Functional Monadic Bounded Algebras

Robert Goldblatt

9 June 2010

Abstract

The variety **MBA** of monadic bounded algebras consists of Boolean algebras with a distinguished element E, thought of as an existence predicate, and an operator $\exists$ reflecting the properties of the existential quantifier in free logic. This variety is generated by a certain class **FMBA** of algebras isomorphic to ones whose elements are propositional functions.

We show that **FMBA** is characterised by the disjunction of the equations $\exists E = \mathbf{1}$ and $\exists E = \mathbf{0}$. We also define a weaker notion of "relatively functional" algebra, and show that every member of **MBA** is isomorphic to a relatively functional one.

In [1], an equationally defined class **MBA** of *monadic bounded algebras* was introduced. Each of these algebras comprises a Boolean algebra $\mathbf{B}$ with a distinguished element E, thought of as an existence predicate, and an operator $\exists$ on $\mathbf{B}$ reflecting the properties of the existential quantifier in logic without existence assumptions. **MBA** was shown to be generated by a certain proper subclass **FMBA** of algebras isomorphic to algebras of Boolean-valued functions.

In this paper we characterise **FMBA** as consisting precisely of those monadic bounded algebras in which $\exists E = \mathbf{0}$ or $\exists E = \mathbf{1}$. So **FMBA** is defined by a disjunction of two equations. We also define a weaker notion of "relativised" functional algebra and show that every monadic bounded algebra is isomorphic to one of these more general functional ones. The paper builds on [1], with which the reader is assumed to be familiar.

We review the definition of **FMBA**. Let $\mathbf{B}$ be a Boolean algebra, X a set, and $X_E \subseteq X$. The set $\mathbf{B}^X$ of all functions from X to $\mathbf{B}$ is a Boolean algebra with respect to the pointwise operations. A Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ with a distinguished member E of $\mathbf{A}$ is called a *functional monadic bounded algebra*, with *domain* (X, X_E) and *distinguished function* E, or more briefly a *functional* **MBA**, iff

(F1) $E(x) = \mathbf{1}^{\mathbf{B}}$ for every $x \in X_E$;

(F2) for every $p \in \mathbf{A}$, both $\bigvee\{p(x) \mid x \in X_E\}$ and $\bigvee\{p(x) \wedge E(x) \mid x \in X\}$ exist in $\mathbf{B}$ and are equal; and

(F3) for every $p \in \mathbf{A}$, $\mathbf{A}$ contains the constant function $\exists p$ on X, defined by

$$\exists p(y) = \bigvee\{p(x) \mid x \in X_E\}.$$

FMBA is the class of all algebras that are isomorphic to some functional algebra meeting this definition. **MBA** on the other hand is a class of abstract algebras $\mathbf{A} = (\mathbf{B}, E, \exists)$ satisfying the equational conditions (ax1)–(ax6) stated in [1, §3]. In §6 of [1] there is an example of a monadic bounded algebra that is not isomorphic to any functional one.

Note that if $\mathbf{A}$ is a functional **MBA** as above, and $\mathbf{A}^*$ is any subalgebra of $\mathbf{A}$, hence $\mathbf{A}^*$ contains E and is closed under $\exists$, then (F2) and (F3) remain true for all $p \in \mathbf{A}^*$, so $\mathbf{A}^*$ is also a functional **MBA** with the same domain and distinguished element.

To build functional algebras we need the notion of a *constant* from the theory of monadic algebras [2, p. 63]. This is motivated by the question of how to represent the concept of a particular individual $x_0 \in X$ within the structure of an abstract algebra. Think of the process of applying each predicate (like "is human") to x_0 to form a proposition ("x_0 is human"). Since predicates correspond to propositional functions $p : X \to \mathbf{B}$, this suggests the definition of a function $c : \mathbf{B}^X \to \mathbf{B}^X$ assigning to each $p \in \mathbf{B}^X$ the function cp defined by $cp(x) = p(x_0)$. This c is an endomorphism of the Boolean algebra $\mathbf{B}^X$. If $\mathbf{B}^X$ is a functional **MBA** with respect to E and $\exists$, then $c \circ \exists = \exists$, since $c(\exists p)(x) = \exists p(x_0) = \exists p(x)$ in general. Note also that if $x_0 \in X_E$, then $cE = \mathbf{1}$ in $\mathbf{B}^X$, as $cE(x) = E(x_0) = \mathbf{1}$ in $\mathbf{B}$.

Now let $\mathbf{A} = (\mathbf{B}, E, \exists)$ be an abstract **MBA**. A *constant* of $\mathbf{A}$ is defined to be a Boolean homomorphism $c : \mathbf{B} \to \mathbf{B}$ such that $c \circ \exists = \exists$. Note that the identity function on $\mathbf{B}$ is a constant, a fact we make significant use of.[1] Let $X^{\mathbf{A}}$ be the set of all constants on $\mathbf{A}$, and

$$X^{\mathbf{A}}_E = \{c \in X^{\mathbf{A}} \mid cE = \mathbf{1}^{\mathbf{A}}\}.$$

$\mathbf{A}$ will be called a *rich* algebra if it satisfies:

(R1) if $p \wedge E \neq \mathbf{0}$, then there is a $c \in X^{\mathbf{A}}_E$ with $\exists p = cp$.

(R2) if $p \neq \mathbf{0}$, then there is a $c \in X^{\mathbf{A}}$ with $cp \neq \mathbf{0}$.

If $\mathbf{A}$ is a monadic algebra, then $E = \mathbf{1}$, $X^{\mathbf{A}}_E = X^{\mathbf{A}}$ and $p \leq \exists p$, so (R2) follows from (R1). So our definition of rich is consistent with that of [2].

Theorem 1. *Every rich* **MBA** *is isomorphic to a functional* **MBA***.*

Proof. Let $\mathbf{A}$ be an **MBA** with $X^{\mathbf{A}}$ and $X^{\mathbf{A}}_E$ as above. For each $p \in \mathbf{A}$, define $\widetilde{p} : X^{\mathbf{A}} \to \mathbf{B}$ by putting $\widetilde{p}(c) = cp$. Then define a function f from $\mathbf{B}$ to $\mathbf{B}^{X^{\mathbf{A}}}$ by putting $f(p) = \widetilde{p}$. It is readily checked that f is a Boolean homomorphism,

[1] Constants on a monadic algebra in [2] are also required to satisfy the condition $\exists \circ c = c$, which would exclude the identity as c unless $\exists$ is the identity. We do not need this condition, and it can fail in the example of the constant defined by x_0 above, e.g. if $X_E = \emptyset$. For that example we have only $\exists cp \leq cp$, and $\exists cp \wedge E = cp \wedge E$.

because each constant preserves the Boolean operations in $\mathbf{B}$, and these operations are defined pointwise in the functional algebra $\mathbf{B}^{X^{\mathbf{A}}}$. Also f is injective by the condition (R2), which implies that if $p \neq \mathbf{0}$ in $\mathbf{B}$, then there exists $c \in X^{\mathbf{A}}$ with $\widetilde{p}(c) \neq \mathbf{0}$, hence $f(p) \neq \mathbf{0}$ in $\mathbf{B}^{X^{\mathbf{A}}}$.

Now let $\widetilde{\mathbf{A}}$ be the range of f, a subalgebra of $\mathbf{B}^{X^{\mathbf{A}}}$ that is isomorphic to $\mathbf{B}$, and contains $\widetilde{E}$. We will demonstrate that $\widetilde{\mathbf{A}}$ is a functional **MBA** with domain $(X^{\mathbf{A}}, X^{\mathbf{A}}_E)$ and distinguished function $\widetilde{E}$ that is isomorphic to $\mathbf{A}$.

For condition (F1) in the definition of a functional **MBA**, if $c \in X^{\mathbf{A}}_E$ then since $cE = \mathbf{1}^{\mathbf{B}}$ it is immediate that $\widetilde{E}(c) = \mathbf{1}^{\mathbf{B}}$ as required. For (F2) and (F3) we show that for each p in $\mathbf{A}$,

$$\exists p = \bigvee\{\widetilde{p}(c) \mid c \in X^{\mathbf{A}}_E\} = \bigvee\{\widetilde{p}(c) \wedge \widetilde{E}(c) \mid x \in X^{\mathbf{A}}\}$$

in $\mathbf{B}$, or equivalently that

$$\exists p = \bigvee_{c \in X^{\mathbf{A}}_E} cp = \bigvee_{c \in X^{\mathbf{A}}} c(p \wedge E). \tag{1}$$

This ensures that (F2) holds for each $\widetilde{p} \in \widetilde{\mathbf{A}}$. Since $\widetilde{\exists p}(c) = c(\exists p) = \exists p$ (as $c \circ \exists = \exists$), it also ensures that $\widetilde{\exists p}$ is the function $\exists\widetilde{p}$ on $X^{\mathbf{A}}$ with constant value $\bigvee\{\widetilde{p}(c) \mid c \in X^{\mathbf{A}}_E\}$, and hence that this function belongs to $\widetilde{\mathbf{A}}$, giving (F3). That makes $\widetilde{\mathbf{A}}$ a functional **MBA**. But then $f(\exists p) = \widetilde{\exists p} = \exists\widetilde{p} = \exists f(p)$, and$f(E) = \widetilde{E}$, so f is an **MBA**-homomorphism making $\mathbf{A}$ isomorphic to $\widetilde{\mathbf{A}}$, completing the proof.

It remains to prove (1). We note that

(i) If $c \in X^{\mathbf{A}}$, then $c(p \wedge E) \leq \exists p$; and

(ii) If $c \in X^{\mathbf{A}}_E$, then $cp = c(p \wedge E)$.

(i) holds as $p \wedge E \leq \exists p$ by (ax3), and c is monotonic, so $c(p \wedge E) \leq c\exists p$; but $c\exists p = \exists p$ as c is a constant of $\mathbf{A}$. (ii) holds because $cE = \mathbf{1}$, so $cp = cp \wedge \mathbf{1} = cp \wedge cE = c(p \wedge E)$.

There are two cases for (1). The first is that $p \wedge E = \mathbf{0}$, i.e. $p \leq E'$. Recall from [1] that $\exists$ takes the value $\mathbf{0}$ on the ideal generated by E' in any **MBA**, so $\exists p = \mathbf{0}$ here. Hence for any $c \in X^{\mathbf{A}}$, we get $c(p \wedge E) = \mathbf{0}$ by (i). But then if $c \in X^{\mathbf{A}}_E$, we get $cp = \mathbf{0}$ by (ii). So (1) holds in this case because all elements referred to in (1) are equal to $\mathbf{0}$.

The other case is when $p \wedge E \neq \mathbf{0}$. Then by richness condition (R1) for $\mathbf{A}$, there is some $c^* \in X^{\mathbf{A}}_E$ with $\exists p = c^* p$. Now (ii) implies that

$$\{cp \mid c \in X^{\mathbf{A}}_E\} \subseteq \{c(p \wedge E) \mid c \in X^{\mathbf{A}}\}. \tag{2}$$

(i) states that $\exists p$ is an upper bound of the larger of these two sets. But $\exists p$ is $c^* p$, which belongs to the smaller set. Thus $\exists p$ belongs to both sets and is an upper bound of both, hence is the least upper bound of both, i.e. (1) holds. $\square$

This proof provides the additional information that for any $q \in \widetilde{\mathbf{A}}$,

> if $q \wedge \widetilde{E} \neq \mathbf{0}$, then there is some $c \in X_E^{\mathbf{A}}$ with $q(c) =$ the constant value of $\exists q$.

For if $q = \widetilde{p}$ and $q \wedge \widetilde{E} \neq \mathbf{0}$ in $\widetilde{\mathbf{A}}$, then $p \wedge E \neq \mathbf{0}$ in $\mathbf{A}$, so by (R1) there is some $c \in X_E^{\mathbf{A}}$ with $cp = \exists p$, which says that $\widetilde{p}(c) = \exists \widetilde{p}(c)$.

We turn now to results about the existence of richness. First we show that when $\exists E = \mathbf{1}$, then condition (R1) can be strengthened.

Lemma 2. *Let* $\mathbf{A}$ *be a rich* **MBA** *having* $\exists E = \mathbf{1}$. *Then for every element* p *of* $\mathbf{A}$, *there is some constant* c *of* $\mathbf{A}$ *with* $cE = \mathbf{1}$ *and* $\exists p = cp$.

Proof. If $E = \mathbf{0}$, then $\mathbf{1} = \exists E = \exists \mathbf{0} = \mathbf{0}$. Hence $\mathbf{A}$ is a one-element algebra, and the conclusion of the Lemma holds simply by taking c as the identity function on $\mathbf{A}$.

So we may assume $E \neq \mathbf{0}$. Then putting $p = E$ in (R1), there is some $c \in X_E^{\mathbf{A}}$ with $cE = \exists E = \mathbf{1}$. Hence $c(E') = (cE)' = \mathbf{0}$. Now for any $p \in \mathbf{A}$, if $p \wedge E = \mathbf{0}$, then $p \leq E'$, so $\exists p = \mathbf{0}$ and $cp \leq c(E') = \mathbf{0}$, giving $\exists p = cp$ $(= \mathbf{0})$ to fulfil the Lemma in this case.

But if $p \wedge E \neq \mathbf{0}$, the desired conclusion is directly given by (R1). □

Theorem 3. *If* $\{\mathbf{A}_i \mid i \in I\}$ *is a collection of rich* **MBA***'s that satisfy* $\exists E = \mathbf{1}$, *then the direct product* $\prod_I \mathbf{A}_i$ *is rich.*

Proof. Let $\mathbf{A}_i = (\mathbf{B}_i, E_i, \exists_i)$ with greatest and least elements $\mathbf{1}_i$ and $\mathbf{0}_i$, so $\exists_i E_i = \mathbf{1}_i$. Let $\mathbf{A}$ be $\prod_I \mathbf{A}_i$. For each i, let $\pi_i : \mathbf{A} \to \mathbf{A}_i$ be the projection homomorphism, and write $p_i = \pi_i(p)$ for each $p \in \mathbf{A}$. Then p is the tuple $\langle p_i \mid i \in I \rangle$. In particular, the distinguished element E of $\mathbf{A}$ is $\langle E_i \mid i \in I \rangle$.

To prove $\mathbf{A}$ satisfies (R1), we prove the stronger version from Lemma 2. Let $p \in \mathbf{A}$. Then for each $i \in I$, by the Lemma applied to $\mathbf{A}_i$ there is some constant c_i of $\mathbf{A}_i$ with $c_i E_i = \mathbf{1}_i$ and $\exists_i p_i = c_i p_i$. Let $c : \mathbf{A} \to \mathbf{A}$ be the product of all these c_i's, defined by $(cq)_i = c_i(q_i)$. Then c is a Boolean homomorphism, as each c_i is. Also $(c\exists q)_i = c_i \exists_i q_i = \exists_i q_i$ for all $i \in I$, so $c\exists q = \exists q$ in general. Hence c is a constant of $\mathbf{A}$. Similarly $(cE)_i = c_i E_i = \mathbf{1}_i$ for all i, so $cE = \mathbf{1}$ in $\mathbf{A}$. Finally, $(\exists p)_i = \exists_i p_i = c_i p_i = (cp)_i$ in general, so $\exists p = cp$ as required.

To prove $\mathbf{A}$ satisfies (R2), let $p \neq \mathbf{0}$. Then for some i, $p_i \neq \mathbf{0}_i$, so by (R2) in $\mathbf{A}_i$, there is some constant c_i of $\mathbf{A}_i$ with $c_i p_i \neq \mathbf{0}_i$. For each $j \neq i$, let c_j be the identity constant on $\mathbf{A}_j$. Then let c be the product of $\{c_j \mid j \in I\}$. As above, c is a constant on $\mathbf{A}$. But $(cp)_i = c_i p_i \neq \mathbf{0}_i$, so $cp \neq \mathbf{0}$ as required. □

Theorem 4. *Every basic* **MBA** *is rich.*

Proof. Recall that in a basic **MBA**, the quantifier takes the value $\mathbf{1}$ outside the ideal $\{p \mid p \leq E'\}$. In any **MBA** it takes the value $\mathbf{0}$ on this ideal, as noted earlier.

It follows that if $\mathbf{A}$ is basic, then any Boolean homomorphism $c : \mathbf{A} \to \mathbf{A}$ is a constant of $\mathbf{A}$, since $\exists p \in \{\mathbf{0}, \mathbf{1}\}$ and c fixes $\mathbf{0}$ and $\mathbf{1}$, so $c\exists p = \exists p$ for all p. In particular, if U is an ultrafilter of $\mathbf{A}$, then the characteristic function of U, of the form $\mathbf{A} \to \{\mathbf{0}, \mathbf{1}\} \subseteq \mathbf{A}$, is a constant of $\mathbf{A}$.

To prove (R1) for $\mathbf{A}$, suppose $p \wedge E \neq \mathbf{0}$. Then there is an ultrafilter U of $\mathbf{A}$ with $p \wedge E \in U$. Let c be the characteristic function of U. Then $E \in U$, so $cE = \mathbf{1}$ and hence $c \in X_E^{\mathbf{A}}$. Also $p \in U$, so $cp = \mathbf{1}$. But $\exists p = \mathbf{1}$ as $\mathbf{A}$ is basic, so $cp = \exists p$ as required.

For (R2), if $p \neq \mathbf{0}$, there is an ultrafilter U with $p \in U$. Again let c be the characteristic function of U. Then $c \in X^{\mathbf{A}}$ and $cp = \mathbf{1}$. But $p \neq \mathbf{0}$ implies $\mathbf{1} \neq \mathbf{0}$, so $cp \neq \mathbf{0}$ as required. □

We are now ready to prove our main result.

Theorem 5. **FMBA** *is precisely the class of all monadic bounded algebras in which* $\exists E$ *is* $\mathbf{0}$ *or* $\mathbf{1}$*.*

Proof. Theorem 2.3 of [1] showed that every functional **MBA** has $\exists E \in \{\mathbf{0}, \mathbf{1}\}$, hence so does every algebra isomorphic to a functional **MBA**, i.e. every member of **FMBA**.

For the converse, let $\mathbf{A} = (\mathbf{B}, E^{\mathbf{A}}, \exists^{\mathbf{A}})$ be any **MBA** having $\exists E^{\mathbf{A}} \in \{\mathbf{0}^{\mathbf{A}}, \mathbf{1}^{\mathbf{A}}\}$. If in fact $\exists E^{\mathbf{A}} = \mathbf{0}^{\mathbf{A}}$, then $E^{\mathbf{A}} = \mathbf{0}^{\mathbf{A}}$, and $\exists^{\mathbf{A}} p = \mathbf{0}^{\mathbf{A}}$ for all p. By the Stone representation of $\mathbf{B}$ there is a set X and a Boolean monomorphism $f : \mathbf{B} \to \mathbf{2}^X$ making $\mathbf{B}$ isomorphic to a subalgebra $\widetilde{\mathbf{A}}$ of the functional Boolean algebra $\mathbf{2}^X$. Let $E = fE^{\mathbf{A}} = f\mathbf{0}^{\mathbf{A}} = \mathbf{0}$ in $\widetilde{\mathbf{A}}$. Put $X_E = \emptyset$. Then it is readily checked that $\widetilde{\mathbf{A}}$ is a functional **MBA** with domain (X, X_E) and distinguished function E. The condition (F1) holds vacuously as $X_E = \emptyset$. For each p in $\mathbf{A}$, the sets $\bigvee\{fp(x) \mid x \in X_E\}$ and $\bigvee\{fp(x) \wedge E(x) \mid x \in X\}$ both have join $\mathbf{0}^{\mathbf{B}}$, and the function $\exists fp$ on X defined by $\exists fp(y) = \bigvee\{fp(x) \mid x \in X_E\}$ has constant value $\mathbf{0}^{\mathbf{B}}$, so is equal to $f\mathbf{0}^{\mathbf{A}} \in \widetilde{\mathbf{A}}$. This proves (F2) and (F3) for $\widetilde{\mathbf{A}}$. But also $f\exists^{\mathbf{A}} p = f\mathbf{0}^{\mathbf{A}} = \exists fp$, so f is an **MBA**-homomorphism making $\mathbf{A}$ isomorphic to the functional **MBA** $\widetilde{\mathbf{A}}$.

Alternatively, $\exists E^{\mathbf{A}} = \mathbf{1}^{\mathbf{A}}$. Now by Theorem 5.1 of [1], every **MBA** is isomorphic to a subdirect product of basic **MBA**'s. Hence there is a collection $\{\mathbf{A}_i \mid i \in I\}$ of basic **MBA**'s and an injective homomorphism $f : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$. For each $i \in I$, composing f with the projection from $\prod_{i \in I} \mathbf{A}_i$ shows there is a homomorphism $\mathbf{A} \to \mathbf{A}_i$, implying $\exists_i E_i = \mathbf{1}_i$. Also $\mathbf{A}_i$ is rich by Theorem 4. Hence by Theorem 3, $\prod_{i \in I} \mathbf{A}_i$ is rich, and so is isomorphic to some functional **MBA** $\widetilde{\mathbf{A}}$ by Theorem 1. Let $\widetilde{f}$ be the composition of f with the isomorphism from $\prod_{i \in I} \mathbf{A}_i$ to $\widetilde{\mathbf{A}}$. The range of $\widetilde{f}$ is then a subalgebra of $\widetilde{\mathbf{A}}$, hence a functional **MBA** with the same domain and distinguished element as $\widetilde{\mathbf{A}}$, to which $\mathbf{A}$ is isomorphic under $\widetilde{f}$.

So in both cases we get that $\mathbf{A}$ is isomorphic to a functional **MBA** and hence belongs to **FMBA**. □

The other topic of this paper is the development of a weaker notion of functional algebra, in terms of which every **MBA** can be represented. Given a set X and a Boolean algebra $\mathbf{B}$, then a Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ with a distinguished element E is called a *relatively functional* **MBA** if for each $p \in \mathbf{A}$, the join $\bigvee\{p(x) \wedge E(x) \mid x \in X\}$ exists in $\mathbf{B}$, and $\mathbf{A}$ contains the constant function $\exists p$ on X with this join as value. In this definition, we have abandoned the notion of the set X_E, but have retained enough structure to ensure that $\mathbf{A}$ is an **MBA**.

Note that any subalgebra of a relatively functional **MBA** is a relatively functional **MBA** with the same distinguished element.

One way to obtain algebras of this kind is to apply the notion of *relativised monadic algebra* from Example 3.1 of [1]. A *functional monadic algebra* based on X and $\mathbf{B}$ is a Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ such that for every $p \in \mathbf{A}$, the join $\bigvee\{p(x) \mid x \in X\}$ exists in $\mathbf{B}$, and $\mathbf{A}$ contains the constant function $\exists p$ on X with this join as value. Here we have no E as well as no X_E. Any monadic algebra (i.e. any **MBA** with $E = \mathbf{1}$) is isomorphic to such a functional monadic algebra [2, p. 70]. But if $\mathbf{A}$ is a functional monadic algebra as described, and E is an arbitrary element of $\mathbf{A}$, we can define an operation $\exists^E$ on $\mathbf{A}$ by putting $\exists^E p = \exists(p \wedge E) \in \mathbf{A}$ for all $\in \mathbf{A}$. Then for any $y \in X$ we have

$$\exists^E p(y) = \bigvee\{p(x) \wedge E(x) \mid x \in X\} \text{ in } \mathbf{B}.$$

So this creates from $\mathbf{A}$ a relatively functional **MBA** $\mathbf{A}^E$ with distinguished element E and quantifier $\exists^E$. The notion of relatively functional **MBA** is itself more general than this, as it does not assume the existence of any background functional monadic algebra.

Next we define an abstract **MBA** $\mathbf{A}$ to be *relatively rich* if it satisfies richness condition (R2), and in place of (R1) it has

(R1′) for any p there is a $c \in X^{\mathbf{A}}$ with $\exists p = c(p \wedge E)$.

Lemma 6. *Every rich* **MBA** *is relatively rich.*

Proof. Let $\mathbf{A}$ satisfy (R1) and (R2). To prove (R1′), suppose first that $p \wedge E \neq \mathbf{0}$. Then by (R1) there is a $c \in X_E^{\mathbf{A}}$ with $\exists p = cp$. But $cE = \mathbf{1}$, so $c(p \wedge E) = cp \wedge cE = cp = \exists p$.

But if $p \wedge E = \mathbf{0}$, i.e. $p \leq E'$, then $\exists p = \mathbf{0}$. Let $c \in X^{\mathbf{A}}$ be the identity constant on $\mathbf{A}$. Then $c(p \wedge E) = c\mathbf{0} = \mathbf{0} = \exists p$. □

Theorem 7. *Any direct product of relatively rich* **MBA***'s is relatively rich.*

Proof. Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ with each $\mathbf{A}_i$ relatively rich. We use the notation of the proof of Theorem 3. For each $p \in \mathbf{A}$ and each $i \in I$, by (R1′) in $\mathbf{A}_i$ there is a constant c_i on $\mathbf{A}_i$ with $\exists p_i = c_i(p_i \wedge E_i)$. Let c be the product of these c_i's. Then c is a constant on $\mathbf{A}$, as in Theorem 3, with $\exists p = c(p \wedge E)$.

This shows that $\mathbf{A}$ satisfies (R1′). The proof that it satisfies (R2) is unchanged from Theorem 3. □

Theorem 8. *Every relatively rich* **MBA** *is isomorphic to a relatively functional* **MBA**.

Proof. Let $\mathbf{A} = (\mathbf{B}, E, \exists)$ be relatively rich. We repeat the construction used in Theorem 1. For each $p \in \mathbf{A}$, define $\widetilde{p} : X^{\mathbf{A}} \to \mathbf{B}$ by putting $\widetilde{p}(c) = cp$; and then $f : \mathbf{B} \to \mathbf{B}^{X^{\mathbf{A}}}$ by $f(p) = \widetilde{p}$. Let $\widetilde{\mathbf{A}}$ be the range of f. f is a Boolean homomorphism, and is injective because $\mathbf{A}$ satisfies (R2).

To show that $\widetilde{\mathbf{A}}$ is a relatively functional **MBA** with distinguished function $\widetilde{E}$, it suffices to show that for each p in $\mathbf{A}$,

$$\exists p = \bigvee \{\widetilde{p}(c) \wedge \widetilde{E}(c) \mid x \in X^{\mathbf{A}}\}$$

in $\mathbf{B}$, i.e. that

$$\exists p = \bigvee_{c \in X^{\mathbf{A}}} c(p \wedge E). \tag{3}$$

This ensures that $\widetilde{\exists p}$ is the function $\exists\widetilde{p}$ on $X^{\mathbf{A}}$ with constant value $\bigvee\{\widetilde{p}(c) \wedge \widetilde{E}(c) \mid x \in X^{\mathbf{A}}\}$, and hence that this function belongs to $\widetilde{\mathbf{A}}$. Then $f(\exists p) = \exists f(p)$, and f is an **MBA**-homomorphism making $\mathbf{A}$ isomorphic to the relatively functional algebra $\widetilde{\mathbf{A}}$, completing the proof.

To prove (3), note that for a given p, by (R1$'$) there is some $c^* \in X^{\mathbf{A}}$ with $\exists p = c^*(p \wedge E)$. But for all $c \in X^{\mathbf{A}}$, we have $c(p \wedge E) \leq c\exists p = \exists p$. Thus $\exists p$ is an upper bound of $\{c(p \wedge E) \mid c \in X^{\mathbf{A}}\}$ and also belongs to this set, which implies (3). □

We can now show that these functional algebras encompass all monadic bounded algebras.

Theorem 9. *Every* **MBA** *is isomorphic to a relatively functional* **MBA**.

Proof. If $\mathbf{A}$ is any **MBA** , by [1, Theorem 5.1] there is an injective homomorphism $f : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i$ into a direct product for which every $\mathbf{A}_i$ is basic, hence rich, hence relatively rich (Lemma 6). Then $\prod_{i \in I} \mathbf{A}_i$ is relatively rich (Theorem 7), so is isomorphic to a relatively functional **MBA** $\widetilde{\mathbf{A}}$ (Theorem 8). Let $\widetilde{f}$ be the composition of f with this isomorphism from $\prod_{i \in I} \mathbf{A}_i$ to $\widetilde{\mathbf{A}}$. The range of $\widetilde{f}$ is then a subalgebra of $\widetilde{\mathbf{A}}$, hence a relatively functional **MBA** with the same distinguished element as $\widetilde{\mathbf{A}}$, to which $\mathbf{A}$ is isomorphic under $\widetilde{f}$. □

References

[1] Galym Akishev and Robert Goldblatt. Monadic bounded algebras.

[2] Paul R. Halmos. *Algebraic Logic.* Chelsea, New York, 1962.