FREE ABELIAN LATTICE-ORDERED GROUPS

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ABSTRACT. Let n be a positive integer and $FA\ell(n)$ be the free abelian lattice-ordered group on n generators. We prove that $FA\ell(m)$ and $FA\ell(n)$ do not satisfy the same first-order sentences in the language $\mathcal{L}=\{+,-,0,\wedge,\vee\}$ if $m \neq n$. We also show that $Th(FA\ell(n))$ is decidable iff $n \in \{1,2\}$. Finally, we apply a similar analysis and get analogous results for the free finitely generated vector lattices.

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1. INTRODUCTION.

The class of lattice-ordered abelian groups forms a variety. We will consider the theories of the *free* lattice-ordered abelian groups $FA\ell(n)$ on n generators, for $n \in \mathbb{Z}_+$, in the language $\mathcal{L}:=\{+,-,0,\wedge,\vee\}$. We are concerned with the issue of decidability and elementary equivalence as n varies; i.e., with the analogue of Tarski's famous problem for the category of groups.

Recently Sela, and Kharlampovich & Miasnikov, have lectured on their work on Tarski's problem for free groups on finitely many generators, and posted various papers on the world-wide web ([9] and [8]). The papers will take time to referee, but we are cautiously optimistic that solutions to some parts of Tarski's problem for abstract free groups have been found.

If one considers the analogue of Tarski's problem for free abelian groups on finitely many generators, a complete answer can be deduced from Szmielew's classification of the theories of abelian groups (see [10]). (The essential idea to distinguish between their theories is the observation that if G is a free abelian group, then G is freely generated by exactly n elements iff the index of the subgroup of the 2-divisible elements in the whole group is equal to 2^n (i.e., $[G : 2.G] = 2^n$).)

We show that, as in the case of free abelian groups, one can distinguish between the theories of free lattice-ordered abelian groups by the number of generators, but one gets undecidable theories for $n > 2$. Note that $FA\ell(1)$ is just the direct product $\mathbb{Z} \times \mathbb{Z}$ with the usual addition and lattice order: $(m,n) \geq 0$ iff $m \geq 0$ and $n \geq 0$; it is generated by $(1,-1)$ since $(1,-1) \vee (0,0) = (1,0)$. This structure is decidable by

the Feferman-Vaught Theorem [3] (Proposition 6.3.2) and the decidability result of Presburger for $(\mathbb{Z}, +, 0, 1, <)$ (see for instance [4]).

Using the decidability of Presburger arithmetic, we will first establish that $FA\ell(2)$ is decidable. Next, we will note that $FA\ell(2) \not\equiv FA\ell(n)$ for any $n > 2$. We will then prove that $FA\ell(m) \not\equiv FA\ell(n)$ if $n \neq m$. As a consequence of our proof and an undecidability result (due to Grzegorczyk) of some topological theories [6], we will derive that the theory of $FA\ell(n)$ is undecidable if $n > 2$.

Finally, as one might expect, we will show that one can prove parallel results for the free finitely generated vector lattices.

2. Preliminaries.

Let n be a positive natural number. Consider the additive group of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the pointwise ordering, and let $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq n$, be the projection functions: $\pi_i(x_1, \cdots, x_n) = x_i$. Then, the lattice-ordered sublattice subgroup generated by these n projections is (isomorphic to) the free lattice-ordered abelian group $FA\ell(n)$ on n generators [5] Theorem 5.A. It has the following representation: $FA\ell(n) := \{f = \bigwedge_i \bigvee_j f_{ij} : f_{ij} \in Hom(\mathbb{Z}^n, \mathbb{Z})\}$. Note that any $g \in Hom(\mathbb{Z}^n, \mathbb{Z})$ is equal to $\sum_{i \in S} z_i.\pi_i$, where $z_i \in \mathbb{Z}$, $S = \{1 \leq j \leq n : g(e_j) \neq 0\}$ and $g(e_i) = z_i$. To take full advantage of the geometry of Euclidean n-space, we will often regard the elements of $FA\ell(n)$ as functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $FA\ell(n)^+$ be the set of elements of $FA\ell(n)$ such that $g(x) \geq 0$ for all $x \in \mathbb{R}^n$. Let $FA\ell(n)_+ = FA\ell(n)^+ \setminus \{0\}$.

Definition 2.1. A subspace $\sum_{i=1}^n m_i x_i = 0$ (with all $m_i \in \mathbb{Z}$) will be called an *integral hyperspace*, and the corresponding n-dimensional subsets $\sum_{i=1}^n m_i x_i > 0$, $\sum_{i=1}^n m_i x_i < 0$, $\sum_{i=1}^n m_i x_i \geq 0$, $\sum_{i=1}^n m_i x_i \leq 0$ (with all $m_i \in \mathbb{Z}$) will be called *integral half spaces*. A *cone* in $\mathbb{R}^n$ is a subset which is invariant under multiplication by elements of $\mathbb{R}_+$. A closed *cone* is a cone which is closed in the topology of $\mathbb{R}^n$ and which contains the origin. We will always confine ourselves to such cones defined by integral half spaces. A closed (or open) *integral polyhedral cone* is a cone obtainable by finite unions and intersections from closed (or open) integral half spaces. It is convex if it is obtained using only intersections.

Zero sets of the elements of $FA\ell(n)$ will play a crucial role in our solution.

Definition 2.2. For $f \in FA\ell(n)$, let $Z(f)$ be the zero set of f; i.e.,

$$Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}.$$

Let $S(f)$ be the support of f; i.e.,

$$S(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}.$$

Let K be a subset of $\mathbb{R}^n$; then $S_K(f)$ is the support of f on K ($=\{x \in K : f(x) \neq 0\}$).

We will make constant use of elementary ideas from linear geometry, particularly concerning dimension. Convex integral polyhedral cones can be thought of as (irreducible) *varieties* [7] and zero sets as *algebraic sets*. As in classical algebraic geometry, each algebraic set is a finite union of varieties. If $Z(f) = \bigcup\{Z(f_j) : j \in J\}$

with each $Z(f_j)$ a variety, then we will call these $Z(f_j)$ the *constituent varieties* of $Z(f)$. We define the *dimension* of $Z(f)$ to be the maximum of the dimensions of its constituent varieties. For example, if $f_1 = \pi_n \vee 0$ and $f_2 = -\pi_n \vee 0$, then $Z(f_1) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \leq 0\}$ and $Z(f_2) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, both of which have dimension n. These are the constituent varieties of $Z(f)$ if $f = f_1 \wedge f_2$. However, f is the zero function whose zero set is the entire space $\mathbb{R}^n$. So, viewing f as 0 one obtains a single constituent variety; but viewing f as $f_1 \wedge f_2$ yields two constituent varieties (whose union is $\mathbb{R}^n$). Thus the definition is dependent on the representation of the function as an infimum of a supremum of group expressions.

Definition 2.3. If $f \in FA\ell(n)$ and $Z(f)$ has dimension k, then $Z(f)$ is said to have *only one piece of dimension k* if the union of the constituent varieties of dimension k is itself a single closed convex polyhedral cone of dimension k. In this case, there is $g \in FA\ell(n)$ such that $g = f$ and g has exactly one constituent variety of dimension that of $Z(f)$. In this sense, if $Z(f)$ can be written so as to have only one piece of its dimension, then it can be written so as to have exactly one constituent variety of this dimension.

We recall some results and notions that appear in [1].

Proposition 2.1. ([1], Lemma 3.2) *The zero sets $Z(f)$, $f \in FA\ell(n)$, are precisely the closed integral polyhedral cones in $\mathbb{R}^n$.*

Given an element f of a lattice-ordered abelian group, we define $|f| = f_+ + f_-$ where $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$. Then $f_+ \perp f_-$ where we write $a \perp b$ for $a \wedge b = 0$.

Proposition 2.2. ([1], Lemma 3.3) *Let $f, g \in FA\ell(n)$ and let K be a closed integral polyhedral cone in $\mathbb{R}^n$. Suppose that $S_K(f) \subset S_K(g)$. Then there is an natural number m such that $|f| \leq m.|g|$ on K.*

A subset C of $FA\ell(n)$ is called *convex* if $c_1, c_2 \in C$ and $g \in FA\ell(n)$ with $c_1 \leq g \leq c_2$ always implies that $g \in C$. The convex sublattice subgroups of $FA\ell(n)$ are called *ℓ-ideals*. They are the kernels of homomorphisms of $\mathcal{L}$-structures.

Corollary 2.3. *Let J be an ℓ-ideal of $FA\ell(n)$. Suppose that $g \in J$ and $S(f) \subset S(g)$. Then $f \in J$.*

Given an element $f \in FA\ell(n+1)$, let $< f >_{cl}$ be the ℓ-ideal generated by f; i.e., the subgroup generated by all elements g with $|g| \leq m.|f|$, for some natural number m.

So, in particular, we have that if $S(f) = S(g)$, then $< f >_{cl}$ is equal to $< g >_{cl}$.

Finally, we have the Baker-Beynon Duality Theorem (see Theorem 5.B in [5]).

Proposition 2.4. (Beynon [2]) *Let $f \in FA\ell(n)$ and $g \in FA\ell(m)$. Then $FA\ell(n)/ < f >_{cl}$ is isomorphic to $FA\ell(m)/ < g >_{cl}$ (as $\mathcal{L}$-structures) iff there is a piecewise integral linear homeomorphism θ from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that $\theta(Z(g))$ is equal to $Z(f)$.*

3. DECIDABILITY RESULTS FOR $FA\ell(2)$.

Let $f \in FA\ell(2)$. Then, if $r \in \mathbb{R}_+$, for any $(x_1, x_2) \in \mathbb{R}^2$, we have $r.f(x_1, x_2) = f(r.x_1, r.x_2)$.

Let C be the circle in $\mathbb{R}^2$ of radius 1 and centre $(0,0)$, and $C_0 := C \cap \{(cos\ \theta, sin\ \theta) : tan\ \theta \in \mathbb{Q} \cup \{\pm\infty\}\}$. Let $[0, 2\pi)_0 = \{\theta \in [0, 2\pi) : (cos\ \theta, sin\ \theta) \in C_0\}$. Each element f of $FA\ell(2)$ has the form $\bigvee_{j \in J} \bigwedge_{i \in I} m_{ij}.\pi_1 + n_{ij}.\pi_2$, with $m_{ij},\ n_{ij} \in \mathbb{Z}$. There are $(0 = r_{\ell+1} = r_0 < r_1 < \cdots < r_\ell)$, with $r_k \in [0, 2\pi)_0$ such that f is linear on each arc $[e^{i.r_k},\ e^{i.r_{k+1}}]$ — we abuse notation and also write the points of C in the simpler form e^{ir} $(0 \leq r < 2\pi)$. That is, on each arc f has the form $m_k.\pi_1 + n_k.\pi_2$, and moreover satisfies the obvious compatibility conditions at the $\{e^{i.r_k} : k = 0, \dots, n\}$. Conversely, any finite sequence of linear functions satisfying these compatibility conditions gives rise to an element of $FA\ell(2)$.

Definition 3.1. Let $z_1,\ z_2 \in C_0$ and $(g_1, \cdots, g_m)$ be a tuple of elements of $FA\ell(2)$. We say that the *order type of $(g_1, \cdots, g_m)$ is the same at z_1 and z_2* iff

$$g_{\sigma(1)}(z_1) \leq \cdots \leq g_{\sigma(m)}(z_1) \longleftrightarrow g_{\sigma(1)}(z_2) \leq \cdots \leq g_{\sigma(m)}(z_2),$$

for any permutation σ of $\{1, \cdots, m\}$.

Now given a tuple $\bar{g} = (g_1, \cdots, g_m)$ of elements of $FA\ell(2)$, there is a cell-like decomposition for it in the following sense. There exists a finite subset of C_0 such that each g_i is linear on each interval of the corresponding subdivision and the order type of $\bar{g}$ is the same at any two points of the same interval.

Notation. Let $\mathcal{L} := \{+, -, 0, \wedge, \vee\}$, $\mathcal{L}_0 := \{+, -, 0, \wedge, \vee, ./n : n \in \mathbb{Z}_+\}$, and $\mathcal{L}_{\leq} := \{+, -, 0, 1, \leq, ./n : n \in \mathbb{Z}_+\}$ where the unary functions $/n$ are defined by: $z/n := z'$ iff $\text{OR}_{i=0}^{n-1}\ z = z'.n + i$.

Remarks:

(1) If $|w| = w \vee -w$, then (in any abelian lattice-ordered group) $|w| \geq 0$ with equality iff $w = 0$ (see, [5], Corollary 2.3.9). Indeed, *op. cit.*, $w_1 = 0$ & ... & $w_n = 0$ iff $(|w_1| \vee \cdots \vee |w_n|) = 0$. Thus every open $\mathcal{L}$-formula can be written as a conjunction of formulae each of which is a disjunction of atomic formulae and at most one negation of an atomic formula.

(2) The composition of the unary functions $/n$ is well behaved: $(u/n)/m = u/(n.m)$, and $(a + b)/n = a/n + b/n + 1$ if $(a - n.(a/n)) + (b - n.(b/n)) \geq n$, and $(a + b)/n = a/n + b/n$ otherwise. If we consider the discretely ordered abelian group $\mathbb{Z}$, we can either view it as an $\mathcal{L}_0$-structure or an $\mathcal{L}_{\leq}$-structure: we will denote these by $\mathbb{Z}_{\mathcal{L}_0}$ and $\mathbb{Z}_{\mathcal{L}_{\leq}}$, respectively.

Recall

Proposition 3.1. [Presburger (see [4] chapter 3, (paragraph 2)] $\mathbb{Z}_{\mathcal{L}_{\leq}}$ *admits quantifier elimination.*

Note that, in $\mathbb{Z}_{\mathcal{L}_{\leq}}$ we have $x \geq j$ is equivalent to $x/j > 0$ and $x \in j\mathbb{Z}$ is equivalent to $x - j.(x/j) = 0$ $(j \in \mathbb{Z}_+)$. Thus we have

Proposition 3.2. *Let $\theta(y, \bar{y})$ be an open $\mathcal{L}_0$-formula. Then one can effectively construct an open $\mathcal{L}_{\leq}$-formula $\phi(y, \bar{y})$ (each of whose disjuncts has the form $\max\{t_i(\bar{y})\} \leq n.y \leq \min\{s_j(\bar{y})\}$, where $s_j(\bar{y})$, $t_i(\bar{y})$ are $\mathcal{L}_{\leq}$-terms and $/n$ occurs in θ) such that for any $\bar{g}, f \subseteq \mathbb{Z}$, we have*

$$\mathbb{Z}_{\mathcal{L}_0} \models \theta(f, \bar{g}) \quad \text{iff} \quad \mathbb{Z}_{\mathcal{L}_{\leq}} \models \phi(f, \bar{g}).$$

$\square$

Proposition 3.3. *Let $g_1, \cdots, g_m \in FA\ell(2)$, and let θ, θ_j, $(j \in J)$ be open $\mathcal{L}_0$-formulae. Then one can effectively construct open $\mathcal{L}_0$-formulae θ', θ'_j $(j \in J)$ such that*

$$\exists y \in FA\ell(2)[(\forall u \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta(y(u), \bar{g}(u)) \ and \ \&_{j \in J}(\exists u_j \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta_j(y(u_j), \bar{g}(u_j))]$$

$$\text{iff} \quad [(\forall u \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta'(\bar{g}(u)) \ and \ \&_{j \in J}(\exists u_j \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta'_j(\bar{g}(u_j))]$$

Proof: We first show that, for $n \in \mathbb{Z}_+$, the existential formula

$$(\exists f)[(\bigvee_i t_{1i}) \leq n.f \leq (\bigwedge_j t_{2j})]$$

is equivalent in $\mathbb{Z}$ to the open $\mathcal{L}_0$-formula

$$\max\{t_{1i}\} \leq \min\{t_{2j}\} \text{ and } [(\min\{t_{2j}\} - \max\{t_{1i}\})/n > 0 \text{ or } ((\min\{t_{2j}\} - \max\{t_{1i}\})/n$$
$$= 0 \text{ and } \max\{t_{1i}\} + (n - (\max\{t_{1i}\} - n \cdot (\max\{t_{1i}\}/n))) \leq \min\{t_{2j}\})].$$

Let $t = \max\{t_{1i}\}$ and $u = \min\{t_{2j}\}$. Clearly, there is $z \in \mathbb{Z}$ with $t \leq n.z \leq t + n \leq u$ if $(u - t)/n > 0$ (and hence a solution to $(\exists f)(t \leq n.(f/m) \leq u)$). If $(u - t)/n = 0$, then there is a solution iff $n\mathbb{Z} \cap \{t, t+1, \ldots, u\} \neq \emptyset$. These are the above conditions since if $t = nk + r$, then $t - (n \cdot (t/n)) = r$.

Now consider the existential formula in parameters $\bar{g} = (g_1, \ldots, g_m) \in FA\ell(2)^m$:

$$(\exists f)[(\bigvee_i t_{1i}(\bar{g})) \leq d \cdot f \leq (\bigwedge_j t_{2j}(\bar{g}))].$$

Its satisfaction in $FA\ell(2)$ is equivalent to:

$$(\exists f)\forall x \in C_0 \ [h_1(x) \leq d \cdot f(x) \leq h_2(x)],$$

where $h_1 := \bigvee_i t_{1i}(\bar{g})$ and $h_2 := \bigwedge_j t_{2j}(\bar{g})$ (so $h_1, \ h_2 \in FAl(2)$).

Let $(r_0, r_1, \cdots, r_\ell)$ be a subdivision of C_0 such that $g_1, \ldots, g_m$ are linear on each interval $(r_k \ r_{k+1})$ $(k = 0, \ldots, \ell - 1)$ and of the same order type (whence h_1 and h_2 are linear on each interval, too). Thus the original inequality is equivalent to

$$(\exists f) \bigwedge_{k=0}^{\ell-1} \forall x \in (r_k \ r_{k+1}) \ [h_1(x) \leq d.f(x) \leq h_2(x)].$$

Provided that $\lim_{x \to r_k-} f_k(x) = \lim_{x \to r_k+} f_k(x)$ for all k, this is equivalent to

$$\bigwedge_{k=0}^{\ell-1} (\exists f_k)\forall x \in (r_k \ r_{k+1}) \ [h_1(x) \leq d.f_k(x) \leq h_2(x)].$$

Claim: This is equivalent to

$$\bigwedge_{k=0}^{\ell-1} \forall x \in (r_k \ r_{k+1}) \ (\exists z)[h_1(x) \le d.z \le h_2(x)],$$

(which in turn is equivalent to:

$$\forall x \in C_0 \ (\exists z)[h_1(x) \le d.z \le h_2(x)]).$$

Proof of Claim: Fix $k \in \{0, \ldots, \ell - 1\}$ and let $h_j(x) = m_j x_1 + n_j x_2$ on the interval $(r_k, \ r_{k+1})$ $(j = 1, 2)$. We assume, without loss of generality, that π is one of the r_j.

We will consider two cases separately: firstly, the interval $[r_k, \ r_{k+1}]$ does not contain $\pi/2$ or $3\pi/2$; secondly, the interval $[r_k, r_{k+1}]$ contains either $\pi/2$ or $3\pi/2$.

Case 1: We suppose that the closed interval $[r_k, r_{k+1}]$ is included in $[0, \ \pi/2)$, the other three possibilities being similar. Now $(cos \ r_k, sin \ r_k) = (cos \ r_k)(1, q_1)$ and $(cos \ r_{k+1}, sin \ r_{k+1}) = (cos \ r_{k+1})(1, q_2)$, where $q_1, \ q_2 \in \mathbb{Q}_+$ and $q_2 > q_1$. Let M be the linear transformation mapping $(1, 0)$ to $(1, q_1)$ and $(0, 1)$ to $(1, q_2)$; it is represented by the matrix

$$M = \begin{pmatrix} 1 & 1 \\ q_1 & q_2 \end{pmatrix}.$$

Let $q_1 = a_1/b$ and $q_2 = a_2/b$ where $a_1, \ a_2, \ b \in \mathbb{Z}_+$. Then

$$M^{-1} = 1/(a_2 - a_1). \begin{pmatrix} a_2 & -b \\ -a_1 & b \end{pmatrix}.$$

Note that M^{-1} is a matrix with integral coefficients if $a_2 - a_1 = 1$.

The interval $[r_k, \ r_{k+1}]$ determines an integral cone D_k of $\mathbb{Z}^2$; any element $\bar{u}$ of D_k has the form $M. \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $(v_1, v_2) \in D := \{(x_1, x_2) \in \mathbb{Z}^2 : x_1, x_2 \ge 0\}$.

Our formula (restricted to $[r_k, r_{k+1}]$) is equivalent to

$$\forall \bar{x} \in D_k \ \exists y \ h_1(x_1, x_2) = m_1.x_1 + n_1.x_2 \le d.y \le h_2(x_1, x_2) = m_2.x_1 + n_2.x_2.$$

Replacing $\bar{x}$ by $M\bar{v}$ with $(v_1, v_2) \in D$, we get that

$$h_1(M\bar{v}) = (m_1 + n_1.q_1).v_1 + (m_1 + n_1.q_2).v_2 \ \text{ and}$$
$$h_2(M\bar{v}) = (m_2 + n_2.q_1).v_1 + (m_2 + n_2.q_2).v_2.$$

Assume that $m', \ n' \in \mathbb{Z}$ are such that

$$\forall (v_1, v_2) \in D \ h_1(M\bar{v}) \le d.m'.(v_1 + v_2) + d.n'.(q_1.v_1 + q_2.v_2) \le h_2(M\bar{v}).$$

Evaluating the functions at $(1, 0)$, we get that

$$(m_1 + n_1.q_1) \le d.(m' + n'.q_1) \le (m_2 + n_2.q_1)$$

— or equivalently that

$$(m_1.b + n_1.a_1) \le d.(m'.b + n'.a_1) \le (m_2.b + n_2.a_1);$$

and at $(0, 1)$ we get that

$$(m_1 + n_1.q_2) \le d.(m' + n'.q_2) \le (m_2 + n_2.q_2)$$

— or equivalently that

$$(m_1.b + n_1.a_2) \leq d.(m'.b + n'.a_2) \leq (m_2.b + n_2.a_2).$$

To establish the claim, we show how to choose $m', n' \in \mathbb{Z}$ independent of the point in $[r_k, r_{k+1}]$. To achieve this we need to consider the sign of $(m_1 + n_1.q_1)$ and $(m_2 + n_2.q_2)$. Suppose first that they are both positive. Let m' (respectively, n') be the least integer greater than or equal to $(m_1.b + n_1.a_1)/d$ (respectively $(m_1.b + n_1.a_2)/d$). Then for all $(v_1, v_2) \in D$, we have that

$$h_1(v_1, v_2) \leq d.(m'.v_1 + n'.v_2) \leq h_2(v_1, v_2).$$

Next suppose that $(m_1 + n_1.q_1) \geq 0$ and $(m_2 + n_2.q_2) \leq 0$. Let m' be as above and $n' := \lfloor (m_1.b + n_1.a_2)/d \rfloor$. Then for all $(v_1, v_2) \in D$, we have that

$$h_1(v_1, v_2) \leq d.(m'.v_1 + n'.v_2) \leq h_2(v_1, v_2).$$

The remaining sign distinction cases are similar.

Finally, let $\bar{v} = M^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, with $\bar{u} \in D_k$. Then

$$h_1(a_2.u_1 - b.u_2, -a_1.u_1 + b.u_2) \leq d.[(m'.a_2 - n'.a_1).u_1 + (-b.m' + n'.b).u_2] \leq$$

$$\leq h_2(a_2.u_1 - b.u_2, -a_1.u_1 + b.u_2).$$

This completes Case 1.

Case 2: Suppose that the interval $[r_k, r_{k+1}]$ contains $\pi/2$ or $3\pi/2$.
We consider only the case that $\pi/2 \in [r_k, r_{k+1}]$, the other case being similar.
Evaluating h_1, h_2 at the point $(0, 1)$, we suppose first that

$$n_1 \leq d.z \leq n_2$$

and evaluating them at the point $(z'_1, z'_2) \in C_0$ (with the additional hypotheses that $0 < z'_1 \leq 1$, $z'_2 > 0$), that

$$m_1.z'_1 + n_1.z'_2 \leq d.z' \leq m_2.z'_1 + n_2.z'_2.$$

If n_1 is not divisible by d, we seek a coefficient a such that

$$m_1.x_1 + n_1.x_2 \leq a.x_1 + d.z.x_2 \leq m_2.x_1 + n_2.x_2$$

holds for $0 \leq x_1 \leq z'_1$ and $1 \geq x_2 \geq z'_2$.
This is equivalent (for $x_1 \neq 0$) to

$$m_1 + (n_1 - d.z).x_2/x_1 \leq a \leq m_2 + (n_2 - d.z).x_2/x_1.$$

Now the left hand side tends to $-\infty$ and the right hand side to $+\infty$ if x_2/x_1 goes to $+\infty$. So we can always find (z''_1, z''_2) and a with a divisible by d for which the inequalities hold for $0 \leq x_1 \leq z''_1 \leq z'_1$ and $1 \geq x_2 \geq z''_2 \geq z'_2$. Therefore, letting r be such that $e^{i.r} = (z''_1, z''_2)$, we refine our interval $[r_k, r_{k+1}]$ into $[r_k, r]$, $[r, \pi/2]$, $(\pi/2, r_{k+1}]$. We have just described what to do for the second of these intervals; for the first and third of these intervals proceed as in Case 1.

Now suppose that n_1 is divisible by d. We seek a coefficient a such that

$$m_1.x_1 \leq a.x_1 \leq m_2.x_1 + (n_2 - d.z).x_2$$

for $0 \le x_1 \le z_1'$ and $1 \ge x_2 \ge z_2'$. Provided that $x_1 \ne 0$, this is equivalent to

$$m_1 \le a \le m_2 + (n_2 - d.z).x_2/x_1,$$

and so to finding a coefficient a which is divisible by d. We refine our interval as in the previous case and proceed as before.

We deal with strict inequations at a finite number of points similarly.

This completes the proof of the claim.

For the general case, write θ and θ_j in conjunctive normal form. Then, with the same subintervals of C_0 as above, the satisfaction of

$$\exists y \in FA\ell(2)[(\forall u \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta(y(u), \bar{g}(u)) \text{ and } \&_{j \in J}(\exists u_j \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta_j(y(u_j), \bar{g}(u_j))]$$

is equivalent to the solution of a finite system of "basic" $\mathcal{L}_0$-inequalities and strict inequalities at the endpoints of these intervals each of which is handled as above. The proposition follows.   □

Proposition 3.4. *For any $\mathcal{L}$-formula ϕ, one can effectively construct open $\mathcal{L}_0$-formulae θ_i ($i \in I$), and θ_j' ($j \in J_i$) such that for any tuple of elements $\bar{g}$ of $FA\ell(2)$,*

$$FA\ell(2) \models \phi(\bar{g}) \longleftrightarrow OR_{i \in I} \left\{ \begin{array}{c} (\forall u \in \mathbb{Z}^2) \quad \mathbb{Z} \models \theta_i(\bar{g}(u)) \\ \&_{j \in J_i} \ (\exists u_j \in \mathbb{Z}^2) \quad \mathbb{Z} \models \theta_j'(\bar{g}(u_j)). \end{array} \right.$$

Proof: By induction on the complexity of ϕ. The proposition is clear for atomic $\mathcal{L}$-formulae and negated atomic $\mathcal{L}$-formula $s(\bar{x}) \ne 0$. By Remark (1) above, the proposition follows for all quantifier-free open $\mathcal{L}$-formulae.

If the proposition holds for ϕ, then it clearly holds for $\neg\phi$, and similarly for the disjunction of ϕ_1 or ϕ_2 if it holds for ϕ_1 and ϕ_2.

Suppose now that the induction hypothesis holds for $\psi(f, \bar{x})$ and that $\phi(\bar{x})$ is of the form $\exists f \psi(f, \bar{x})$. Let θ_i, θ_j' be open $\mathcal{L}_0$-formulae ($i \in I$, $j \in J_i$) such that
$FA\ell(2) \models \exists f \psi(f, \bar{g}) \longleftrightarrow$
$OR_i \ (\exists f \in FA\ell(2)) \ \ [(\forall u \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta_i(f(u), \bar{g}(u)) \ \& \ \&_{j \in J_i}(\exists u_j \in \mathbb{Z}^2) \ \ \mathbb{Z} \models \theta_j'(f(u_j), \bar{g}(u_j))]$.

By Proposition 3.3, we can effectively construct open $\mathcal{L}_0$-formulae θ_i'', θ_j''' ($i \in I'$, $j \in J_i'$) such that this last equivalent is equivalent to
$OR_i \ [(\forall u \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta_i''(\bar{g}(u)) \ \& \ \&_{j \in J_i'}(\exists u_j \in \mathbb{Z}^2) \ \mathbb{Z} \models \theta_j'''(\bar{g}(u_j))]$.   □

Corollary 3.5. *Let σ be an $\mathcal{L}$-sentence. Then, one can effectively obtain an open $\mathcal{L}_0$-sentence θ such that $FA\ell(2) \models \sigma \longleftrightarrow \mathbb{Z} \models \theta$.*

Corollary 3.6. *$FA\ell(2)$ is decidable.*

Proof: This follows from the results of this section since $\mathbb{Z}$ is decidable as a lattice-ordered group and the procedure to associate θ with σ is effective by the preceding corollary.   □

4. UNDECIDABILITY RESULTS.

We will prove that the lattice-ordered abelian groups $FA\ell(n)$ are not elementarily equivalent if $n \neq m$, and (as a consequence of the method) that for $n > 2$ their theories are undecidable.

The proof relies on the fact that one can obtain formulae in our language which express the dimension of the zero set of an element of $FA\ell(n)$. We will get these formulae by induction on n.

If one factors out $\langle \pi_{n+1} \rangle_{cl}$ from $FA\ell(n+1)$, then the quotient is isomorphic (as an $\mathcal{L}$-structure) to $FA\ell(n)$. And one can hope to use this correspondence to inductively define the dimension of a zero set in $FA\ell(n+1)$. This we will achieve in a series of lemmata. One cannot capture π_{n+1} by automorphisms of $\mathcal{L}$-structures, and so certainly not by first order sentences in the language. However, one can capture (using the first order language $\mathcal{L}$) elements in $FA\ell(n+1)$ which are "π_{n+1}-like". We will then use the dimension of zero sets in the resulting quotient $FA\ell(n)$ to define the dimension of a zero set in $FA\ell(n+1)$ correctly.

The key idea to do this is to cover the n-sphere with two n-dimensional pieces (the zero sets of $f_1, f_2 > 0$ intersected with the n-sphere), so that $f_1 \wedge f_2 = 0$ and the overlap of these restricted zero sets (the zero set of $f = f_1 \vee f_2$ intersected with the n-sphere) is of dimension $n-1$. This will be characterised by the formula $\chi(f, f_1, f_2)$ in Lemma 4.5. If f, f_1, f_2 are such that the restricted zero set of f is minimal with this property, then we next obtain relativised formulae $\xi^r(\bar{x}, y)$ so that $FA\ell(n) \cong FA\ell(n+1)/\langle f \rangle_{cl} \models \xi(\bar{h} + \langle f \rangle_{cl})$ iff $FA\ell(n+1) \models \xi^r(\bar{h}, f)$ (Lemma 4.6). Finally, we will use these relativised formulae to transfer (the expressibility of) the dimension of zero sets in $FA\ell(n)$ to (the expressibility of) the corresponding dimension theory for all zero sets of elements in $FA\ell(n+1)$, and thereby complete the proofs of the theorems.

Let S^n be the n-sphere; i.e., $\{\bar{y} \in \mathbb{R}^{n+1} : d(\bar{y}, 0) = 1\}$. To any nonzero $\bar{x} \in \mathbb{R}^{n+1}$, we associate the point $\bar{p} \in S^n$ with $\exists r \in \mathbb{R}^+(r \cdot \bar{x} = \bar{p})$. Let ϕ be this map, the projection of $\bar{x}$ from the origin onto S^n. Now for each $f \in FA\ell(n+1)$ and $\bar{x} \in \mathbb{R}^{n+1}$, we have $f(\bar{x}) = r \cdot f(\phi(\bar{x}))$, where r is the norm of $\bar{x}$. So for inductive purposes it is enough to consider the images of $Z(f)$ and $S(f)$ under ϕ; we denote these by $\mathcal{Z}(f)$ and $\mathcal{S}(f)$, respectively. We will regard $\mathcal{Z}(f)$ as a finite union of irreducible varieties $\mathcal{Z}(f_j)$, analogously to $Z(f)$.

Since $Z(g) = Z(|g|)$, we need only consider the zero sets of positive elements.

By our conventions, we have

$$Z(f) \cup Z(g) = Z(f \wedge g) \text{ and } Z(f) \cap Z(g) = Z(f \vee g), \text{ and so}$$

$$\mathcal{Z}(f) \cup \mathcal{Z}(g) = \mathcal{Z}(f \wedge g) \text{ and } \mathcal{Z}(f) \cap \mathcal{Z}(g) = \mathcal{Z}(f \vee g).$$

Our first result (Theorem 4.2) is that we can easily distinguish $Th(FA\ell(2))$ from all $Th(FA\ell(n))$ with $n > 2$ (in the language $\mathcal{L}$). (We can distinguish $FA\ell(1)$ from all the others by noting that the index of the subgroup $2.FA\ell(1)$ in $FA\ell(1)$ is equal to 4.)

For convenience, we begin with a lemma in which we express, in $FA\ell(n)$, that a zero-set has dimension n.

We will check that our formulae are invariant under the equivalence relation on $FA\ell(n)_+$ given by $f \sim f'$ iff $Z(f) = Z(f')$.

Lemma 4.1. *For every n, there are formulae $\psi_{n,n-1}(x)$, $\phi_{n,n-1}(x)$ such that for all $f \in FA\ell(n)_+$,*
  (1) $FA\ell(n) \models \psi_{n,n-1}(f)$ *iff* $dim(Z(f)) = n$.
  (2) $FA\ell(n) \models \phi_{n,n-1}(f)$ *iff* $Z(f)$ *has only one piece of dimension n.*

Proof:
   We first express (in the language $\mathcal{L}$) that the zero set of one element contains the support of another. Then we go on to express (in the language $\mathcal{L}$) that the zero set of an element has dimension n. Let

$$\psi_{n,n-1}(f,g) := (g \perp f \text{ and } g \neq 0), \text{ and } \psi_{n,n-1}(f) := (\exists g)\psi_{n,n-1}(f,g).$$

Note that $dim(\mathcal{S}(g)) = n-1$ for all $g \neq 0$; and ($g \neq 0$ and $f \perp g$) implies $\mathcal{S}(g) \subseteq \mathcal{Z}(f)$ (or, equivalently, $S(g) \subseteq Z(f)$). Since $dim(\mathcal{Z}(f)) = n-1$ implies that $Z(f)$ includes a closed convex polyhedral cone of dimension n (which, necessarily, contains the support of an element of $FA\ell(n)_+$), we get (for $f, g > 0$)

$$FA\ell(n) \models \psi_{n,n-1}(f,g) \text{ iff } \mathcal{S}(g) \subseteq \mathcal{Z}(f),$$

and

$$FA\ell(n) \models \psi_{n,n-1}(f) \text{ iff } dim(\mathcal{Z}(f)) = n-1.$$

Observe that $\psi_{n,n-1}$ is an existential formula and that it is invariant under the equivalence relation $\sim$.

We next express that the support, $S(h)$, of an element $h > 0$, is connected. Let $\theta(h)$ be the formula:

$$(h > 0) \text{ and } \neg((\exists h_1, h_2 > 0)(h_1 \perp h_2, \text{ and } h_1 \vee h_2 = h)).$$

Clearly, for $f > 0$, the algebraic set $Z(f)$ has only *one* n-dimensional piece iff

$$FA\ell(n) \models \psi_{n,n-1}(f) \text{ and } (\forall g > 0)(g \perp f \to (\exists h \geq g)(h \perp f \text{ and } \theta(h))).$$

Denote this formula by $\phi_{n,n-1}(f)$.
   Again, it is invariant under the equivalence relation $\sim$. The complexity of this formula is $\forall \exists \forall$. $\square$

We can now distinguish $FA\ell(2)$ from the other $FA\ell(n)$.

Theorem 4.2. $FA\ell(2) \not\equiv FA\ell(n)$ *for any $n > 2$.*

Proof:
   Consider the sentence:

$$(\forall f_1, f_2 > 0)\bigwedge_{i=1}^{2}(\forall h_{i,1}, h_{i,2})[(f_1 \perp f_2 \text{ and } \bigwedge_{i=1}^{2} f_i = h_{i,1} \vee h_{i,2} \text{ and } h_{i,1} \perp h_{i,2} \text{ and }$$

$$\theta(h_{i,1}) \text{ and } \theta(h_{i,2})] \rightarrow [\bigwedge_{i=1,2 \ k=1,2} (\exists h'_{i,k} > 0)(\ h'_{i,k} \le h_{i,k} \text{ and } \theta(h'_{i,k}) \text{ and }$$

$$(\bigwedge_{i=1,2} \exists g_i \ge (h'_{i,1} \vee h'_{i,2}) \text{ and } (g_1 \perp g_2) \text{ and } \theta(g_1) \text{ and } \theta(g_2)))].$$

It holds in $FA\ell(n)$ if $n > 2$, but not in $FA\ell(2)$. □

We next wish to prove

Theorem 4.3. *The structures* $(FA\ell(n), +, -, \wedge, \vee, 0)$ *and* $(FA\ell(m), +, -, \wedge, \vee, 0)$ *with* $2 < m < n$, $m, n \in \mathbb{Z}_+$, *are not elementarily equivalent.*

Proof:

Our goal is to find formulae in the language $\mathcal{L}$ which encode in $FA\ell(n)$ the usual notion of the dimension of a variety in Euclidean space $\mathbb{E}_n$. This we achieve by induction.

Induction hypothesis: for every $m \in \{2, \ldots, n\}$ and $m - 1 \ge k \ge -1$, we have

$$FA\ell(m) \models \psi_{m,k}(f)$$

iff

$$dim(\mathcal{Z}(f)) = k$$

iff

$Z(f)$ contains the positive span of $(k + 1)$ $\mathbb{R}$-linearly independent vectors in $\mathbb{E}_m$ (but not $(k + 2)$ such).

The case $n = 2$ of the induction is summed up in the following lemma:

Lemma 4.4. *Let* $f \in FA\ell(2)_+$. *Then*
  (1) $FA\ell(2) \models \psi_{2,1}(f)$ *iff* $\mathcal{Z}(f)$ *contains an interval;*
  (2) $FA\ell(2) \models \psi_{2,0}(f)$ *iff* $\mathcal{Z}(f)$ *is a non-empty finite set of points;*
  (3) $FA\ell(2) \models \psi_{2,-1}(f)$ *iff* $\mathcal{Z}(f) = \emptyset$.

Proof: We have already defined the formula $\psi_{2,1}(f)$ in Lemma 4.1.
  Define $\psi_{2,0}(f)$ by:

$$\neg\psi_{2,1}(f) \text{ and } (\exists f_1, f_2 > 0)(f = f_1 \vee f_2 \text{ and } \bigwedge_{i=1,2} \psi_{2,1}(f_i) \text{ and } \phi_{2,1}(f_1 \wedge f_2)).$$

Note that there is nothing specific to the case $n = 2$ in this formula.
  (In fact, one can show that the formula $\psi_{n,n-2}(f)$ obtained by replacing $\psi_{2,1}$ by $\psi_{n,n-1}$ and $\phi_{2,1}$ by $\phi_{n,n-1}$ encodes in $FA\ell(n)$ the zero sets of dimension $n - 2$.)
  It remains to express that a zero set is empty. This can simply be done with the formula

$$\neg\psi_{2,1}(f) \text{ and } \neg\psi_{2,0}(f).$$

Alternatively, we can use the equivalent (in $FA\ell(2)$) formula $\psi_{2,-1}(f)$:

$$\neg\psi_{2,1}(f) \text{ and } \neg\psi_{2,0}(f) \text{ and } (\forall f_1, \ f_2 > 0)[(f = f_1 \vee f_2 \text{ and } \phi_{2,1}(f_1) \text{ and } \phi_{2,1}(f_2))$$

$$\rightarrow (\forall h_1, \ h_2)[(\bigwedge_{j=1}^{2} h_j \perp f_j) \rightarrow (\forall h \geq h_1 \vee h_2)(\neg(\theta(h)) \text{ or } h \not\perp (f_1 \wedge f_2))]].$$

Note that in this case we use the geometry of the circle to ensure that this formula has the desired meaning. □

Assume our induction hypothesis up to $n \geq 2$. We wish to deduce it for $FA\ell(n+1)$. This will be undertaken in the three following lemmata. Our main technical tool will be the Baker-Beynon Duality Theorem (see [2] Theorem 4.1 or Section 2 above). We will apply it as follows:

Suppose that an element $f \in FA\ell(n+1)_+$ behaves like π_{n+1} in the sense that

$$FA\ell(n+1)/ < f >_{cl} \cong FA\ell(n).$$

This property holds iff $Z(f)$ and $\mathbb{R}^n$ are piecewise homogeneous integral linear homeomorphic, which entails that the above isomorphism can be described quite explicitly. Namely, let θ be a piecewise linear integral homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ sending $\mathbb{Z}^n$ onto $Z(f)$. It is of the form: $\theta(\bar{x}) = (u_1(\bar{x}), \cdots, u_{n+1}(\bar{x}))$ with $u_1, \cdots, u_{n+1} \in FA\ell(n)$. Let $T(\theta)$ be the induced map from $FA\ell(n+1)$ to $FA\ell(n)$ sending h to $h \circ \theta$. One may identify $h \circ \theta$ with $h|_{Z(f)}$. The kernel of $T(\theta)$ is the ℓ-ideal $< f >_{cl}$. For convenience we will denote the image $T(\theta)(f)$ by f^θ and write $T^*(\theta)$ for the induced ℓ-isomorphism between $FA\ell(n+1)/ < f >_{cl}$ and $FA\ell(n)$. (see Corollary 5.2.2 in [5].)

Lemma 4.5. *There is a formula $\chi(y)$ such that if $f \in FA\ell(n+1)_+$ and $FA\ell(n+1) \models \chi(f)$, then there exists $f' \geq f$ such that $FA\ell(n+1) \models \chi(f')$ and, say, $\vartheta: FA\ell(n+1)/ < f' >_{cl} \simeq FA\ell(n)$. Moreover, this element f' satisfies the following minimality condition: for any $g \in FA\ell(n+1)_+$ with $Z(g) \subsetneq Z(f')$, then $FA\ell(n+1)_+ \models \neg\chi(g)$. In addition, we have that, for $k \in \{-1, 0, \ldots, n-1\}$*

$$dim\mathcal{Z}(\vartheta(h+ < f' >_{cl})) = k \ \ iff \ \ dim(\mathcal{Z}(h) \cap \mathcal{Z}(f')) = k.$$

Proof: Let

$$\chi(f, f_1, f_2) := f > 0 \text{ and } f = f_1 \vee f_2 \text{ and } \neg\psi_{n+1,n}(f) \text{ and } \bigwedge_{i=1}^{2} \phi_{n+1,n}(f_i) \text{ and } f_1 \perp f_2.$$

Let

$$\chi(f) := (\exists f_1 > 0, \exists f_2 > 0)\chi(f, f_1, f_2).$$

Again note that this formula is invariant under the equivalence relation $\sim$.

We first check that $\chi(|\pi_{n+1}|)$ holds. Let $p_1 := (-\pi_{n+1} \vee 0)$ and $p_2 := (\pi_{n+1} \vee 0)$. Then we have that $|\pi_{n+1}| = p_1 \vee p_2$, $p_1 \perp p_2$ and $\bigwedge_{i=1}^{2} \phi_{n+1,n}(p_i)$.

Next we show the minimality property for $|\pi_{n+1}|$. Take any element $h > 0$ such that $Z(h) \subsetneq Z(\pi_{n+1})$. We claim that $\neg\chi(h)$ holds.

Suppose that $\chi(h)$ holds. Then there are positive elements h_1 and h_2 (each $\mathcal{Z}(h_j)$ having exactly one n-dimensional piece) such that $\mathcal{Z}(h_1) \cup \mathcal{Z}(h_2) = S^n$ and $\mathcal{Z}(h_1) \cap \mathcal{Z}(h_2) = \mathcal{Z}(h)$. Denote the n-dimensional piece of $\mathcal{Z}(h_j)$ by $\mathcal{Z}_n(h_j)$ $(j = 1, 2)$. Since $\mathcal{Z}(h_1) \cup \mathcal{Z}(h_2) = S^n$ and $\mathcal{Z}(h_1) \cap \mathcal{Z}(h_2) \subseteq \mathcal{Z}(\pi_{n+1})$, we cannot have both

$\mathcal{Z}_n(h_1) \cap \mathcal{Z}(p_1) \nsubseteq \mathcal{Z}(\pi_{n+1})$ and $\mathcal{Z}_n(h_2) \cap \mathcal{Z}(p_1) \nsubseteq \mathcal{Z}(\pi_{n+1})$. *Mutatis mutandis* with p_2 in place of p_1. Thus one may assume that $\mathcal{Z}_n(h_j) \subseteq \mathcal{Z}(p_j)$ ($j = 1, 2$). On the other hand, the positive span of $u_1, \cdots, u_m$ is strictly included in $Z(\pi_{n+1})$ (by assumption), so either $\mathcal{Z}_n(h_1) \subsetneq \mathcal{Z}(p_1)$ or $\mathcal{Z}_n(h_2) \subsetneq \mathcal{Z}(p_2)$. This contradicts the fact that $\mathcal{Z}(h_1) \cup \mathcal{Z}(h_2) = S^n$. Hence $\neg\chi(h)$ holds.

Let θ be the embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ sending $(x_1, \cdots, x_n)$ to $(x_1, \cdots, x_n, 0)$ and $T^*(\theta)$ be the induced ℓ-isomorphism between $FA\ell(n+1)/ < |\pi_{n+1}| >_{cl}$ and $FA\ell(n)$.

Let $f, f_1, f_2 \in FA\ell(n+1)_+$ be such that $FA\ell(n+1) \models \chi(f, f_1, f_2)$. Then, since $\phi_{n+1,n}(f_i)$ hold for $i = 1, 2$, we get that (to within lower dimensional varieties) $Z(f_1)$ has one piece of dimension $n+1$ (that is equal to a finite union of $(n+1)$-dimensional polyhedral cones) and that $Z(f_2)$, also up to varieties of lower dimension, is equal to the the closure of the complement of $Z(f_1)$. In particular, $Z(f_1)$ contains an $(n+1)$-dimensional closed polyhedral cone C_1 whose boundary is included in $Z(f_1 \vee f_2)$, positively generated by at least $n+1$ $\mathbb{R}$-linearly independent elements $u_1, \cdots, u_{n+1}$. Define $f_1' \geq f_1$ to be a function whose zero set is equal to C_1 and $f_2' \geq f_2$ a function whose zero set is the closure of the complement of C_1. Let $f' = f_1' \vee f_2'$. We have that $Z(f') = Z(f_1' \vee f_2') \subseteq Z(f_1 \vee f_2)$. The boundary of the cone C_1 is piecewise linearly homeomorphic to $Z(|\pi_{n+1}|)$; so denoting this homeomorphism by τ, we have that $< f' >_{cl}$ is equal to $< |\pi_{n+1}|^\tau >_{cl}$. Let $\hat{\tau}$ be the induced ℓ-isomomorphism of $FA\ell(n+1)$. Thus $\chi(|\pi_{n+1}|) \leftrightarrow \chi(|\pi_{n+1}|^\tau)$. Moreover, χ is invariant under $\sim$, and so we get that $\chi(f')$ holds. Indeed, this function f' is minimal such. For let g be such that $Z(g) \subsetneq Z(f')$,. Then $Z(g^{\tau^{-1}}) \subsetneq Z(|\pi_{n+1}|)$, so $FA\ell(n+1) \models \neg\chi(g^{\tau^{-1}})$. But $\hat{\tau}$ is an ℓ-isomorphism and so we also get that $FA\ell(n+1) \models \neg\chi(g)$.

Let ϑ be the composition $T^*(\theta) \circ \hat{\tau}^{-1}$. Then, ϑ is an ℓ isomorphism between: $FA\ell(n+1)/ < f' >_{cl}$ and $FA\ell(n)$. Moreover,

$$h+ < f' >_{cl} \mapsto h|_{\mathcal{Z}(f')}$$

is well-defined since $h \in< f' >_{cl}$ iff $\mathcal{Z}(h) \supseteq \mathcal{Z}(f')$.

The last assertion of the lemma follows from the fact that we may identify h^ϑ with $h|_{Z(f')}$. $\quad\square$

Lemma 4.6. *There is a definable relation $R(h, f)$, such that whenever $f \in FA\ell(n+1)_+$ is such that $\chi(f)$ holds and if $Z(f') \subsetneq Z(f)$ then $\neg\chi(f')$, then $R(h, f)$ iff $Z(h)$ contains the n-dimensional-part of $Z(f)$. Further, for any formula $\xi(\bar{x})$ one can explicitly construct a formula $\xi^r(\bar{x}, y)$ such that for any $\bar{h} \subseteq FA\ell(n+1)$,*

$$FA\ell(n+1)/ < f >_{cl} \models \xi(\bar{h}+ < f >_{cl}) \text{ iff } FA\ell(n+1) \models \xi^r(\bar{h}, f).$$

Proof: Define $R(h, f)$ by

$$(\exists f_1 > 0)(\exists f_2 > 0)(\chi(f, f_1, f_2) \text{ and}$$

$$(\exists h_1 \geq 0)(\exists h_2 \geq 0)[h = h_1 \vee h_2 \text{ and } \bigwedge_{i=1}^{2} \forall k((k \perp f_i) \leftrightarrow (k \perp h_i))]).$$

This relation R expresses that the $(n+1)$-dimensional part of $Z(h_i)$ is equal to that of $Z(f_i)$ for $i = 1, 2$. Now the minimality hypothesis of $Z(f)$ implies that its

n-dimensional part is equal to the intersection of the $(n + 1)$-dimensional part of $Z(f_1)$ and $Z(f_2)$.

We define ξ^r by induction on the complexity of the formula ξ. For an atomic formula $\xi(\bar{x}) := (t(\bar{x}) = 0)$, we define $\xi^r(\bar{x}, y)$ as $R(t(x), y)$. For a quantifier-free formula $\xi(\bar{x})$ (i.e., a Boolean combination of atomic formulae $\xi_i(\bar{x}) := t_i(\bar{x}) = 0$), we define $\xi^r(\bar{x}, y)$ to be the same Boolean combination of $\xi_i^r(\bar{x}, y)$. Finally, if $\xi(\bar{x})$ is in prenex form $Q(\bar{z})\xi(\bar{z}, \bar{x})$, where $Q(\bar{z})$ is a block of quantifiers, define $\xi^r(\bar{x}, y)$ as $Q(\bar{z})\xi^r(\bar{z}, \bar{x}, y)$.

We now prove the equivalence: for any $\bar{h} \in FA\ell(n+1)$,

$$FA\ell(n+1)/ < f >_{cl} \models \xi(\bar{h} + < f >_{cl}) \text{ iff } FA\ell(n+1) \models \xi^r(\bar{h}, f),$$

by induction on the complexity of the formula. It suffices to prove it for atomic formulae. This is immediate as $t(\bar{h} + < f >_{cl}) = 0$ iff $t(\bar{h}) \in < f >_{cl}$ iff $t(\bar{h})|_{Z(f)} = 0$ iff $Z(t(\bar{h})) \supseteq Z(f)$. By the minimality of $Z(f)$ (subject to $\chi(f)$), the zero set of f comprises a single piece of dimension n; hence the last equivalent condition holds iff $R(t(\bar{h}), f)$. $\square$

Let $f > 0$ be such that $FA\ell(n+1)/ < f >_{cl} \cong FA\ell(n)$ and let ϑ be this isomorphism. By the induction hypothesis on $FA\ell(n)$, we get that

$$FA\ell(n) \models \psi_{n,k}(\vartheta(h + < f >_{cl}) \text{ iff } \dim(\mathcal{Z}(h) \cap \mathcal{Z}(f)) = k,$$

for $k = -1, \cdots, n - 1$. Since ϑ is an isomorphism, we also get that

$$FA\ell(n+1)/ < f >_{cl} \models \psi_{n,k}(h + < f >_{cl}) \text{ iff } \dim(\mathcal{Z}(h) \cap \mathcal{Z}(f)) = k.$$

We are now ready to define the formulae $\psi_{n+1,i}(x)$, for $i = -1, \cdots, n$.

Definition 4.1. Recall that $\psi_{n+1,n}(x)$ has been defined by:

$$\exists g \ (g \perp x \text{ and } g \neq 0).$$

Then for $-1 \leq i \leq n - 1$, define $\psi_{n+1,i}(x)$ to be the formula

$$\neg\psi_{n+1,n}(x) \text{ and } \exists f \ [\chi(f) \text{ and } \forall f' \geq f \ (\chi(f') \rightarrow \psi_{n,i}^r(x, f'))] \text{ and}$$

$$\neg[\exists f(\chi(f) \text{ and } \bigvee_{0 < j < n-i} \psi_{n,i+j}^r(x, f))].$$

Remarks: (1) If we denote the block of quantifiers of maximal complexity in the formulae $\psi_{m,k}$ in prenex forms by Q (where $0 \leq k \leq m - 1$ and $2 \leq m \leq n$), then we get that the complexity of the formulae $\psi_{n+1,k}$ is at most $\max\{\exists\forall\exists\forall\exists, \exists\forall Q, \forall\neg Q\}$.

(2) By construction, these formulae $\psi_{n+1,i}, -1 \leq i \leq n$, are pairwise inconsistent.

It remains to prove that they really capture the notion of dimension (in $\mathbb{E}_{n+1}$).

Lemma 4.7. Let $h \in FA\ell(n+1)_+$. Suppose that $\dim(\mathcal{Z}(h)) = i$, for some $-1 \leq i \leq n - 1$. Then there is $f \in FA\ell(n+1)_+$ such that $\chi(f)$ and $\dim(\mathcal{Z}(h) \cap \mathcal{Z}(f)) = i$ and $FA\ell(n+1) \models \psi_{n+1,i}(h)$.

Conversely, if $FA\ell(n+1) \models \psi_{n+1,i}(h)$, then $\dim(\mathcal{Z}(h)) = i$.

Proof: Let $h > 0$ and assume that $\dim(\mathcal{Z}(h)) = i$. So its zero set contains the positive span of $(i + 1)$ $\mathbb{R}$-linearly independent elements: $u_1, \cdots, u_{i+1}$. Without loss of generality, we may assume that this span is maximal with these properties. Complete this set to a basis of $\mathbb{Z}^{n+1}$ and obtain $u_1, \cdots, u_{n+1}$. Let σ be the transformation sending $(u_1, \cdots, u_{n+1})$ to $(e_1, \cdots e_{n+1})$ the canonical basis of $\mathbb{Z}^{n+1}$. It is a piecewise linear integral homeomorphism. Let $f \in FA\ell(n + 1)_+$ with zero set the $\mathbb{R}$-span of $u_1, \cdots, u_n$. Then, $FA\ell(n + 1)/ < f >_{cl} \cong FA\ell(n + 1)/ < \pi_{n+1} >_{cl} \cong FA\ell(n)$. Denote by ς the isomorphism between $FA\ell(n + 1)/ < f >_{cl}$ and $FA\ell(n)$. As in the proof of Lemma 4.5, we have that $FA\ell(n + 1) \models \chi(f)$ and $FA\ell(n + 1) \models \neg\chi(f')$ whenever $Z(f') \subsetneq Z(f)$.

By the induction hypothesis applied to $FA\ell(n)$, we have $\dim \mathcal{Z}(\varsigma(h+ < f >_{cl})) = i$ iff $FA\ell(n) \models \psi_{n,i}(\varsigma(h+ < f >_{cl}))$ iff $FA\ell(n + 1)/ < f >_{cl} \models \psi_{n,i}(h+ < f >_{cl})$ iff, by the preceding Lemma, $FA\ell(n + 1) \models \psi_{n,i}^r(h, f)$. Finally, we have that $\dim \mathcal{Z}(h) \cap \mathcal{Z}(f) = i$ iff $\dim \mathcal{Z}(\varsigma(h+ < f >_{cl})) = i$.

Now assume that $FA\ell(n+1) \models \psi_{n+1,i}(h)$; so there is $f > 0$ such that for all $f' \geq f$ with $\chi(f')$ we have $\psi_{n,i}^r(h, f')$. Choose f' as in Lemma 4.5; i.e., such that

$$FA\ell(n + 1)/ < f' >_{cl} \cong FA\ell(n + 1)/ < \pi_{n+1} >_{cl} \cong FA\ell(n)$$

and f' also satisfies the minimality assumption. Let ς be the above isomorphism. We may apply Lemma 4.6. By hypothesis, we have that $FA\ell(n + 1) \models \psi_{n,i}^r(h, f')$; it follows that $FA\ell(n + 1)/ < f' >_{cl} \models \psi_{n,i}(h+ < f' >_{cl})$. Hence we get that $FA\ell(n) \models \psi_{n,i}(\varsigma(h+ < f' >_{cl}))$. By induction hypothesis, we have that $\dim(\mathcal{Z}(\varsigma(h+ < f' >_{cl})))=i$. So, by Lemma 4.5 again, we have that $\dim(\mathcal{Z}(f') \cap \mathcal{Z}(h)) = i$. Therefore, $dim(\mathcal{Z}(h)) \geq i$. Suppose that $dim(\mathcal{Z}(h)) = j > i$. Then by the first part of the lemma, there would be $g > 0$ such that $\chi(g)$ and $FA\ell(n + 1) \models \psi_{n,j}^r(h, g)$. This contradicts $FA\ell(n + 1) \models \psi_{n+1,i}(h)$. $\square$

Proof of Theorem 4.3 (continued): We apply Lemmata 4.5, 4.6 and 4.7.

Observe first that the formulae $\psi_{n,n-1}(x, y)$ and $\theta(x)$ are independent of n. So if $m \leq n$ and $h \in FA\ell(m)_+$, then

$$FA\ell(m) \models \psi_{n,n-1}(h) \quad \text{iff} \quad FA\ell(m) \models \psi_{m,m-1}(h)$$

and

$$FA\ell(m) \models \phi_{n,n-1}(h) \quad \text{iff} \quad FA\ell(m) \models \phi_{m,m-1}(h).$$

It follows that $\chi(x, y, z)$ and $\chi(x)$ are likewise independent of n, whence so is the passage from a formula $\xi(\bar{x})$ to $\xi^r(\bar{x}, y)$.

Let $n > j$ and let $\sigma_{n,j}$ be the sentence $(\exists h > 0)\psi_{n,j}(h)$. By Lemmata 4.5, 4.6 and 4.7, $FA\ell(n) \models \sigma_{n,j}$.

So to prove the theorem, it is enough to establish (by induction on $m \geq 2$)

$$(\forall n > m) \ FA\ell(m) \models \neg\sigma_{n,0} \qquad\qquad (Hyp(m)).$$

We first establish $Hyp(2)$.

Let $h \in \mathbb{Z}_+^k$. We write $Z(h)$ for $\{i \in \{1, \ldots, k\} : h(i) = 0\}$. By the definitions of $\psi_{2,1}, \phi_{2,1}$ and χ,

(I) $\mathbb{Z}^k \models \psi_{2,1}(h)$ iff $Z(h) \neq \emptyset$, and

(II) $Z(h) = \emptyset$ if $\mathbb{Z}^k \models \chi(h)$.

Hence

(III) $\mathbb{Z}^2 \models \chi(h)$ iff $Z(h) = \emptyset$, and

(IV) $\mathbb{Z}^k \not\models \chi(h)$ if $k \geq 3$.

Further, by taking $g = (1, 1)$ in the formula $\phi_{2,1}$ and noting that any (x, y) with $x, y > 0$ can be written as $(x, 0) \vee (0, y)$, it follows that

(V) $\mathbb{Z}^2 \not\models \phi_{2,1}(0)$

Note that if $f \in FA\ell(2)_+$ and $FA\ell(2) \models \chi(f)$, then $\mathcal{Z}(f)$ is a finite set of points, say k, with $k \geq 2$. Thus by (I) — (V), we have $FA\ell(2)/\langle f \rangle_{cl} \cong \mathbb{Z}^k$. Therefore

(VI) $FA\ell(2) \models \psi_{2,1}^r(h, f)$ and $\chi(f)$ iff $dim(\mathcal{Z}(h)) \geq 0$.

Moreover if $f, f' \in FA\ell(2)_+$ are such that $FA\ell(2) \models \chi(f)$ and $\chi(f')$, then

(VII) $FA\ell(2) \models \chi^r(f, f')$ iff $\mathcal{Z}(f) \cap \mathcal{Z}(f') = \emptyset$.

Now assume $m = 2$ and $n = 3$. By the way of contradiction, suppose that $FA\ell(2) \models \psi_{3,0}(h)$ for some $h \in FA\ell(2)_+$. By the last clause of the definition of $\psi_{3,0}$ and (VI), we know that $dim(\mathcal{Z}(h)) = -1$. Further, by the previous clause in that definition, there is $f \in FA\ell(2)_+$ with $|\mathcal{Z}(f)| = 2$ such that $FA\ell(2) \models \chi(f)$ and $\psi_{2,0}^r(h, f)$. Hence $\mathbb{Z}^2 \cong FA\ell(2)/\langle f \rangle_{cl} \models \psi_{2,0}(h + \langle f \rangle_{cl})$. By the definition of $\psi_{2,0}$ and (I) we have $h + \langle f \rangle_{cl} = (a, b)$ with $a, b > 0$ (since $\mathbb{Z}^2 \models \neg\psi_{2,1}(h + \langle f \rangle_{cl})$), whence $f_1 = (a, 0)$ and $f_2 = (0, b)$ in the definition. Since $f_1 \wedge f_2 = 0$, it follows that $\mathbb{Z}^2 \models \phi_{2,1}(0)$. This contradicts (V) and establishes the $m = 2$, $n = 3$ case.

To complete the proof of $Hyp(2)$, let $n > 3$. We prove that $FA\ell(2) \models \neg\sigma_{n,j}$ for all j with $-1 \leq j \leq n - 4$. Indeed, we will show that for any $h \in FA\ell(2)_+$,

$FA\ell(2) \models \psi_{n,n-1}(h)$ or $(\exists f > 0)$ $[\chi(f)$ and $(\psi_{n-1,n-2}^r(h, f)$ or $\psi_{n-1,n-3}^r(h, f))]$.

Suppose that $h \in FA\ell(2)_+$.

Case 1. $dim(\mathcal{Z}(h)) = 1$.

By Lemma 4.4, $FA\ell(2) \models \psi_{2,1}(h)$ iff $\mathcal{Z}(h)$ is 1-dimensional. Since $\psi_{n-1,n-2}(x)$ is exactly the same formula as $\psi_{2,1}(x)$, when we take the relativized formula with respect to $\mathcal{Z}(f)$, we get that $FA\ell(2) \models \psi_{n-1,n-2}^r(h, f)$.

Case 2. $dim(\mathcal{Z}(h)) = 0$.

Choose $f \in FA\ell(2)_+$ such that $|\mathcal{Z}(f)| = 2$ (whence $FA\ell(2) \models \chi(f)$). Then $FA\ell(2) \models \psi_{n-1,n-2}^r(h, f)$ by (VI).

Case 3. $dim(\mathcal{Z}(h)) = -1$.

Thus $FA\ell(2) \models \neg\psi_{n,n-1}(h)$ and $\neg[\exists f(\chi(f)$ and $\psi_{n-1,n-2}^r(h, f))]$. Choose $f, f' \in FA\ell(2)_+$ with $|\mathcal{Z}(f)| = 2 = |\mathcal{Z}(f')|$ and $\mathcal{Z}(f') \cap \mathcal{Z}(f) = \emptyset$. Therefore, by (VII),

$$FA\ell(2) \models \chi^r(f', f) \text{ and } \forall g \geq f' \ (\chi^r(g, f) \to \psi_{n-2,n-3}^{r^{(2)}}(h, g, f)).$$

Hence Hyp(2) as

$$FA\ell(2) \models (\exists f > 0) \ [\chi(f) \ \text{and} \ (\psi_{n-1,n-3}^r(h, f))].$$

Now assume $Hyp(k)$ holds but $Hyp(k+1)$ fails. Then for some $h \in FA\ell(k+1)_+$ and $n + 1 > k + 1$, we have $FA\ell(k+1) \models \psi_{n+1,0}(h)$. By definition, we have that

$$FA\ell(k+1) \models \neg\psi_{n+1,n}(h) \ \text{and} \ \exists f \ [\chi(f) \text{ and } \forall f' \geq f \ (\chi(f') \to \psi_{n,0}^r(h, f'))] \ \text{and}$$

$$\neg[\exists f(\chi(f) \text{ and } \bigvee_{0 < i < n} \psi_{n,i}^r(h, f))].$$

Since the formula $\psi_{n+1,n}$ does not depend on n, it is equivalent to $\psi_{k+1,k}$; so $\mathcal{Z}(h)$ has dimension k_0, say, with $k_0 \leq k - 1$. Since any $f \in FA\ell(k+1)_+$ satisfying χ has $dim(\mathcal{Z}(f)) = k - 1$, we can choose $f \in FA\ell(k+1)_+$ so that
(i) $\mathcal{Z}(f) \cap \mathcal{Z}(h) = \emptyset$, and
(ii) f satisfies χ with the additional property that

$$FA\ell(k+1)/ < f >_{cl} \cong FA\ell(k).$$

Hence $FA\ell(k) \models \psi_{n,0}(\bar{h})$, where $\bar{h}$ is the image of $h+ < f >_{cl}$ in $FA\ell(k)_+$. This contradiction to $Hyp(k)$ completes the induction step, and hence the proof of the theorem. $\square$

By the above, one can express by formulae (though of different complexities in each $FA\ell(n)$) the fact that a zero set is empty. We now show that we can interpret in each $FA\ell(n)$ the lattice of zero sets of elements of $FA\ell(n)$ for $n \geq 2$. For $n > 2$, this will imply the desired undecidability results. The idea of the proof is to use a result of A. Grzegorczyk on the undecidability of some topological theories (see Theorems 5 and 6 in [6]). By interpreting Peano arithmetic, he showed that (assuming certain separation axioms) the theory of a class of closed subsets is undecidable.

Define a topology on $\phi(\mathbb{Z}^{n+1}) \cap S^n$ as follows. The closed sets are the images under ϕ of the closed polyhedral cones in $\mathbb{Z}^{n+1}$, the zero sets of the elements of $FA\ell(n+1)$ (by Proposition 2.1). Let $\mathcal{Z}er(S^n)$ denote this set of closed subsets of S^n. We next observe that for $n \geq 2$, the topological space $\mathcal{Z}er(S^n)$ satisfies Grzegorczyk's conditions and furthermore, in any $FA\ell(n+1)$, we can interpret the lattice of these closed subsets of S^n. To prove this we recall the separation and connectedness conditions required on the space:

The topological space must be Hausdorff, connected, and satisfy the axiom of normality (namely, two disjoint closed sets have disjoint neighbourhoods), the second axiom of countability (namely it has a countable basis), any non-empty closed subset has to contain an atom (i.e. a closed subset which is minimal). Further, if A and B are two finite closed subsets, then
(i) if $A \cap B = \emptyset$ and $A \cup B$ is included in a connected open subset E, then there exist two connected open sets $C \supset A$ and $D \supset B$ such that the intersection of their closures is empty and their union included in E; and
(ii) if there exists a bijection between A and B, then there exists a closed set C such

that $A \cup B \subset C$ and every component D of C contains exactly one point of A and one point of B (see paragraphs 2 and 3 in [6]). (Recall that a component is the union of all connected subsets of C containing a given element of C.)

Note that properties (i) and (ii) fail in S^1.

Theorem 4.8. $(\mathscr{Z}er(S^n), \cap, \cup, -, =, \mathbf{0}, \mathbf{1})$ *is interpretable in* $(FA\ell(n+1), +, -, \wedge, \vee, 0)$.

Proof: In the proof of Theorem 4.3 we obtained a formula expressing that the zero set of a positive element f is empty; for simplicity, we will write $\mathscr{Z}(v) = \emptyset$ for the formula expressing this. We also have the definability of the lattice operations in $\mathscr{Z}er(S^n)$. The constant $\mathbf{1}$ is given by $\mathbf{1} = \mathscr{Z}(0)$, the constant $\mathbf{0} = \mathscr{Z}(f)$ for any function f for which $\psi_{n+1,-1}(f)$ holds; i.e., for which $\mathscr{Z}(f) = \emptyset$.

We have already defined the union and intersection of zero sets of positive elements; they correspond to the lattice operations in $FA\ell(n+1)$.

We next define the complement; this is done using the fact that we can express the fact that a zero set is empty.

Assume that f, $g > 0$. The relative complement $\mathscr{Z}(g)$ in $\mathscr{Z}(f)$ will be defined as $\mathscr{Z}(h)$, where $h > 0$ and satisfies the formula $\theta''(h, f, g)$; this latter will be a formula $\theta'(h, f, g)$ with an additional minimality assumption. Let $\theta'(h, f, g)$ be:

$$h \geq f \text{ and } (\forall k \geq f)\mathscr{Z}(k \vee [h \wedge (f \vee g)]) \neq \emptyset,$$

and $\theta''(h, f, g)$ be $\theta'(h, f, g)$ and

$$\neg[(\exists h_1 \geq h)[\theta'(h_1, f, g) \text{ and } (\exists h_2 > 0)(h \geq h_1 \wedge h_2 \text{ and } \mathscr{Z}(h_1 \vee h_2) = \emptyset \neq \mathscr{Z}(h_2)).$$

Finally, observe that the equivalence relation $\sim$ is now definable in $FA\ell(n+1)$ since $\mathscr{Z}(f) = \mathscr{Z}(g)$ iff their symmetric difference is empty. $\square$

Corollary 4.9. $(FA\ell(n), +, -, \wedge, \vee, 0)$ *is undecidable, for* $n \geq 3$. $\square$

Remark: We can now express the decomposition of a zero set into its "component pieces", the (irreducible) varieties for an appropriate choice of element with this zero set. Let $h > 0$ and suppose that for some k we have that $dim(\mathscr{Z}(h)) = k$. We say that this zero set is *k-irreducible* if there do not exist h_1, $h_2 > 0$ such that $h = h_1 \wedge h_2$, $\bigwedge_{i=1}^{2} dim(\mathscr{Z}(h_i)) = k$ and $dim(\mathscr{Z}(h_1 \vee h_2)) < k - 1$.

A zero set is an (irreducible) variety if there do not exist h_1, $h_2 > 0$ such that $h \leq h_1 \wedge h_2$, $\bigvee_{i,j=1}^{n} dim(\mathscr{Z}(h_1)) = i, dim(\mathscr{Z}(h_2)) = j$ and $dim(\mathscr{Z}(h_1 \vee h_2)) < min\{i, j\}$.

5. Free vector lattices.

Let $FV\ell(n)$, $n \in \mathbb{Z}_+$, be the free vector lattice (over $\mathbb{R}$) on n generators. We will consider it as an $\mathcal{L}_{\mathbb{R}}$-structure where $\mathcal{L}_{\mathbb{R}} := \{+, -, 0, \wedge, \vee, r\cdot : r \in \mathbb{R}\}$, and $r\cdot$ is an unary function symbol that will be interpreted by scalar multiplication by $r \in \mathbb{R}$. $FV\ell(n)$ can be represented as the real vector lattice of all continuous piecewise-linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ with pointwise operations and scalar multiplication by elements of $\mathbb{R}$. All the technical results we developed for $FA\ell(n)$ also hold in $FV\ell(n)$. [Usually, they are first proved for free vector lattices and then adapted to $FA\ell(n)$.]

So, applying the same techniques as previously, we get:

Theorem 5.1. *For $n = 1, 2$, the structures $FV\ell(n)$ have decidable theories, and for $n > 2$, the theories are undecidable. Moreover, they are pairwise non-elementarily equivalent.*

Proof: The decidability result for $n = 1, 2$ relies this time on the quantifier elimination and decidability results for the theory of divisible totally ordered abelian groups.   □

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