On Max Born's
Vorlesungen über Atommechanik, Erster Band

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Abstract

A little more than half a year before Matrix Mechanics was born, Max Born finished his book *Vorlesungen über Atommechanik, Erster Band*, which is a state-of-the-art presentation of Bohr-Sommerfeld quantisation. This book, which today seems almost forgotten, is remarkable for its epistemological as well as technical aspects. Here I wish to highlight one aspect in each of these two categories, the first being concerned with the rôle of axiomatisation in the heuristics of physics, the second with the problem of quantisation proper before Heisenberg and Schrödinger. This paper is a contribution to the project *History and Foundations of Quantum Physics* of the Max Planck Institute for the History of Sciences in Berlin and will appear in the book *Research and Pedagogy. The History of Quantum Physics through its Textbooks*, edited by M. Badino and J. Navarro.

Contents

1 Outline

Max Born's monograph *Vorlesungen über Atommechanik, Erster Band,* was published in 1925 by Springer Verlag (Berlin) as volume II in the Series *Struktur der Materie* [3]. The second volume appeared in 1930 as *Elementare Quantenmechanik*, coauthored by Pascual Jordan, as volume IX in the same series. Here the authors attempt to give a comprehensive and self-contained account of Matrix Mechanics [4]. The word "elementare" in the title alludes, in a sense, to the logical hierarchy of mathematical structures and is intended to mean "by algebraic methods (however sophisticated) only", as opposed to Schrödinger's wave mechanics, which uses (non elementary) concepts from calculus. Since by the end of 1929 (the preface is dated December 6th 1929) several comprehensive accounts of wave mechanics had already been published[1], the authors felt that it was time to do the same for Matrix Mechanics.

Here I will focus entirely on the first volume, which gives a state-of-the-art account of Bohr-Sommerfeld quantisation[2] from the analytic perspective. One might therefore suspect that the book had almost no impact on the post-1924 development[3] of Quantum Mechanics proper, whose 1925-26 breakthrough did not originate from yet further analytical refinements of Bohr-Sommerfeld theory.[4] But this would be a fruitless approach to Born's book, which is truly remarkable in at least two aspects: First, for its presentation of analytical mechanics, in particular Hamilton-Jacobi theory and its applications to integrable systems as well as perturbation theory and, second, for its epistemological orientation; and even though it is very tempting indeed to present some of the analytic delicacies that Born's book has to offer, I feel equally tempted to highlight some of the epistemological aspects, since the latter do not seem to we widely appreciated. In contrast, Born's book is often cited and praised in connection with Hamilton and Hamilton-Jacobi theory, like e.g. in the older editions of Goldstein's book on classical mechanics.[5]

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[1] Born and Jordan mention the following four books: A. Haas' *Materiewellen und Quantenmechanik*, A. Somerfeld's *Atombau und Spektrallinien Vol.2 (Wellenmechanischer Ergänzungsband)*, L. de Broglie's *Einführung in die Wellenmechanik*, and J. Frenkel's *Einführung in die Wellenmechanik.*

[2] As usual, I use the term "Bohr-Sommerfeld quantisation" throughout as shorthand for what probably should be called Bohr-Ishiwara-Wilson-Planck-Sommerfeld-Epstein-Schwarzschild$\cdots$ quantisation.

[3] The preface is dated November 1924.

[4] A partial revival and refinement of Bohr-Sommerfeld quantisation set in during the late 1950s, as a tool to construct approximate solutions to Schrödinger's equation, even for non-separable systems [16]; see also [11]. Ever since it remained an active field of research in atomic and molecular physics.

[5] In the latest editions (2002 English, 2006 German) the author's seem to have erased all references to Born's book.

2   Structure of the Book

The book is based on lectures Born had given in the winter semester 1923/24 at the University of Göttingen and written with the help of Born's assistant Friedrich Hund, who wrote substantial parts and contributed important mathematical results (uniqueness of action-angle variables). Werner Heisenberg outlined some paragraphs, in particular the final ones dealing with the Helium atom. The text is divided into 49 Sections, grouped into 5 chapters, and a mathematical appendix, which together amount to almost 350 pages. It may be naturally compared and contrasted with Sommerfeld's *Atombau and Spektrallinien I* , which has about twice the number of pages. As already said, Born's text is today largely cited and remembered (if at all!) for its presentation of Hamilton-Jacobi theory and perturbation theory (as originally developed for astronomical problems), which is considered comprehensive and most concise, though today one would approach some of the material by more geometric methods (compare Arnold's book [2] or that of Abraham & Marsden [1]).

The list of contents on the level of chapters is as follows:

Intro.: Physical Foundations (3 sections, 13 Pages)

Ch.1: Hamilton-Jacobi Theory (5 sections, 23 pages)

Ch.2: Periodic and multiple periodic motions (12 sections, 81 pages)

Ch.3: Systems with a single valence ('light') electron (19 sections, 129 pages)

Ch.4: Perturbation theory (10 sections, 53 pages)

Both *Vorlesungen über Atommechanik* were reviewed by Wolfgang Pauli for *Die Naturwissenschaften*. In his Review of the first volume, young Pauli emphasised in a somewhat pointed fashion its strategy to apply mechanical principles to special problems in atomic physics, of which he mentioned the following as essential ones: Keplerian motion and the influence it receives from relativistic mass variations and external fields, general central motion (Rydberg-Ritz formula), diving orbits ["Tauchbahnen"], true principal quantum numbers of optical terms, construction of the periodic system according to Bohr, and nuclear vibrations and rotation of two-atomic molecules. He finally stresses the elaborateness of the last chapter on perturbation theory,

> "...of which one cannot say, that the invested effort corresponds to the results achieved, which are, above all, mainly negative (invalidity of mechanics for the Helium atom). Whether this method can be the foundation of the true quantum theory of couplings, as the author believes, has to be shown by future developments. May this work itself accelerate the development of a simpler and more unified theory of atoms with more than one electron, the manifestly unclear character as of today is clearly pictured in this chapter." ([19], p. 488)

As an amusing aside, this may be compared with Pauli's review of the second volume, which showed already considerably more of his infamous biting irony. Alluding to Born's as well as Born's & Jordan's own words in the introductions to volume 1 and 2 respectively, Pauli's review starts with:

> "This book is the second volume of a series, in which each time the aim and sense [Ziel und Sinn] of the nth volume is made clear by the virtual existence of the (n+1)st." ([18], p. 602)

Having given no recommendation, the review then ends with:

> "The making [Ausstattung] of the book with respect to print and paper is excellent [vortrefflich]". ([18], p. 602)

VORLESUNGEN
ÜBER ATOMMECHANIK

VON

DR. MAX BORN

PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN

HERAUSGEGEBEN
UNTER MITWIRKUNG VON

DR. FRIEDRICH HUND

ASSISTENT AM PHYSIKALISCHEN INSTITUT
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ERSTER BAND

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Fig. 1: Title Page

Inhaltsverzeichnis.

Fig. 2: Table of Contents

3 Born's pedagogy and the heuristic rôle of the deductive/axiomatic method

3.1 Sommerfeld versus Born

Wilhelm von Humboldt's early 19th-century programmatic vision of an intimate coexistence and cross fertilisation of teaching and research soon became a widely followed paradigm for universities in Prussia, other parts of Germany, and around the World. And even though it is clear from experience that there cannot be a general rule saying that the best researchers make the best teachers and vice versa, Humboldt's programme has nevertheless proven extremely successful. In fact, outstanding examples for how to suit the action to the word are provided by the Munich and Göttingen schools of Quantum Physics during the post-World-War-I period. Their common commitment to the "Humboldian Ideal", with action speaking louder than words, resulted in generations of researchers and teachers of highest originality and quality. What makes this even more convincing is the impression that this was not achieved on account of personal individuality; quite the contrary. Sommerfeld in Munich, for example, is well known to have had an extraordinary fine sense for the gifts of each individual students and how to exploit it in an atmosphere of common scientific endeavour [22]. Similar things can be said of Max Born in Göttingen, though perhaps not quite as emphatic. Born's style was slightly less adapted to the non-systematic approaches of scientific greenhorns, whereas Sommerfeld would appreciate any new ideas and tricks, if only for the purpose of problem solving. For Sommerfeld, teaching the art of problem solving was perhaps the single most important concern in classes and seminars [22]. Overly tight and systematic expositions are not suited for that purpose. This point was often emphasised by Sommerfeld, for example right at the beginning of his classic five-volume "Lectures on Theoretical Physics". The first volume is called "Mechanics", not "Analytical Mechanics" as Sommerfeld stresses in a one-page preliminary note that follows the preface, since

> "This name [analytical mechanics] originated in the grand work of Lagrange's of 1788, who wanted to cloth all of mechanics in a uniform language of formulae and who was proud that one would not find a single figure throughout his work. We, in contrast, will resort to intuition [Anschauung] whenever possible and consider not only astronomical but also physical and, to a certain extent, technical applications."

The preface already contains the following programmatic paragraph, which clearly characterises Sommerfeld's approach to teaching in general:

> "Accordingly, in print [as in his classes; D.G.] I will not detain myself with the mathematical foundations, but proceed as rapidly as possible to the physical problems themselves. I wish to supply the reader with a vivid picture of the highly structured material that comes within

the scope of theory from a suitable chosen mathematical and physical vantage point. May there, after all, remain some gaps in the systematic justification and axiomatic consistency. In any case during my lectures I did not want to put off my students with tedious investigations of mathematical or logical nature and distract them from the physically interesting. This approach has, I believe, proven useful in class and has been maintained in the printed version. As compared to the lectures by Planck, which are impeccable in their systematic structure, I believe I can claim a greater variety in the material and a more flexible handling of the mathematics."

This pragmatic paradigm has been taken over and perfected by generations of theoretical physicists; just think of the 10-volume lecture courses by Landau and Lifshitz, which is still in print in many languages and widely used all over the world.

There are many things to be said in favour of this pragmatic approach. For one thing, it takes account of the fact that understanding is a cyclic process. Every student knows that one has to go over the same material again and again in order to appreciate the details of the statements, its hidden assumptions, and the intended range of validity. Often on one's nth iteration one discovers new aspects, in view of which one's past understanding is revealed as merely apparent and ill based. Given that we can almost never be sure for this not to happen again, one might even be tempted to measure one's own *relative* degree of understanding by the number of times this has already happened in the past. From that perspective, the pragmatic approach seems clearly much better suited, since it does not pretend the fiction of an ultimate understanding. Being able to solve concrete problems sounds then like a reasonable and incorruptible criterion.

However, as Thomas Kuhn pointed out long ago, well characterised (concrete) problems, also called "puzzles" by him, must be supplied by paradigms to which the working scientists adhere. If concrete problems become critically severe, with eventually all hopes for solutions under the current paradigm fading away, further puzzle-solving activities will sooner or later decouple from further progress. The crucial question then is: Where can seeds for further progress be found and how should they be planted?

It is with regard to this question that I see a clear distinction between the approaches of Born and Sommerfeld. Sommerfeld once quite frankly admitted to Einstein:

"Everything works out all right [klappt] and yet remains fundamentally unclear. I can only cultivate [fördern] the techniques of the quanta, you have to provide your philosophy." ([14], p. 97).

The planting of seeds could start with simple axioms in a well defined mathematical framework. But even that might turn out to be premature. Heisenberg is one of the figures who repeatedly expressed the optimistic view that physical problems can be "essentially" solved while being still detached from such a framework. In

8

connection with his later search for a unified field theory of elementary particles he said in the preface to his textbook on that matter:

> "At the current status of the theory it would be premature to start with a system of well defined axioms and then deduce from them the theory by means of exact mathematical methods. What one needs is a mathematical description which adequately describes the experimental situation, which does not seem to contain contradictions and which, therefore, might later be completed to an exact mathematical scheme. History of physics teaches us that, in general, a new theory can be phrased in a precise mathematical language only after all essential physical problems have been solved." [13]

It seems even more obvious that in phases of paradigmatic uncertainty not much help can be expected from attempts to establish an axiomatic framework for the doomed theory. And yet, quite surprisingly, this is precisely what Born did, as we shall see in the next subsection.

In a letter to Paul Ehrenfest from 1925, Einstein divided the community of physicists into the *Prinzipienfuchser* and the *Virtuosi* ([22], p. 186).[6] Einstein saw Ehrenfest, Bohr, and himself in the first category and named Debey and Born as members of the latter one. Virtuosity here refers to the high mathematical and calculational abilities, any encounter of which results in mental depression on the side of the Prinzipienfuchser, as Einstein concedes to Ehrenfest who first complained about this effect. However, Einstein adds that opposite effect exists, too.

This dichotomy is certainly not strictly exclusive. An obvious example of somebody who could with equal right be located in both camps is Wolfgang Pauli. But also Born lives in both camps and can be best described, I think, as a Prinzipienfuchser amongst the Virtuosi. The principles about which he is so much concerned arise within the attempt to find a logical basis from which the physically relevant can be deduced without ambiguity, rather than just apply clever tricks. This difference to the Sommerfeld school has once been expressed by Heisenberg in an interview to Thomas Kuhn from February 15th 1963:

> "In Sommerfeld's institute one learned to solve special problems; one learned the tricks, you know. Born took it much more fundamentally, from a very general axiomatic point of view. So only in Göttingen did I really learn the techniques well. Also in this way Born's seminar was very helpful for me. I think from this Born seminar on I was able really to do perturbation calculations with all the rigour which was necessary to solve such problems." (Quoted in [22], p. 58).

Let us now see how Born himself expresses the heuristic value of the axiomatic method in times of uncertainty.

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[6] As Seth already remarked in Note 29 to Chapter 6 of [22], *Prinzipienfuchser* is nearly untranslatable. Existing compound words are Pfennigfuchser (penny pincher) and Federfuchser (pedant) (not 'Pfederfuchser', as stated in [22], which does not exist).

3.2 A remarkable introduction

One third through the book, Born recalls the basic idea of 'Quantum Mechanics' in the following way (the emphases are his; the German original of various terms and phrases are included in square brackets):

> "Once again, we summarise the basic idea of Quantum Mechanics, as developed so far: *For a given Model [Modell] we calculate the totality of all motions* (which are assumed to be multiply periodic) *according to the laws of Classical Mechanics* (neglecting radiation damping); *the quantum conditions select a discrete subset from this continuum of motions.* The *energies* of the selected motions shall be the *true* [wirkliche] ones, as measurable by electron collision, and the energy differences shall, according to Bohr's frequency condition, correspond [zusammenhängen] with the *true* [wirklichen] *light frequencies*, as observed in the spectrum. Besides frequencies, the emitted light possesses the observable properties of intensity, phase, and state of polarisation, which are only approximately accounted for by the theory (§ 17). These exhaust the observable properties of the motion of the atomic system. However, our computation assigns *additional properties* to it, namely orbital frequencies and distances, that is, the course [Ablauf] of motion in time. It seems that these quantities are, as a matter of principle, not accessible to observation.[7] Therewith we arrive at the following judgement [Urteil], that *for the time being our procedure is just a formal computational scheme* which, for certain cases, allows us to replace the still unknown quantum laws by computations on a classical basis [auf klassischer Grundlage]. Of these true [wahren] laws we would have to require, that they only contain relations between observable quantities, that is, energy, light frequencies, intensities, and phases. As long as these laws are still unknown, we have to always face the possibility that our provisional quantum rules will fail; one of our main tasks will be to delimit [Abgrenzen] the validity of these rules by comparison with experience." ([3], p. 113-114)

As an (obvious) side remark, we draw attention to the similarity between Born's formulations in the second half of the above cited passage and Heisenberg's opening sentences of his Umdeutung paper [12].

Born's book attempts an axiomatic-deductive approach to Bohr-Sommerfeld quantisation. This might seem totally misguided at first, as one should naively think that such a presentation only makes sense *after* all the essential physical notions and corresponding mathematical structures have been identified. Certainly

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[7] Here Born adds the following footnote: "Measurements of atomic radii and the like do not lead to better approximations to reality [Wirklichkeit] as, say, the coincidence between orbital and light frequencies."

none of the serious researchers at the time believed that to be the case for Bohr-Sommerfeld quantisation, with Born making no exception as we have just seen from his outline and judgement cited above. So what is Born's own justification for such an attempt? This he explains in his introduction to the book, where he takes a truly remarkable heuristic attitude. I found it quite inappropriate to interfere with his words, so I will now largely quote from that introduction [the translation is mine]:

> "The title 'Atommechanik' of this lecture, which I delivered in the winter-semester 1923/24 in Göttingen, is formed after the label 'Celestial Mechanics'. In the same way as the latter labels that part of theoretical astronomy which is concerned with the calculation of trajectories of heavenly bodies according to the laws of mechanics, the word 'Atommechanik' is meant to express that here we deal with the facts of atomic physics from the particular point of view of applying mechanical principles. This means that we are attempting a deductive presentation of atomic theory. The reservations, that the theory is not sufficiently mature [reif], I wish to disperse with the remark that we are dealing with a test case [Versuch], a logical experiment, the meaning of which just lies in the determination of the limits to which the principles of atomic- and quantum physics succeed, and to pave the ways which shall lead us beyond that limits. I called this book 'Volume I' in order to express this programme already in the title; the second volume shall then contain a higher approximation to the 'final' mechanics of atoms.
>
> I am well aware that the promise of such a second volume is daring [kühn]; since presently we have only a few hints as to the nature of the deviations that need to be imposed onto the classical laws in order to explain the atomic properties. To these hints I count first of all Heisenberg's rendering of the laws of multiplets and anomalous Zeeman effect, the new radiation theory of Bohr, Kramers, and Slater, the ensuing Ansätze of Kramers for a quantum-theoretic explanation of the phenomena of dispersion, and also some general considerations concerning the adaptation of perturbation theory to the quantum principles, which I recently communicated. But all this material, however extensive it might be, does not nearly suffice to shape a deductive theory from it. Therefore, the planned '2. Volume' might remain unwritten for many years to come; its virtual existence may, for the time being, clarify the aim and sense [Ziel und Sinn] of this book.[...]" ([3], p. V-VI)

Born continues and explicitly refers (and suggests the reading of) Sommerfeld's Atombau und Spektrallinien, almost as a prerequisite for a successful study of his own book. But he also stresses the difference which, in part, lies in the deductive approach:

"For us the mechanical deductive approach always comes first [steht überall obenan]. Details of empirical facts will only be given when they are essential for the clarification, the support, or the refutation of theoretical strings of thought [Gedankenreihen]." ([3], p. VI)

But, Born continues, there is a second difference to Atombau und Spektrallinien, namely with respect to the foundations of Quantum Theory, where

..."differences in the emphasis of certain features [Züge] are present; but I leave it to the author to find these out by direct comparison. As regards the relation of my understanding to that of Bohr and his school, I am not not aware of any significant opposition. I feel particularly sympathetic with the Copenhagen researchers in my conviction, that it is a rather long way to go to a 'final quantum theory'." ([3], p. VI)

It would be an interesting project to try to work out the details of the 'second difference', concerning the foundations of Quantum Theory, by close comparison of Born's text with Atombau und Spektrallinien. Later, as we know, Born conceptually favoured the more abstract algebraic approach (Heisenberg) against the more 'anschauliche' wave-theoretic picture, quite in contrast to Sommerfeld, who took a more pragmatic stance. Born's feeling that this conceptual value should receive a stronger promotion, for it is blurred by the semi-anschauliche picture of waves travelling in (high dimensional) configuration space, is clearly reflected in the second volume, as well as in later publications, like in the booklet by him and Herbert Green of 1968 on "matrix methods in quantum mechanics". This split attitude is still very much alive today, though it is clear that in terms of calculational economy wave mechanics is usually preferred.

Born ends his introduction by acknowledging the help of several people, foremost his assistant Friedrich Hund for his "devoted collaboration":

Here I specifically mention the theorem concerning the uniqueness of action-angle variables which, according to my view, lies at the foundation of *today's* quantum theory; the proof worked out by Hund forms the centre [Mittelpunkt] of the second chapter (§ 15)." ([3], p. VII)

Hund is also thanked for the presentation of Bohr's theory of periodic systems. Heisenberg is thanked for his advice and for outlining particular chapters, like the last one on the Helium atom. L. Nordheim's help with the presentation of perturbation theory is acknowledged and H. Kornfeld for checking some calculations. Finally F. Reiche H. Kornfeld and F. Zeilinger are thanked for helping with corrections.

4   On technical issues: What is quantisation?

A central concern of Born's book is the issue of quantisation rules, that is: How can one *unambiguously* generalise

$$J := \oint p\,dq = nh \tag{1}$$

to systems with more than one degrees of freedom? The history of attempts to answer this question is interesting but also rather intricate, and involves various suggestions by Ishiwara [15], Wilson [26], Planck [20], Sommerfeld [23], Schwarzschild [21], Epstein [10, 9], and, last not least, the somewhat singular paper by Einstein from 1917 on "The Quantum Theorem of Sommerfeld and Epstein" ([24], Vol. 6, Doc. 45, pp. 556-567), to which we turn below. These papers have various logical dependencies and also partially differ in subtle ways. Leaving aside Einstein's paper for the moment, the rule that emerged from the discussions looked innocently similar to (1), namely

$$J_k := \oint p_k\,dq_k = n_k h \qquad \text{(no summation over k)} \tag{2}$$

where $k = 1, 2, \cdots, s$ labels the degrees of freedom to be quantised, which need not necessarily exhaust all physical degrees of freedom, of which there are $f \geq s$, as we shall discuss below.[8] Here we adopt the notation from Born's book, where $(q_1, \cdots, q_f; p_1, \cdots, p_f)$ are the generalised coordinates (configuration variables) and momenta respectively. The apparent simplicity of (2) is deceptive though. One thing that needs to be clarified is the domain of integration, here implicit in the $\oint$-symbol. It indicates that the integration over q_k is to be performed over a full periodicity interval of *that* configuration variable. In Sommerfeld's words (his emphases):

> "*Each coordinate shall be extended over the full range necessary to faithfully label the phase of the system.* For a cyclic azimuth in a plane this range is 0 to 2π, for the inclination in space (geographic latitude θ) twice the range between $\theta_{\min}$ and $\theta_{\max}$, for a radial segment r [Fahrstrahl] likewise twice the covered interval from $r_{\min}$ to $r_{\max}$ for the motion in question." ([23], p. 7)

Another source of uncertainty concerns the choice of canonical coordinates in which (2) is meant to hold. Again in Sommerfeld's words of his comprehensive 1916 account:

> "Unfortunately a general rule for the choice of coordinates can hardly be given; it will be necessary to collect further experience by means of specific examples. In our problems it will do to use (planar and

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[8]   In (2) as well as in all formulae to follow, we shall never make use of the summation convention.

spatial) polar coordinates. We will come back to a promising rule
of Schwarzschild and Epstein for the choice of coordinates in § 10."
([23], p. 6)

The rule that Epstein and independently Schwarzschild formulated in their papers
dealing with the Stark effect ([10] and [21] respectively, compared by Epstein in [9]
shortly after Schwarzschild's death) is based on the assumptions that, first, Hamilton's equations of motion

$$\dot{q}_k = \partial H / \partial p_k, \qquad \dot{p}_k = -\partial H / \partial q_k, \tag{3}$$

for time independent Hamiltonians $H(q_1, \cdots, q_f; p_1, \cdots, p_f)$ are solved by means
of a general solution $S(q_1, \cdots, q_f; \alpha_1, \cdots, \alpha_f)$ for the Hamilton-Jacobi equation

$$H\left(q_1, \cdots, q_f; \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_f}\right) = E, \tag{4}$$

where $p_k = \partial S / \partial q_k$ and $\alpha_1, \cdots, \alpha_f$ are constants of integration on which the
energy E depends. Second, and most importantly, that this solution is obtained by
separation of variables:

$$S(q_1, \cdots, q_f; \alpha_1, \cdots, \alpha_f) = \sum_{i=1}^{f} S_i(p_i; \alpha_1, \cdots, \alpha_f). \tag{5}$$

Note that this in particular implies that $p_k = p_k(q_k; \alpha_1, \cdots, \alpha_f)$, i.e. the kth
momentum only depends on the kth configuration variable and the f constants of
integration $\alpha_1, \cdots, \alpha_f$. This is indeed necessary for (2) to make sense, since the
right hand side is a constant and can therefore not be be meaningfully equated to
a quantity that depends non trivially on phase space. Rather, the meaning of (2) is
to select a subset of solutions through equations for the αs. However, separability
is a very strong requirement indeed which, in particular, requires the integrability
of the dynamical system in question, a fact to which only Einstein drew special
attention to in his paper ([24], Vol. 6, Doc. 45, pp. 556-567), as we will discuss in
more detail below. In fact, integrability is manifest once the $J_1, \cdots, J_f$ have been
introduced as so-called "action variables", which are conjugate to some "angle
variables" $w_1, \cdots, w_f$; for then the action variables constitute the f observables in
involution, i.e. their mutual Poisson brackets obviously all vanish.[9]

But even if we swallow integrability as a *conditio sine qua non*, does separability ensure uniqueness? What is the strongest uniqueness result one can hope for?
Well, for (2) to make sense, any two allowed (by conditions yet to be formulated)
sets of canonical coordinates $(q_i, p_i)_{i=1\cdots n}$ and $(\bar{q}_i, \bar{p}_i)_{i=1\cdots n}$ must be such that the
(J_k/h)s (calculated according to (2)) are integers if and only if the $(\bar{J}_k/h)$s are.

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[9] The implication of integrability for separability is far less clear; compare, e.g., [11]. Classic results
concerning sufficient conditions for separability were obtained by Stäckel; see [6])

This is clearly the case if the allowed transformations are such that among the action variables J_k they amount to linear transformations by invertible integer-valued matrices:[10]

$$\bar{J}_k = \sum_{l=1}^{f} \tau_{lk} J_l \quad (\tau_{lk}) \in \mathrm{GL}(f, \mathbb{Z}) \,. \tag{6a}$$

Here $\mathrm{GL}(f, \mathbb{Z})$ is the (modern) symbol for the group of invertible $f \times f$ – matrices with integer entries. The most general transformations for the angle variables compatible with (6a) are

$$\bar{w}_k = \sum_{l=1}^{f} \tau_{kl}^{-1} w_l + \lambda_k(J_1, \cdots, J_f) \,, \tag{6b}$$

where the λ_k are general (smooth) functions.[11]

The task is now to carefully amend the Epstein-Schwarzschild condition of separability by further technical assumptions under which the transformations (6) are the *only* residual ones. The solution of this problem is presented in § 15 of Born's book, who acknowledges essential help with this by Friedrich Hund.

Born also states that the technical conditions under which this result for multiply periodic systems can be derived were already given in the unpublished thesis by J.M. Burgers [5], who is better known for his works on the adiabatic invariants. The arguments in Burger's thesis to show uniqueness are, according to Born, technically incomplete. The conditions themselves read as follows:

A The position of the system shall periodically depend on the angle variables $(w_1, \cdots, w_f)$ with primitive period 1.

B The Hamiltonian is transformed in to a function W depending only on the $(J_1, \cdots, J_f)$.[12]

C The phase-space function

$$S^* = S - \sum_{k=1}^{f} w_k J_k \,, \tag{7}$$

considered as function of the variables (q, w), which generates the canonical transformation $(q, p) \mapsto (w, J)$ via

$$p_k = \frac{\partial S^*}{\partial q_k} \qquad J_k = -\frac{\partial S^*}{\partial w_k} \,, \tag{8}$$

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[10] Note that the inverse matrices must also be integer valued; hence the matrices must have determinant equal to ± 1.

[11] Our equation (6b) differs in a harmless fashion from the corresponding equation (7) on p. 102 of [3], which reads $w_k = \sum_{l=1}^{f} \tau_{kl} \bar{w}_l + \psi_k(J_1, \cdots, J_f)$, into which our equation turns if we redefine the functions through $\psi_k = -\sum_{l=1}^{f} \tau_{kl} \lambda_l$.

[12] We follow Born's notation, according to which the Hamiltonian, considered as function of the action variables, is denoted by W.

shall also be a periodic function of the ws with period 1.

A and B are immediately clear, but the more technical condition C is not. But, as Born remarks, A and B no not suffice to lead to the desired result. In fact, a simple canonical transformation $(w, J) \mapsto (\bar{w}, \bar{J})$ compatible with A and B is

$$\bar{w}_k = w_k + f_k(J_1, \cdots, J_f), \quad \bar{J}_k = J_k + c_k, \tag{9}$$

where the c_k are arbitrary constants. Their possible presence disturbs the quantisation condition, since J_k and $\bar{J}_k$ cannot generally be simultaneously integer multiples of h. Condition C now eliminates this freedom. After some manipulations the following result is stated on p. 104 of [3]:

Theorem (Uniqueness for non-degenerate systems) *If for a mechanical system variables (w, J) can be introduced satisfying conditions A-C, and if there exist no commensurabilities between the quantities*

$$\nu_k = \frac{\partial W}{\partial J_k}, \tag{10}$$

then the action variables J_k are determined uniquely up to transformations of type (6a) [that is, linear transformations by $\mathrm{GL}(f, \mathbb{Z})$].

For the proof, as well as for the ensuing interpretation of the quantisation condition, the notions of *degeneracy* and *commensurability* are absolutely essential: An f-tuple $(\nu_1, \cdots, \nu_f)$ of real numbers is called r-fold degenerate, where $0 \leq r \leq f$, if there are r but not $r+1$ independent integer relations among them, that is, if there is a set of r mutually independent f-tuples $n_1^{(\alpha)}, \cdots, n_f^{(\alpha)}, \alpha = 1, \cdots, r$ of integers, so that r relations of the form

$$\sum_{k=1}^{f} n_k^{(\alpha)} \nu_k = 0, \quad \forall \alpha = 1, \cdots, r. \tag{11}$$

hold, but there are no $r + 1$ relations of this sort. The f-tuple is simply called degenerate if it is r-fold degenerate for some $r > 0$. A relation of the form (11) is called a commensurability. If no commensurabilities exist, the system called non-degenerate or incommensurable.

It is clear that a relation of the form (11) with $n_k^{(\alpha)} \in \mathbb{Z}$ exists if and only if it exists for $n_k^{(\alpha)} \in \mathbb{Q}$ (rational numbers). Hence a more compact definition of r-fold degeneracy is the following: Consider the real numbers $\mathbb{R}$ as vector space over the rational numbers $\mathbb{Q}$ (which is infinite dimensional). The f vectors $\nu_1, \cdots, \nu_f$ are r-fold degenerate if and only if their span is s-dimensional, where $s = f - r$.

Strictly speaking, we have to distinguish between *proper* [Born: "eigentlich"] and *improper* (or contingent) [Born: "zufällig"] degeneracies. To understand the difference, recall that the frequencies are defined through (10), so that each of them is a function of the action variables $J_1, \cdots, J_f$. A proper degeneracy holds

identical for all considered values $J_1, \cdots, J_f$ (which must at least contain for each J_k an open interval of values around the considered value), whereas an improper degeneracy only holds for singular values of the Js. This distinction should then also be made for the notion of r-fold degeneracy: a proper r-fold degeneracy of frequencies is such that it holds identical for a whole neighbourhood of values $J_1, \cdots, J_f$ around the considered one.

The possibility of degeneracies and their relevance for the formulation of quantisation conditions was already anticipated by Schwarzschild [21], who was of course very well acquainted with the more refined aspects of Hamilton-Jacobi theory, e.g. through Charlier's widely read comprehensive treatise [6, 7]. Schwarzschild stated in §3 of [21] that if action-angle variables could be found for which some of the frequencies ν_k, say $\nu_{s+1}, \cdots, \nu_{s+r}$ where $s + r = f$ vanished, then no quantum condition should be imposed on the corresponding actions $J_{s+1}, \cdots, J_{s+r}$. The rational for that description he gave was that defining equation (10) for the frequencies showed that the energy W was independent of $J_1, \cdots, J_k$. In his words (and our notation):

> "This amendment to the prescription [of quantisation] is suggested by the remark, that for a vanishing mean motion ν_k, the equation $\nu_k = \partial W / \partial J_k$ shows that the energy becomes independent of the variables J_k, that therefore these variables have no relation to the energetic process within the system." ([21], p. 550)

From that it is clear that the independence of the energy W of the J_k for which $\nu_k = 0$ is only given if the system is *properly* degenerate; otherwise we just have a stationary point of W with respect to J_k at that particular J_k-value. So Schwarzschild's energy argument only justifies to not quantise those action variables whose conjugate angles have frequencies that vanish identically in the J_k (for some open neighbourhood).

Now, it is true that for a r-fold degenerate system (proper or improper) a canonical transformation exists so that, say, the first $s = f - r$ frequencies $\nu_1, \cdots, \nu_s$ are non-degenerate, whereas the remaining r frequencies $\nu_{s+1}, \cdots, \nu_{s+r}$ are all zero (for the particular values of Js in the improper case). The number s of independent frequencies is called the *degree of periodicity* of the system ([3], p. 105). Hence Scharzschild's energy argument amounts to the statement, that for *proper degeneracies* only the s action variables $J_1, \cdots, J_s$ should be quantised, but not the remaining $J_{s+1}, \cdots, J_{s+r}$. If the degeneracies are improper, systems for arbitrarily close values of the J_k would have them quantised, so that it would seem physically unreasonable to treat the singular case differently, as Epstein argued in [9] in reaction to Schwarzschild.

Born now proceeds to generalise the uniqueness theorem to degenerate systems. For this one needs to determine the most general transformations preserving conditions A-C and, in addition, the separation into s independent and r mutually dependent (vanishing) frequencies. This can indeed be done, so that the above theorem has the following natural generalisation:

Theorem (Uniqueness for degenerate systems) *If for a mechanical system variables (w, J) can be introduced satisfying conditions A-C, then they can always be chosen in such a way that the first s of the partial derivatives*

$$\nu_k = \frac{\partial W}{\partial J_k}, \tag{12}$$

i.e. the $\nu_1, \cdots, \nu_s$ are incommensurable and the others $\nu_{s+1}, \cdots, \nu_{s+r}$, where $s + r = f$, vanish. Then the first s action variables, $J_1, \cdots, J_s$, are determined uniquely up to transformations of type (6a) [that is, linear transformations by $\mathrm{GL}(s, \mathbb{Z})$].

In the next section ($\S\,16$), Born completes these results by showing that adiabatic invariance holds for $J_1, \cdots, J_s$ but not for J_k for $k > s$, even if the degeneracy is merely improper ([3] p. 111). He therefore arrives at the following

Quantisation rule: *Let the variables (w, J) for a mechanical system satisfying conditions A-C be so chosen that $\nu_1, \cdots, \nu_s$ are incommensurable and $\nu_{s+1}, \cdots, \nu_{s+r}$ ($s + r = f$) vanish (possibly $r = 0$). The stationary motions of this systems are then determined by*

$$J_k = n_k h \qquad for \quad k = 1 \cdots, s. \tag{13}$$

Born acknowledges that Schwarzschild already proposed to exempt those action variables from quantisation whose conjugate angles have degenerate frequencies. But, at this point, he does not sufficiently clearly distinguish between proper and improper degeneracies. This issue is taken up again later in Chapter 4 on perturbation theory, where he states that the (unperturbed) system, should it have improper degeneracies, should be quantised in the corresponding action variables (cf. p. 303 of [3]).

A simple system with (proper) degeneracies

To illustrate the occurrence of degeneracies, we present in a slightly abbreviated form the example of the 3-dimensional harmonic oscillator that Born discusses in $\S\,14$ for the very same purpose. Its Hamiltonian reads

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{m}{2}\left(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2\right). \tag{14}$$

The general solution to the Hamilton-Jacobi equation is ($i = 1, 2, 3$):

$$x_i = \sqrt{\frac{J_i}{2\pi^2 \nu_i^2 m}}\ \sin(2\pi w_i), \tag{15a}$$

$$p_i = \sqrt{2\nu_i m J_i}\ \cos(2\pi w_i), \tag{15b}$$

where

$$\nu_i = \frac{\omega_i}{2\pi} \quad \text{and} \quad w_i = \nu_i t + \delta_i. \tag{15c}$$

The δ_i and J_i are six integration constants, in terms of which the total energy reads

$$W = \sum_{i=1}^{3} \nu_i J_i \,. \tag{16}$$

Now, a one-fold degeneracy occurs if the frequencies ν_i obey a single relation of the form

$$\sum_{i=1}^{3} \tau_i \nu_i = 0 \,, \tag{17}$$

where $\tau_i \in \mathbb{Z}$. This happens, for example, if

$$\omega_1 = \omega_2 =: \omega \neq \omega_3 \,, \tag{18}$$

in which case the Hamiltonian is invariant under rotations around the third axis. The energy then only depends on J_3 and the sum $(J_1 + J_2)$. Introducing coordinates x_i' with respect to a system of axes that are rotated by an angle α around the third axis,

$$x_1' \;=\; x_1 \cos\alpha - x_2 \sin\alpha \,, \tag{19a}$$
$$x_2' \;=\; x_1 \sin\alpha + x_2 \cos\alpha \,, \tag{19b}$$
$$x_3' \;=\; x_3 \,, \tag{19c}$$

under which transformation the momenta transform just like the coordinates[13]. The new action variables, J_i', are given in terms of the old (w_i, J_i) by:

$$J_1' \;=\; J_1 \cos^2\alpha + J_2 \sin^2\alpha - 2\sqrt{J_1 J_2}\,\cos(w_1 - w_2)\sin\alpha\cos\alpha \,, \tag{20a}$$
$$J_2' \;=\; J_1 \sin^2\alpha + J_2 \cos^2\alpha + 2\sqrt{J_1 J_2}\,\cos(w_1 - w_2)\sin\alpha\cos\alpha \,, \tag{20b}$$
$$J_3' \;=\; J_3 \,. \tag{20c}$$

As Born stresses, the J_i' do not just depend on the J_is, but also on the w_is, more precisely on the difference $w_1 - w_2$, which is a constant $(\delta_1 - \delta_2)$ along the dynamical trajectory according to (15c) and (18), as it must be (since the J_i' are constant). It is now clear that, for general α, the conditions $J_{1,2} = n_{1,2}h$ and $J_{1,2}' = n_{1,2}'h$ are mutually incompatible. However, (20) shows that the sums are invariant

$$J_1' + J_2' = J_1 + J_2 \,, \tag{21}$$

hence a condition for the sum

$$J_1' + J_2' = J_1 + J_2 = nh \tag{22a}$$

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[13] Generally, the momenta, being elements of the vector space dual to the velocities, transform via the inverse-transposed of the Jacobian (differential) for the coordinate transformation. But for linear transformations the Jacobian is just the transformation matrix and orthogonality implies that its inverse equals its transpose.

together with

$$J_3' = J_3 = n_3 h \tag{22b}$$

makes sense.

But what about other coordinate changes than just rotations? To see what happens, Born considers instead of (19) the transformation to cylindrical polar coordinates (r, φ, z) with conjugate momenta (p_r, p_φ, p_z) (cf. footnote 13):

$$x_1 = r \cos \varphi \qquad p_r = p_1 \cos \varphi + p_2 \sin \varphi \,, \tag{23a}$$

$$x_2 = r \sin \varphi \qquad p_\varphi = -p_1 r \sin \varphi + p_2 r \cos \varphi \,, \tag{23b}$$

$$x_3 = z \qquad p_z = p_3 \,. \tag{23c}$$

The transformation equations from the old (w_i, J_i) to the new action variables (J_r, J_φ, J_z) are:

$$J_r = \tfrac{1}{2}(J_1 + J_2) - \nu^{-1}\sqrt{J_1 J_2}\, \sin\!\big(2\pi(w_1 - w_2)\big)\,, \tag{24a}$$

$$J_\varphi = 2\nu^{-1}\sqrt{J_1 J_2}\, \sin\!\big(2\pi(w_1 - w_2)\big)\,, \tag{24b}$$

$$J_z = J_3 \,. \tag{24c}$$

The total energy expressed as a function of the new action variables reads:

$$W = \nu(2J_r + J_\varphi) + \nu_z J_z \,, \tag{25}$$

where here and in (24) $\nu := \omega/2\pi$ and $\nu_z := \omega_3/2\pi$ (cf. (18)). Again it is only the combination $2J_r + J_\varphi$ that enters the energy expression, and from (24) we see immediately that that

$$2J_r + J_\varphi = J_1 + J_2 \,, \tag{26}$$

Again, conditions of the form $J_r = n_r h$, $J_\varphi = n_\varphi h$, and $J_r = n_r h$ would pick out different "quantum orbits" [Born speaks of "Quantenbahnen"] than those corresponding to $J_i = n_i h$. The energies, however, are the same.

5   Einstein's view

Already in 1917 Einstein took up the problem of quantisation in his long neglected[14] paper "On the Quantum Theorem of Sommerfeld and Epstein" ([24], Vol. 6, Doc. 45, p. 556-567). Einstein summarised this paper in a letter to Ehrenfest dated June 3rd 1917 ([24], Vol. 8, Part A, Doc. 350, pp. 464-6), in which he

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[14] Einsteins paper was cited by de Broglie in his thesis [8], where he spends slightly more than a page (pages 64-65 of Section II in Chapter III) to discuss the "interpretation of Einstein's quantisation condition", and also in Schrödinger's *Quantisation as Eigenvalue Problem*, where in the Second Communication he states in a footnote that Einstein's quantisation condition "amongst all older versions stands closest to the present one [Schrödinger's]". However, after Matrix- and Wave Mechanics settled, Einstein's paper seems to have been largely forgotten until Keller [16] reminded the community of its existence in 1958.

also makes very interesting comments, as we shall see below. For discussions of its content from a modern viewpoint see, e.g., [11] and [25].)

In this paper Einstein suggested to replace the quantum condition (2) by

$$\oint_\gamma \sum_{k=1}^f p_k dq_k = n_\gamma h, \qquad \forall \gamma. \tag{27}$$

First of all one should recognise that here the sum rather than each individual term $p_k dq_k$ as in (2) forms the integrand. Second, (27) is not just one but many conditions, as many as there are independent paths (loops) γ against which the integrand is integrated.

Let us explain the meaning of all this in a modernised terminology. For this, we first point out that the integrand has a proper geometric meaning, since

$$\theta = \sum_{k=1}^f p_k dq_k \tag{28}$$

is the coordinate expression of a global one-form on phase space (sometimes called the Liouville form)[15], quite in contrast to each individual term $p_k\, dq_k$, which has no coordinate independent geometric meaning. Being a one-form it makes invariant sense to integrate it along paths. The paths γ considered here are all closed, i.e. loops, hence the $\oint$-sign. But what are the loops γ that may enter (27)? For their characterisation it is crucial to assume that the system be integrable. This means that there are f (= number of degrees of freedom) functions on phase space, $F_A(q,p)$ $(A = 1, \cdots, f)$, the energy being one of them, whose mutual Poisson brackets vanish:

$$\{F_A, F_B\} = 0. \tag{29}$$

This implies that the trajectories remain on the level sets for the f-component function $\vec{F} = (F_1, \cdots, F_f)$, which can be shown to be f-dimensional tori $T_{\vec{F}}$ embedded in $2f$-dimensional phase space. From (29) it follows that these tori are geometrically special (Lagrangian) submanifolds: The differential of the one form (27), restricted to the tangent spaces of these tori, vanishes identically. By Stokes' theorem this implies that any two integrals of θ over loops γ and γ' within the same torus T coincide in value (possibly up to sign, depending on the orientation given to the loops) if there is a 2-dimensional surface σ within T whose boundary is just the union of γ and γ'. This defines an equivalence relation on the set of

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[15] In the terminology of differential geometry, phase space is the cotangent bundle T^*Q over configuration space Q with projection map $\pi : T^*Q \to Q$. The one-form θ on T^*Q is defined by the following rule: Let z be a point in T^*Q and X_z a vector in the tangent space of T^*Q at z, then $\theta_z(X_z) := z\big(\pi_*|_z(X_z)\big)$. Here the symbol on the right denotes the differential of the projection map π, evaluated at z and then applied to X_z. This results in a tangent vector at $\pi(z)$ on Q on which $z \in T^*_{\pi(z)}Q$ may be evaluated. In local adapted coordinates $(q_1, \cdots, q_f; p_1, \cdots, p_f)$ the projection map π just projects onto the qs. Then, for $X = \sum_k (Y_k \partial_{q_k} + Z_k \partial_{p_k})$ we have $\pi_*(X) = \sum_k Y_k \partial_{q_k}$ and $z\big(\pi_*(X)\big) = \sum_k p_k Y_k$, so that $\theta = \sum_k p_k\, dq_k$.

loops on T whose equivalence classes are called homology classes (of dimension 1). The homology classes form a finitely generated Abelian group (since the level sets are compact) so that each member can be uniquely written as a linear combination of f basis loops (i.e. their classes) with integer coefficients. For example, if one pictures the f-torus as an f-dimensional cube with pairwise identifications of opposite faces through translations, an f-tuple of basis loops is represented by the straight lines-segments connecting the midpoints of opposite faces. Each such basis is connected to any other by a linear $\mathrm{GL}(f, \mathbb{Z})$ transformation.

Now we can understand how (27) should be read, namely as a condition that selects out of a continuum a discrete subset of tori $T_{\vec{F}}$, which may be characterised by discretised values for the f observables F_A. By the last remark of the previous paragraph it does not matter which basis for the homology classes of loops is picked to evaluate (27). This leads to quantisation condition independent of the need to separate variables.

What remains undecided at this stage is how to proceed in cases where degeneracies occur. In the absence of degeneracies, the torus is *uniquely* determined as the closure of the phase-space trajectory for all times. If degeneracies exist, that closure will define a torus of dimension $s < f$, the embedding of which in a torus of dimension f is ambiguous since the latter is not uniquely determined by the motion of the system. This we have seen by Born's examples above. Even simpler examples would be the planar harmonic oscillator and planar Keplerian motion; cf. Sect, 51 of [2]). In that case one has to decide whether (27) is meant to apply only to the s generating loops of the former or to all f of the latter, thus introducing an $f - s$ fold ambiguity in the determination of "quantum orbits" [Born: "Quantenbahnen"].

The geometric flavour of these arguments are clearly present in Einstein's paper, though he clearly did not use the modern vocabulary. Einstein starts from the f-dimensional configuration space that is coordinatised by the qs and regards the ps as certain 'functions' on it, defined through an f parameter family of solutions. Locally in q-space (i.e. in a neighbourhood or each point) Hamilton's equations guarantee the existence of ordinary (i.e. single valued) functions $p_k(q_1, \cdots, q_f)$. However, following a dynamical trajectory that is dense in a portion of q-space the values p_k need not return to their original values. Einstein distinguishes between two cases: either the number of mutually different p-values upon return of the trajectory in a small neighbourhood U around a point in q-space is finite, or it is infinite. In the latter case Einstein's quantisation condition does not apply. In the former case, Einsteins considers what he in the letter to Ehrenfest called the Riemannianisation ("Riemannisierung") of q-space, that is, a finite-sheeted covering. The components p_k will then be a well defined (single valued) co-vector field over the dynamically allowed portion of q-space (see [25] for a lucid discussion with pictures).

In a most interesting 1.5-page supplement added in proof, Einstein points out that the first type of motion, where q-space trajectories return with infinitely many mutually different p-values, may well occur for simple systems with relatively few

degrees of freedom, like e.g. that of three pointlike masses moving under the influence of their mutual gravitational attractions, as was first pointed out by Poincaré in the 1890s to which Einstein refers. Einstein ends his supplement (and the paper) by stating that for non-integrable systems his condition also fails. In fact, as discussed above, it even cannot be written down.

Hence one arrives at the conclusion that the crucial question concerning the applicability of quantisation conditions is that of *integrability*, i.e. whether sufficiently many constants of motion exist; other degrees of complexity, like the number of degrees of freedom, do not directly matter. As we know from Poincaré's work, non-integrability occurs already at the 3-body level for simple 2-body interactions. But what is the meaning of "Quantum Theory" if "quantisation" is not a universally applicable procedure?[16]

In the letter to Ehrenfest already mentioned above, Einstein stresses precisely this point, i.e. that his condition is only applicable to integrable systems, and ends with a truly astonishing statement (here the emphases are mine):

> "As pretty as this may appear, it is just restricted to the special case where the p_ν can be represented as (multi-valued) functions of the q_ν. It is interesting that this restriction just nullifies the validity of statistical mechanics. The latter presupposes that upon recurrence of the q_ν, the p_ν of a system in isolation assume all values by and by which are compatible with the energy principle. *It seems to me, that the true [wirkliche] mechanics is such that the existence of the integrals (which exclude the validity of statistical mechanics) is already assured by the general foundations.* But how to start??"[17] ([24], Vol. 8, Part A, Doc. 350, p. 465)

Are we just told that Einstein contemplated the impossibility of any rigorous foundation of classical statistical mechanics?

6   Final comments

In his book, Born also mentions Poincaré's work and cites the relevant chapters on convergence of perturbation series and the 3-body problem in Charlier's treatise [7], but he does not seem to make the fundamental distinction between integrable and non-integrable systems in the sense Einstein made it. Born never cites Einstein's paper in his book. He mentions the well known problem (since

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[16] Even today this question has not yet received a unanimously accepted answer.

[17] "So hübsch nun diese Sache ist, so ist sie eben auf den Spezialfall beschränkt, dass die p_ν als (mehrdeutige) Funktion der q_ν dargestellt werden können. Es ist interessant, dass diese Beschränkung gerade die Gültigkeit der statistischen Mechanik aufhebt. Denn diese setzt voraus, dass die p_ν eines sich selbst überlassenen Systems bei Wiederkehr der q_ν nach und nach alle mit dem Energieprinzip vereinbaren Wertsysteme annehmen. *Es scheint mir, dass die wirkliche Mechanik so ist, dass die Existenz der Integrale, (welche die Gültigkeit der statistischen Mechanik ausschliessen), schon vermöge der allgemeinen Grundlagen gesichert ist.* Aber wie ansetzen??."

Bruns 1884) of small denominators (described in Chapter 10, §5 of [7]) and also Poincaré's result on the impossibility to describe the motion for even arbitrarily small perturbation functions in terms of convergent Fourier series. From that Born concludes the impossibility to introduce constant J_ks and hence the impossibility to pose quantisation rules in general. His conclusion from that is that, for the time being, one should take a pragmatic attitude (his emphases):

> "Even though the mentioned approximation scheme does not converge in the strict sense, it has proved useful in celestial mechanics. For it could be shown [by Poincaré] that the series showed a *type of semi-convergence*. If appropriately terminated they represent the motion of the perturbed system with great accuracy, not for arbitrarily long times, but still for practically very long times. *From this one sees on purely theoretical grounds, that the absolute stability of atoms cannot be accounted for in this way*. However, for the time being one will push aside [sich hinwegsetzen] this fundamental difficulty and make energy calculations test-wise, in order to see whether one obtains similar agreements as in celestial mechanics." ([3], p. 292-293)

Ten pages before that passage, in the introduction to the chapter on perturbation theory, Born stressed the somewhat ambivalent situation perturbation theory in atomic physics faces in comparison to celestial mechanics: One one hand, 'perturbations' caused by electron-electron interactions are of the same order of magnitude than electron-nucleus interactions, quite in contrast to the solar system , where the sun is orders of magnitude heavier than the planets. On the other hand, the quantum conditions drastically constrain possible motions and could well act as regulator. As regards the analytical difficulties already mentioned above, he comments in anticipation:

> "Here [convergence of Fourier series] an insurmountable analytical difficulty seems to inhibit progress, and one could arrive at the opinion that it is impossible to gain a theoretical understanding of atomic structures up to Uranium." ([3], p. 282-283)

However,

> "The aim of the investigations of this chapter shall be to demonstrate, that this is difficulty is not essential. It would indeed be strange [sonderbar] if Nature barricaded herself behind the analytical difficulties of the n-body problem against the advancement of knowledge [das Vordringen der Erkenntnis]." ([3], p. 282-283)

In the course of the development of his chapter on perturbation theory very interesting technical points come up, one of them being connected with the apparent necessity to impose quantisation conditions for the unperturbed action variables conjugate to angles whose frequencies are *improperly* degenerate. But the discussion of this is technical and hence I leave it for another occasion.

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