

$$m := \mathcal{E}/c^2 = \frac{\mu_0}{8\pi}\frac{Q^2}{R}\left\{1 + \tfrac{2}{9}\beta^2\right\}. \tag{39}$$

Hence g can be solved for:

$$g = \frac{3}{2}\left\{1 + \tfrac{2}{9}\beta^2\right\}, \tag{40}$$

so that

$$\tfrac{3}{2} < g < \tfrac{11}{6} \qquad \text{if} \qquad 0 < \beta < 1. \tag{41}$$

Even with that simple model we do get quite close to $g = 2$.

3.1.4 Predicting $g = 2$?

It is sometimes stated that Abraham's model somehow 'predicts' $g = 2$ (e.g., [24] p. 39 or [27] p. 206), though this is not at all obvious from [1]. My interpretation for how such a 'prediction' could come about can be given in terms of the present special-relativistic model.[22] It rests on an (inconsistent) combination of the following two observations. First, if we Lorentz transform the purely electric field (24) into constant translational motion with velocity w, we obtain a new electric and also a non-vanishing magnetic field. The integrated Poynting vector then gives the total electromagnetic momentum of the charged shell at speed w:

$$p = \frac{4}{3}\frac{m_e w}{\sqrt{1 - w^2/c^2}}\,, \tag{42}$$

where

$$m_e := \mathcal{E}_e/c^2 = \frac{\mu_0}{8\pi}\frac{Q^2}{R}\,. \tag{43}$$

The infamous factor 4/3 results from the contribution of the (unbalanced) electromagnetic stresses.[23] In this way one is led to assign to the electron a dynamically measurable rest-mass of $m = \frac{4}{3}m_e$ *if one neglects the rotational energy*. Second, we may ask how fast the electron is to spin for (39) to just give $m = \frac{4}{3}m_e$ (rest energy of the spinning electron). The immediate answer is, that this is just the case if and only if $1 + \frac{2}{9}\beta^2 = \frac{4}{3}$, which in view of (40) is equivalent to $g = 2$.

It is now obvious how this argument rests on the conflation of two different notions of mass. The factor $4/3$ will consistently be dealt with by taking into account the stresses that balance electrostatic repulsion, not by trying to account for it in letting the electron spin fast enough.

3.2 A side remark on the kinematics of Faraday lines

In the Introduction we stressed that the emancipation of the notion of angular momentum from the usual kinematical notion of rotation in space had already begun in classical field theory. More precisely this applies to Maxwell's theory, in which the notion of a field differs from that of, say, hydrodynamics in that it is *not* thought of as being attached to a material carrier. This has consequences if we wish to apply kinematical states of motion to the field itself.

At first sight, Faraday's picture of lines of force in space suggests to view them as material entities, capable of assuming different kinematical states of motion. If

[22] Here we ignore Abraham's rigidity condition which would complicate the formulae without changing the argument proper. Also recall footnote 19.

[23] Generally speaking, the factor 4/3 marks the discrepancy between two definitions of 'electromagnetic mass', one through the electromagnetic momentum, the other, called m_e above, through the electrostatic energy. This discrepancy is nothing to get terribly excited about and simply a consequence of the non-conservation of the electromagnetic energy-momentum tensor, i.e., $\nabla_\mu T^{\mu\nu}_{em} \neq 0$, a result of which is that the (unbalanced) electromagnetic stresses contribute to the electromagnetic momentum another third of the expression $p = m_e w/\sqrt{1 - w^2/c^2}$ that one naively obtains from just formally transforming total energy and momentum as time and space components respectively of a four vector. Much discussion in the literature was provoked by getting confused whether this state of affairs had anything to do with Lorentz non-covariance. See, e.g., [6] for a good account and references.

so, the time-dependence of the electromagnetic field might then be interpreted as, and possibly explained by, the motions of such lines (given by some yet unknown equations of motion, of which the Maxwell equations might turn out to be some coarse grained version). That this is not possible has been stressed by Einstein in his 1920 Leiden address "Ether and the Theory of Relativity", where he writes

> If one wishes to represent these lines of force as something material in the usual sense, one is tempted to interpret dynamical processes [of the em. field] as motions of these lines of force, so that each such line can be followed in time. It is, however, well known that such an interpretation leads to contradictions.
> In general we have to say that it is possible to envisage extended physical objects to which the notion of motion [in space] does not apply. ([34], Vol. 7, Doc. 38, p. 315)

The reason why we mention this is that the notion of an "electromagnetic moment of inertia", introduced in (32), nicely illustrates this point. Assume that the electrostatic energy density ρ_e of the Coulomb field of charge Q corresponded to a mass density according to a local version of $E = mc^2$, i.e.,

$$\rho_m(\vec{x}) := \rho_e(\vec{x})/c^2 = \left(\frac{\mu_0}{32\pi^2}\right)\frac{Q^2}{r^4}\,. \tag{44}$$

If the electrostatic energy is now thought of as being attached to the somehow individuated lines of force, a moment of inertia for the shell $R < r < R'$ would result, given by

$$I(R') = \int_{R<r<R'} \rho_m(\vec{x})\,(r\sin\theta)^2\,d^3x = \left(\frac{2\mu_0}{27\pi}\right)\,Q^2\,(R'-R)\,. \tag{45}$$

But this diverges as $R' \to \infty$, in contrast to (32), showing that we may not think of the energy distribution of the electromagnetic field as rigidly rotating in the ordinary sense.

3.3 An electron model with Poincaré stresses

In this section we will modify the previous model for the electron in the following three aspects

1. The infinitesimally thin spherical shell is given a small rest-mass of constant surface density $m_0/4\pi R^2$.

2. Stresses in the shell are taken into account which keep the electron from exploding. They are called "Poincaré stresses" since Poincaré was the first in 1906 to discuss the dynamical need of balancing stresses [28][23].

3. The rotational velocity is small, so that $(R\omega/c)^n$ terms are neglected for $n \geq 2$.

3.3.1 Poincaré stress

The second modification needs further explanation. If we view the surface $r = R$ as a kind of elastic membrane, there will be tangential stresses in the surface of that membrane that keep the charged membrane from exploding. In the present approximation, which keeps only linear terms in ω, these stresses need only balance the electrostatic repulsion, which is constant over the surface $r = R$. In quadratic order the stresses would, in addition, need to balance the latitude dependent centrifugal forces, which we neglect here.

To calculate the surface stress that is needed to balance electrostatic repulsion we recall the expression (30a) for the electrostatic energy as function of radius R:

$$\mathcal{E}_e = \frac{Q^2}{8\pi\varepsilon_0 R}\,. \tag{46}$$

Varying R gives us the differential of work that we need to supply in order to change the volume through a variation of R. Equating this to $-p\,dV = -p\,4\pi R^2\,dR$ gives the pressure inside the electron:

$$p = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q^2}{8\pi\,R^4}\,. \tag{47}$$

Now, imagine the sphere $r = R$ being cut into two hemispheres along a great circle. The pressure tries to separate these hemispheres by acting on each with a total force of strength $p\pi R^2$ in diametrically opposite directions.[24] This force is distributed uniformly along the cut (the great circle), whose length is $2\pi R$. Hence the force per length is just $pR/2$. The surface stress, σ, (force per length) that is needed to prevent the electron from exploding is just the negative of that. Using (47), we therefore get

$$\sigma = -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q^2}{16\pi R^3}\,. \tag{48}$$

3.3.2 Energy-momentum tensor

The energy-momentum tensor now receives a contribution that accounts for the presence of the surface stress (48) that acts tangential to the surface $r = R$ *in the local rest frame corresponding to each surface element of the rotating sphere*. The four-velocity of each surface element is given by[25]

$$u = \partial_t + \omega\,\partial_\varphi\,, \tag{49}$$

which is normalised ($g(u,u) = c^2$) up to terms ω^2 (which we neglect). Recall that the space-time metric of Minkowski space in spatial polar coordinates is (we use the "mostly plus" convention for the signature)

$$g = -c^2\,dt\otimes dt + dr\otimes dr + r^2\,d\theta\otimes d\theta + r^2\sin^2\theta\,d\varphi\otimes d\varphi\,. \tag{50}$$

[24] This follows immediately from the general fact that the total force along a given direction that a constant pressure exerts on a surface is given by the pressure times the area of the planar projection of that surface perpendicular to the given direction. Alternatively it may be verified directly through integrating the element of force in polar direction (i.e. perpendicular to the surface spanned by the great circle), $dF = (p\cos\theta)(R^2\sin\theta\,d\theta\,d\varphi)$, over a hemisphere.

[25] I use spacetime coordinates (t, r, θ, φ) where the latter three are standard spherical polar coordinates. I also employ the notation $\partial_\mu := \partial/\partial x^\mu$ for the chart-induced vector fields, so that, e.g., $\partial_\varphi := \partial/\partial\varphi$.

The energy-momentum tensor has now three contributions, corresponding to the matter of the shell (subscript m), the Poincaré stresses within the shell (subscript σ), and the electromagnetic field (subscript em):

$$\mathbf{T} = \mathbf{T}_m + \mathbf{T}_\sigma + \mathbf{T}_{em}\,. \tag{51a}$$

The first two comprise the shell's contribution and are given by

$$\mathbf{T}_m \;=\; \frac{m_0}{4\pi R^2}\,\delta(r-R)\,u\otimes u\,, \tag{51b}$$

$$\mathbf{T}_\sigma \;=\; -\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q^2}{16\pi R^3}\,\delta(r-R)\,\mathbf{P}\,. \tag{51c}$$

Here $\mathbf{P}$ is the orthogonal projector onto the 2-dimensional subspace orthogonal to u and ∂_r, which is the subspace tangential to the sphere in each of its local rest frames. It can be written explicitly in terms of local orthonormal 2-legs, n_1 and n_2, spanning these local 2-planes. For example, we may take $n_1 := \frac{1}{r}\partial_\theta$ and write (since n_2 must be orthogonal to ∂_r and ∂_θ) $n_2 = a\partial_t + b\partial_\varphi$, where the coefficients a, b follow from $g(u, n_2) = 0$ and normality. This gives

$$\mathbf{P} = n_1\otimes n_1 + n_2\otimes n_2\,, \tag{52a}$$

where

$$n_1 \;:=\; \tfrac{1}{r^2}\,\partial_\theta\,, \tag{52b}$$

$$n_2 \;:=\; c^{-2}\omega\, r\sin\theta\,\partial_t + (r\sin\theta)^{-1}\partial_\varphi\,. \tag{52c}$$

Note that $g(n_1, n_1) = g(n_2, n_2) = 1$ and $g(n_1, n_2) = 0$. Equation (52a) may therefore be be written in the form (again neglecting ω^2 terms)

$$\mathbf{P} = r^{-2}\,\partial_\theta\otimes\partial_\theta + (r\sin\theta)^{-2}\,\partial_\varphi\otimes\partial_\varphi + c^{-2}\omega\left(\partial_t\otimes\partial_\varphi + \partial_\varphi\otimes\partial_t\right). \tag{53}$$

For us the crucial term will be the last one, which is off-diagonal, since it will contribute to the total angular momentum. More precisely, we will need to invoke the integral of $\partial_t\cdot\mathbf{P}\cdot\partial_\varphi$ (the dot $(\cdot)$ refers to the inner product with respect to the Minkowski metric) over the sphere $r = R$:

$$\int(\partial_t\cdot\mathbf{P}\cdot\partial_\varphi)\,R^2\sin\theta\,d\theta\,d\varphi = \int c^{-2}\,\omega g_{tt}g_{\varphi\varphi}\,R^2\sin\theta\,d\theta\,d\varphi = -\tfrac{8\pi}{3}\omega R^4\,. \tag{54}$$

where we used $g_{tt} := g(\partial_t, \partial_t) = -c^2$ and $g_{\varphi\varphi} := g(\partial_\varphi, \partial_\varphi) = R^2\sin^2\theta$ from (50).

3.3.3 A note on linear momentum and von Laue's theorem

The addition of the stress part has the effect that the total energy-momentum tensor is now conserved (here in the slow-rotation approximation):

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{55}$$

as one may explicitly check. Note that since we use curvilinear coordinates here we need to invoke the covariant derivative.[26] Indeed, writing the shell's energy momentum tensor as $\mathbf{T}_s := \mathbf{T}_m + \mathbf{T}_\sigma$, it is not difficult to show that $\nabla_\mu T_s^{\mu\nu}$ is zero for $\nu \neq r$, and for $\nu = r$ is given by $p\,\delta(r-R)$ with p as in (47). But this clearly equals $-\nabla_\mu T_{em}^{\mu\nu}$ since, according to Maxwell's equations, this quantity equals minus the electromagnetic force density on the charge distribution, which is obviously $-p\,\delta(r-R)$. In fact, this is precisely the interpretation that we used to determine p in the first place.

The conservation equation (55) generally ensures that total energy and total momentum form, respectively, the time- and space component of a four vector. Let us now show explicitly that T_σ removes the factor $4/3$ in the calculation of the linear momentum when the system is boosted in ,say, the z direction. To do this we need to calculate the integral of $\partial_z \cdot \mathbf{T}_\sigma \cdot \partial_z$ over all of space and show that it precisely cancels the corresponding integral of the electromagnetic part, i.e. the integral over $\partial_z \cdot \mathbf{T}_{em} \cdot \partial_z$. Noting that $g(\partial_\theta, \partial_z) = r\sin\theta$, we have

$$\int dV\,\big(\partial_z \cdot \mathbf{T}_\sigma \cdot \partial_z\big) = \int dr d\theta\, d\varphi\big(\sigma\,\delta(r-R)\,r^2\sin^3\theta\big) = \tfrac{8\pi}{3}\sigma R^2 = -\tfrac{1}{3}\mathcal{E}_e\,, \quad (56)$$

whereas the tracelessness of $\mathbf{T}_{em}$ together with isotropy immediately imply

$$\int dV\,\big(\partial_z \cdot \mathbf{T}_{em} \cdot \partial_z\big) = \tfrac{1}{3}\int dV\,c^{-2}\big(\partial_t \cdot \mathbf{T}_{em} \cdot \partial_t\big) = \tfrac{1}{3}\mathcal{E}_e\,. \quad (57)$$

That the sum of (56) and (57) vanishes is a consequence of Laue's theorem, which basically states that the integral over all of space of the space-space components of a time-independent conserved energy-momentum tensor vanish. Here this was achieved by including stresses, which subtracted one third of the electromagnetic linear momentum.[27] Similarly, the stresses will also subtract from the electromagnetic angular momentum, this time even the larger portion of three quarters of it. Moreover, since the magnetic moment is the same as before, the stresses will have the tendency to increase the gyromagnetic ratio. This we will see next in more detail

3.3.4 Angular momentum

The total angular momentum represented by (51) is calculated by the general formula

$$J = -\frac{1}{c^2}\int \partial_t \cdot \mathbf{T} \cdot \partial_\varphi\, d^3x = J_m + J_\sigma + J_{em}\,. \quad (58a)$$

The matter part, J_m, corresponding to (51b), yields the standard expression for a mass-shell of uniform density:

$$J_m = \tfrac{2}{3}m_0\omega R^2\,. \quad (58b)$$

[26] We have $\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\lambda}T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda}$, where $\Gamma^\nu_{\mu\lambda} := \frac{1}{2}g^{\nu\sigma}\big(-\partial_\sigma g_{\mu\lambda} + \partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma}\big)$, with $g_{\mu\nu}$ taken from (50). The Γ's are most easily computed directly from the geodesic equation.

[27] The requirement on the stress part $\mathbf{T}_\sigma$ to be such that the total energy and momentum derived from $\mathbf{T}_{em} + \mathbf{T}_\sigma$ should transform as a four vector clearly still leaves much freedom in the choice of $\mathbf{T}_\sigma$. The choice made here is such that the total rest energy equals the electrostatic self energy. But other values for the rest energy (like, e.g., 4/3 of the electrostatic contribution) would also have been possible. In particular, the 'covariantisation through stresses' does not as such prefer any of the 'electromagnetic masses' mentioned above (footnote 23), as has also been demonstrated in an elegant and manifestly covariant fashion in [31].

The electromagnetic part is the same as that already calculated, since the electromagnetic field is the same. Therefore we just read off (37) and (32) that

$$J_{em} = \tfrac{2}{3} \cdot \tfrac{2}{3}\, m_e \omega R^2 \,. \tag{58c}$$

Finally, using (54), the contribution of the stresses can also be written down:

$$J_\sigma = -\tfrac{1}{2} \cdot \tfrac{2}{3}\, m_e \omega R^2 = -\tfrac{3}{4}\, J_{em} \,. \tag{58d}$$

Adding the last two contributions shows that the inclusion of stresses amounts to reducing the electromagnetic contribution from the value given by (58b) to a quarter of that value:

$$J_{em} + J_\sigma = J_{em} - \tfrac{3}{4} J_{em} = \tfrac{1}{4} J_{em} \tag{58e}$$

In total we have

$$J = \left(m_0 + \tfrac{1}{6} m_e \right) \tfrac{2}{3} \omega R^3 \,. \tag{58f}$$

To linear order in ω the kinetic energy does not contribute to the overall mass, m, which is therefore simply given by the sum of the bare and the electrostatic mass

$$m = m_0 + m_e \,. \tag{59}$$

Using this to eliminate m_e in (58f) gives

$$J = \left(\frac{1 + 5 m_0/m}{6} \right) \left(\frac{2}{3} m \omega R^2 \right) \,. \tag{60}$$

3.3.5 The gyromagnetic factor

Since the electromagnetic field is exactly as in the previous model, the magnetic moment in the present case is that given by (28). The gyromagnetic factor is defined through

$$\frac{M}{J} = g \frac{Q}{2m} \,, \tag{61}$$

which leads to

$$g = \frac{6}{1 + 5 m_0/m} \,. \tag{62}$$

This allows for a range of g given by

$$1 \le g \le 6 \,, \tag{63}$$

where $g = 1$ corresponds to $m = m_0$, i.e., no electromagnetic contribution and $g = 6$ corresponds to $m_0 = 0$, i.e., all mass is of electromagnetic origin. The interval (63) looks striking, given the modern experimental values for the electron and the proton:

$$g_{\text{electron}} = 2.0023193043622 \quad \text{and} \quad g_{\text{proton}} = 5.585694713 \,. \tag{64}$$

However, we have not yet discussed the restrictions imposed by our slow-rotation assumption. This we shall do next.

3.3.6 Restrictions by slow rotation

Our model depends on the four independent parameters, $P = (m_0, Q, R, \omega)$. On the other hand, there are four independent physical observables, $O = (m, Q, g, J)$ (M is dependent through (61)). Our model provides us with a functional dependence expressing the observables as functions of the parameters: $O = O(P)$. Since Q is already an observable, it remains to display m, g, J in terms of the parameters:

$$m(m_0, Q, R) = m_0 + \frac{\mu_0}{8\pi}\frac{Q^2}{R} =: m_0 + m_e(Q, R)\,, \tag{65a}$$

$$g(m_0, Q, R) = \frac{6}{1 + 5m_0/m(m_0, Q, R)}\,, \tag{65b}$$

$$J(m_0, Q, R, \omega) = \left(m_0 + \tfrac{1}{6}m_e(Q, R)\right)\tfrac{2}{3}\omega R^2\,. \tag{65c}$$

These relations can be inverted so as to allow the calculation of the values of the parameters from the values of the observables. If we choose to display $\beta := R\omega/c$ rather than ω, this gives

$$m_0(m, g) = m\,\frac{6-g}{5g}\,, \tag{66a}$$

$$m_e(m, g) = m - m_0 = m\,\frac{6(g-1)}{5g}\,, \tag{66b}$$

$$R(m, Q, g) = \frac{\mu_0}{8\pi}\frac{Q^2}{m_e} = \frac{\mu_0}{8\pi}\frac{Q^2}{m}\frac{5g}{6(g-1)} \tag{66c}$$

$$\beta(J, Q, g) = 2J\left[\frac{Q^2}{4\pi\varepsilon_0 c}\right]^{-1}\frac{9(g-1)}{5}\,, \tag{66d}$$

where the last equation (66d) follows from (65c) using (66a-66c). It is of particular interest to us since it allows to easily express the slow-rotation assumption $\beta \ll 1$. For this it will be convenient to measure Q in units of the elementary charge e and J in units of $\hbar/2$. Hence we write

$$Q = n_Q\, e \qquad \text{and} \qquad 2J = n_J\hbar\,. \tag{67}$$

Then, using that the fine-structure constant in SI units reads $\alpha = e^2/(4\pi\varepsilon_0 c\hbar) \approx 1/137$, we get

$$\beta = \frac{n_J}{n_Q^2}\,\alpha^{-1}\,\frac{9(g-1)}{5}\,. \tag{68}$$

This nicely shows that the slow-rotation approximation constrains the given combination of angular momentum, charge, and gyromagnetic factor. In particular, any gyromagnetic factor up to $g = 6$ can be so obtained, given that the charge is sufficiently large. If we set $g = 2$ and $n_J = 1$ (corresponding to the electron's values), we get

$$n_Q \gg \sqrt{n_J(g-1)247} \approx 16\,. \tag{69}$$

This means that indeed we cannot cover the electrons values with the present model while keeping the slow-rotation approximation, though this model seems to be able to accommodate values of g up to six if the charge is sufficiently high. However, we did not check whether the assumption that the matter of the shell provided the stabilising stresses is in any way violating general conditions to be imposed on any energy-momentum tensor. This we shall do next.

3.3.7 Restrictions by energy dominance

Energy dominance essentially requires the velocity of sound in the stress-supporting material to be superluminal. It is conceivable that for certain values of the physical quantities (m, Q, g, J) the stresses would become unphysically high. To check that, at least for the condition of energy dominance, we first note from (51c) and (43) that the stress part of the energy-momentum tensor can be written in the form

$$\mathbf{T}_\sigma = -\frac{1}{2}\frac{m_e}{4\pi R^2}c^2\delta(r-R)\mathbf{P}\,. \tag{70}$$

Hence the ratio between the stress within the shell (in any direction given by the unit spacelike vector n tangent to the shell, so that $n\cdot\mathbf{P}\cdot n = 1$) and its energy density, as measured by a locally co-rotating observer, is given by

$$\left|\frac{n\cdot\mathbf{T}\cdot n}{u\cdot\mathbf{T}\cdot u}\right| = \frac{m_e}{2m_0} = \frac{3(g-1)}{6-g}\,, \tag{71}$$

where we used (66a) and (66b) in the last step. The condition of energy dominance now requires this quantity to be bounded above by 1, so that

$$\frac{3(g-1)}{6-g} \le 1 \Longleftrightarrow g \le \frac{9}{4}\,. \tag{72}$$

Interestingly this depends on g only. Hence we get, after all, an upper bound for g, though from the condition of energy dominance, i.e. a subluminal speed of sound in the shell material, and not from the condition of a subluminal rotational speed.

3.3.8 The size of the electron

What is the size of the electron? According to (66c), its radius comes out to be

$$R = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{e^2}{2m}\,\frac{5}{3}\,, \tag{73}$$

where we set $Q = -e$ and $g = 2$. On the other hand, in Quantum Mechanics, the Compton wavelength of the electron is

$$\lambda = \frac{2\pi\hbar}{mc}\,, \tag{74}$$

so that their quotient is just

$$\frac{R}{\lambda} = \frac{5}{6}\frac{\alpha}{2\pi} \approx 2\cdot 10^{-3}\,. \tag{75}$$

This might first look as if the classical electron is really small, at least compared to its Compton wavelength. However, in absolute terms we have (fm stands for the length scale "Fermi")

$$R \approx 2\cdot 10^{-15}\mathrm{m} = 2\,\mathrm{fm}\,, \tag{76}$$

which is very large compared to the scale of 10^{-3} fm at which modern high-energy experiments have probed the electron's structure, so far without any indication for

substructures. At that scale the model discussed here is certainly not capable of producing any reasonable values for the electron parameters, since the electrostatic mass (and hence the total mass, if we assume the weak energy-condition, $m_0 > 0$, for the shell matter) comes out much too large and the angular momentum much too small (assuming $\beta < 1$).

One might ask whether the inclusion of gravity will substantially change the situation. For example, one would expect the gravitational binding to reduce the electrostatic self-energy. An obvious and answerable questions is whether the electron could be a Black Hole? What is particularly intriguing about spinning and charged Black Holes in General Relativity is that their gyromagnetic factor is $g = 2$, always and exactly![28] For a mass M of about 10^{-30} kg to be a Black Hole it must be confined to a region smaller than the Schwarzschild radius $R_s = 2GM/c^2 \approx 10^{-57}$ m, which is almost 40 orders of magnitude below the scale to which the electron structure has been probed and found featureless. Hence, leaving alone Quantum Theory, it is certainly a vast speculation to presumes the electron to be a Black Hole. But would it also be inconsistent from the point of view of General Relativity? The Kerr-Newman family of solutions for the Einstein-Maxwell equations allow any parameter values for mass (except that it must be positive), charge, and angular momentum. As already stated, $g = 2$ automatically. Hence there is also a solution whose parameter values are those of the electron. However, only for certain restricted ranges of parameter values do these solutions represent Black Holes, that is, possess event horizons that cover the interior singularity; otherwise they contain naked singularities.

More precisely, one measures the mass M, angular momentum per unit mass A, and charge Q of a Kerr-Newman solution in geometric units, so that each of these quantities acquires the dimension of length. If we denote these quantities in geometric units by the corresponding lower case letters, m, a, and q respectively, we have

$$m = M\,\frac{G}{c^2}\,, \tag{77a}$$

$$a = \frac{A}{c}\,, \tag{77b}$$

$$q = Q\,\sqrt{\frac{\mu_0}{4\pi}\,\frac{G}{c^2}}\,. \tag{77c}$$

The necessary and sufficient condition for an event horizon to exist is now given by

$$\left(\frac{a}{m}\right)^2 + \left(\frac{q}{m}\right)^2 \leq 1\,. \tag{78}$$

The relevant quantities to look at are therefore the dimensionless ratios[29]

$$\frac{a}{m} = \frac{A}{M}\cdot\frac{c}{G} \approx \frac{A[\mathrm{m}^2\cdot\mathrm{s}^{-1}]}{M[\mathrm{kg}]}\cdot 5.5\cdot 10^{18}\,, \tag{79a}$$

$$\frac{q}{m} = \frac{Q}{M}\cdot\sqrt{\frac{\mu_0}{4\pi}\,\frac{c^2}{G}} \approx \frac{Q[\mathrm{C}]}{M[\mathrm{kg}]}\cdot 10^{10}\,. \tag{79b}$$

[28] It is known that $g = 2$ is already a preferred value in special-relativistic electrodynamics [4], a fact on which modern precision measurements of $g - 2$ rest. See [27] and [13] for instructive discussions as to what makes $g = 2$ also a special value in General Relativity.

[29] We write $P[X]$ to denote the number that gives the physical quantity P in units of X.

Now, if we insert the parameter values for the electron[30] (we take for Q the modulus e of the electron charge) we arrive at the preposterous values

$$\left.\frac{a}{m}\right|_{\text{electron}} \approx \left(5\cdot 10^{25}\right)\left(5.5\cdot 10^{18}\right) \approx 2.5\cdot 10^{44}\,, \tag{80a}$$

$$\left.\frac{q}{m}\right|_{\text{electron}} \approx \left(1.6\cdot 10^{11}\right)\cdot 10^{10} \approx 1.6\cdot 10^{21}\,, \tag{80b}$$

so that we are indeed very far from a Black Hole. Classically one would reject the solution for the reason of having a naked singularity. But note that this does not exclude the possibility that this exterior solution is valid up to some finite radius, and is then continued by another solution that takes into account matter sources other than just the electromagnetic field.[31]

4 Summary

Understanding the generation of new ideas and the mechanisms that led to their acceptance is a common central concern of historians of science, philosophers of science, and the working scientists themselves. The latter might even foster the hope that important lessons can be learnt for the future. In any case, it seems to me that from all perspectives it is equally natural to ask whether a specific argument is actually true or just put forward for persuasive reasons.

Within the history of Quantum Mechanics the history of spin is, in my opinion, of particular interest, since it marks the first instance where a genuine quantum degree of freedom without a classically corresponding one were postulated to exist. If this were the general situation, our understanding of a quantum theory as the quantisation of a classical theory cannot be fundamentally correct.[32] On the other hand, modern theories of quantisation can explain the quantum theory of a spinning particle as the result of a quantisation applied to some classical theory, in which the notion of spin is already present.[33] Hence, from a modern perspective, it is simply not true that spin has no classical counterpart. That verdict (that is has no classical counterpart), which is still

[30] We have $A = S/M$ with $S = \frac{1}{2}\hbar$ (modulus of electron spin) and use the approximate values $\hbar[\mathrm{J}\cdot\mathrm{s}] \approx 10^{-34}$, $M[\mathrm{kg}] = 10^{-30}$, and $Q[\mathrm{C}] = 1.6\cdot 10^{-19}$.

[31] Even in mesoscopic situations $a < m$ means a very small angular momentum indeed. Recall that in Newtonian approximation the angular momentum of a homogeneous massive ball of radius R is $2MR^2\omega/5$, so that $a/m \leq 1$ translates to the following inequality for the spin period $T = 2\pi/\omega$:

$$T \geq \frac{4\pi}{5}\,\frac{R}{c}\,\frac{R}{m} \approx \frac{R^2[\mathrm{m}]}{M[\mathrm{kg}]}\cdot 10^{19}\ \mathrm{sec}\,, \tag{81}$$

which for a ball of radius one meter and mass 10^3 kilogrammes sets an upper bound for T of $3\cdot 10^8$ years! In fact, (81) is violated by all planets in our solar system.

[32] I take this to be an important and very fundamental point. Perhaps with the exception of Axiomatic Local Quantum Field Theory, any quantum theory is in some sense the quantisation of a classical theory. Modern mathematical theories of 'quantisation' understand that term as 'deformation' (in a precise mathematical sense) of the algebra of observables over *classical* phase space; cf. [40].

[33] Namely in the sense that it has a corresponding classical state space given by a two-sphere, which is a symplectic manifold. However, this state space is not the phase space (i.e. cotangent bundle) over some space of classical configurations, so that one might feel hesitant to call it a classical *degree of freedom*.

often heard and/or read[34], is based on a narrow concept of 'classical system', which has been overcome in modern formulations, as was already mentioned in footnote 18 to which I refer at this point. From that point of view, spin is no less natural in classical physics than in Quantum Theory, which has now become the standard attitude in good textbooks on analytical mechanics, e.g. [33][19] as well as in attempts to formulate theories of quantisation [40][41].

In the present contribution I concentrated on another aspect, namely whether it is actually true that classical models for the electron (as they were already, or could have been, established around 1925) are not capable to account for the actual values of the four electron parameters: mass, charge, angular momentum, and the gyromagnetic factor. This criticism was put forward from the very beginning (Lorentz) and was often repeated thereafter. It turns out that this argument is not as clear cut as usually implied. In particular, $g = 2$ is by no means incompatible with classical physics. Unfortunately, explicit calculations seem to have been carried out only in a simplifying slow-rotation approximation, in which the Poincaré stresses may be taken uniform over the charged shell. In the regime of validity of this approximation $g = 2$ is attainable, but not for small charges. I do not think it is known whether and, if so, how an exact treatment improves on the situation. In that sense, the answer to the question posed above is not known. An exact treatment would have to account for the centrifugal forces that act on the rotating shell in a latitude dependent way. As a result, the Poincaré stresses cannot retain the simple (constant) form as in (51c) but must now also be latitude dependent. In particular, they must be equal in sign but larger in magnitude than given in (48) since now they need in addition to balance the outward pushing centrifugal forces. On one hand, this suggests that their effect is a still further reduction of angular momentum for fixed magnetic moment, resulting in still larger values for g. On the other hand, fast rotational velocities result in an increase of the inertial mass according to (8) and hence an increase of angular momentum, though by the same token also an increase in the centrifugal force and hence an increase in stress. How the account of these different effects finally turns out to be is unclear (to me) without a detailed calculation.[35] It would be of interest to return to this issue in the future.

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[34] Even in critical historical accounts, e.g.: "Indeed, there were unexpected results from quantum theory such as the fact that the electron has a fourth degree of freedom, namely, a spin which has no counterpart in a classical theory" ([23], p. 319).

[35] In that respect the corresponding statements made in Sect. 4.7.1 of [29] seem to me premature.

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