Persistence and Atomic Generation for Varieties of Boolean Algebras with Operators

Robert Goldblatt
School of Mathematical and Computing Sciences
Victoria University, P. O. Box 600, Wellington, New Zealand
`Rob.Goldblatt@vuw.ac.nz` `http://www.mcs.vuw.ac.nz/~rob`

20 April 2001

Abstract

A variety V of Boolean algebras with operators is *singleton-persistent* if it contains a complex algebra whenever it contains the subalgebra generated by the singletons. V is *atom-canonical* if it contains the complex algebra of the atom structure of any of the atomic members of V.

This paper explores relationships between these "persistence" properties and questions of whether V is generated by its complex algebras or its atomic members, or is closed under canonical embedding algebras or completions. It also develops a general theory of when operations involving complex algebras lead to the construction of elementary classes of relational structures.

Contents

1 Introduction and Overview

This paper explores relationships between certain *generation* and *persistence* properties of varieties of Boolean algebras with operators (BAO's). The most well known generation property is *completeness*: a variety is complete if it is generated by *complex algebras* $\mathsf{Cm}\,\mathfrak{S}$ of relational structures $\mathfrak{S}$. Here $\mathsf{Cm}\,\mathfrak{S}$ is a BAO based on the Boolean algebra of all subsets of $\mathfrak{S}$. Completeness of a variety of modal algebras corresponds to the notion of a modal logic that is characterised by its Kripke frames. A variety is *elementarily generated* if it is generated by the class $\mathsf{Cm}\,K = \{\mathsf{Cm}\,\mathfrak{S} : \mathfrak{S} \in K\}$ for some elementary (i.e. first-order definable) class K of structures. This corresponds to a modal logic being characterised by some elementary class of frames.

Persistence properties refer to closure of a variety V under passage from a given member $\mathfrak{A}$ to some "larger" algebra $\mathfrak{A}^*$. The most well known persistence property is *canonicity*, in which $\mathfrak{A}^*$ is the canonical embedding algebra (or perfect extension) of $\mathfrak{A}$, a complex algebra based on the Stone space of $\mathfrak{A}$. It is known that every elementarily generated variety is *canonical*, i.e. is closed under canonical embedding algebras. The question of whether, conversely, every canonical variety is elementarily generated remains unanswered.

A partial answer to this question was given by the author in [6, Theorem 5.8] by invoking another persistence property that concerns the *atom structure* $\mathbb{A}\mathsf{t}\,\mathfrak{A}$ of an atomic BAO $\mathfrak{A}$. As the name suggests, $\mathbb{A}\mathsf{t}\,\mathfrak{A}$ is a certain relational structure based on the set of atoms of $\mathfrak{A}$. A variety V is *atom-canonical* if it contains the complex algebra $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ whenever it contains $\mathfrak{A}$. If $\mathbb{A}\mathsf{t}\,V$ is the class of all atom structures of atomic members of V, then atom-canonicity amounts to the requirement that $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,V \subseteq V$. The result of [6] was that if V is atom-canonical then $\mathbb{A}\mathsf{t}\,V$ is an elementary class, and if V is also canonical then $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,V$ generates V.

Further insight into this situation was provided by Venema [14] through consideration of the *singleton algebra* of a structure $\mathfrak{S}$. This algebra, which we denote $\mathsf{Sn}\,\mathfrak{S}$, is the subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ generated by the singleton subsets of $\mathfrak{S}$. $\mathfrak{S}$ is a *weak structure for* V if $\mathsf{Sn}\,\mathfrak{S} \in V$. Venema showed that the class $\mathbb{W}\mathsf{st}\,V = \{\mathfrak{S} : \mathsf{Sn}\,\mathfrak{S} \in V\}$ of all weak structures for V is always elementary. If V is atom-canonical, then $\mathbb{A}\mathsf{t}\,V$ is equal to $\mathbb{W}\mathsf{st}\,V$, and moreover both are equal to the class $\mathbb{S}\mathsf{tr}\,V = \{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in V\}$ of all *structures for* V.

A number of problems were listed in [14] concerning the relationship between canonicity and atom-canonicity, and the question of whether every variety is generated by its atomic members. In this paper we address some of these issues and relate them to what will be called *singleton-persistence* of V. This means that $\mathbb{W}\mathsf{st}\,V = \mathbb{S}\mathsf{tr}\,V$: every weak structure for V is a structure for V, i.e. $\mathsf{Cm}\,\mathfrak{S}$ belongs to V whenever $\mathsf{Sn}\,\mathfrak{S}$ does. Every atom-canonical variety is singleton-persistent.

The main contributions of the paper are as follows.

- Let Φ be an operation assigning to each structure $\mathfrak{S}$ a subalgebra $\Phi\mathfrak{S}$ of $\mathsf{Cm}\,\mathfrak{S}$. Conditions on Φ are given which guarantee that $\{\mathfrak{S} : \Phi\mathfrak{S} \in V\}$ is an elementary class whenever V is a variety, or more generally a universal class, of BAO's. These conditions apply when $\Phi = \mathsf{Sn}$, and lead to a structural proof that $\mathbb{W}\mathsf{st}\,V$ is elementary (see 3.2, 3.3).
- Any canonical singleton-persistent variety is elementarily generated (3.4).
- Examples are constructed of a variety that is singleton-persistent but not

atom-canonical (4.1), and one that is atom-canonical but not complete and therefore not canonical (4.2). On the other hand is it shown that if an atom-canonical variety is generated by its finite members, then it is canonical.

- If a variety of BAO's with completely additive operators is closed under *completions* in the sense of [12], then it is atom-canonical, and is canonical if and only if it is complete (5.1).
- In the completely additive setting, a singleton-persistent variety is generated by its atomic members if and only if it is complete, and this holds if and only if it is canonical (5.2).

2 Background

Familiarity will be assumed with the general theory of varieties of BAO's [4, 6, 7, 10], but the main concepts needed here will now be reviewed.

An *operator* on a Boolean algebra $\mathfrak{B}$ is a finitary function on $\mathfrak{B}$ which is *additive*, meaning that it preserves the join operation of $\mathfrak{B}$ in each of its arguments. Such a function is always *monotonic*, i.e. preserves the order relation $\leq$, in each argument. An operator is *normal* if it preserves the least element of $\mathfrak{B}$ in each argument. A BAO (Boolean algebra with operators) is an algebra of the form $\mathfrak{A} = (\mathfrak{B}, \{f_i : i \in I\})$ where each f_i is an operator on Boolean algebra $\mathfrak{B}$.

Extensive use will be made of notation for operations on classes of algebras and relational structures that produce new classes closed under isomorphism. If V is a class of algebras then $\mathsf{S}V$ denotes the closure of V under isomorphic copies of subalgebras of its members. We write $\mathfrak{A} \rightarrowtail \mathfrak{A}'$ to mean that there exists an injective homomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$, so we can say that

$$\mathsf{S}V = \{\mathfrak{A} : \mathfrak{A} \rightarrowtail \mathfrak{A}' \text{ for some } \mathfrak{A}' \in V\}.$$

Similarly $\mathsf{H}V$, $\mathsf{P}V$, and $\mathsf{Pu}V$ denote the closure of V under isomorphic copies of homomorphic images, direct products, and ultraproducts of its members, respectively. A famous theorem of Birkhoff states that V is a *variety*, i.e. is the class of all models of some set of equations, if and only if V is closed under subalgebras, homomorphic images and direct products: $V = \mathsf{S}V = \mathsf{H}V = \mathsf{P}V$. The variety *generated by a class* V is the smallest variety including V (and is equal to $\mathsf{HSP}V$). An algebra $\mathfrak{A}$ belongs to the variety generated by V if and only if every equation satisfied by all members of V is also satisfied by $\mathfrak{A}$.

A class V is *universal*, i.e. is the class of all models of some set of universal first-order sentences, if and only if it is closed under isomorphism, subalgebras and ultraproducts: $V = \mathsf{S}V = \mathsf{Pu}V$ (see [2, Theorem V.2.20]).

We will assume throughout this paper that

- all BAO's discussed are normal, i.e. have only normal operators; and

- any class V of BAO's under consideration is closed under isomorphic images of subalgebras, i.e. $\mathsf{S}V = V$, and thus is closed under isomorphism.

If K is a class of relational structures, then $\mathbb{P}\mathsf{u}K$ and $\mathbb{P}\mathsf{w}K$ denote the closure of K under isomorphic copies of ultraproducts and ultrapowers of its members, respectively. K is an *elementary class*, i.e. is the class of all models of some set of first-order sentences, if and only if it is closed under isomorphism, ultraproducts and *ultraroots*. The latter means that if an ultrapower $\mathfrak{S}^J/U$ belongs to K then so does the structure $\mathfrak{S}$.

Associated with any relational structure $\mathfrak{S} = (X, \{R_i : i \in I\})$ is its *complex algebra*

$$\mathsf{Cm}\,\mathfrak{S} = (Sb(X), \{f_{R_i} : i \in I\}).$$

This is the normal BAO in which $Sb(X)$ is the Boolean powerset algebra of X and, if R_i is an $n+1$-ary relation on X, then f_{R_i} is the normal n-ary operator on $Sb(X)$ defined by

$$f_{R_i}(Y_1, \dots, Y_n) = \{x : R_i(x, y_1, \dots, y_n) \text{ for some } y_i \in Y_i\}.$$

Associated with each class K of structures is its class

$$\mathsf{Cm}\,K = \{\mathfrak{A} : \mathfrak{A} \cong \mathsf{Cm}\,\mathfrak{S} \text{ for some } \mathfrak{S} \in K\}$$

of complex algebras and their isomorphic images. Associated with each class V of BAO's is the class

$$\mathbb{S}\mathsf{tr}\,V = \{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in V\}$$

of *structures for* V. Since V is closed under isomorphism, $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V \subseteq V$. Also $\mathbb{S}\mathsf{tr}\,V$ is closed under isomomorphism, because V is so closed and $\mathfrak{S}_1 \cong \mathfrak{S}_2$ implies $\mathsf{Cm}\,\mathfrak{S}_1 \cong \mathsf{Cm}\,\mathfrak{S}_2$. A variety V is called *complete* if it is generated by the class $\{\mathsf{Cm}\,\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in V\}$ of complex algebras in V, or equivalently is generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$.

Associated with any BAO $\mathfrak{A}$ is a certain relational structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$, the *canonical structure of* $\mathfrak{A}$, based on the set of ultrafilters of $\mathfrak{A}$ [4, 6]. The *canonical embedding algebra*, or *perfect extension*, of $\mathfrak{A}$ is the complex algebra $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$. This construction was introduced in [11] where it was shown that $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$. A class V is called *canonical* if it contains the canonical embedding algebras of all its members. If this holds then $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\,V$ whenever $\mathfrak{A} \in V$, and so $V \subseteq \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$. Consequently, any canonical variety is generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$ and so is complete. The converse is false: algebraic semantics for modal logics provide a number of significant examples of complete varieties that are not canonical, such as the variety of diagonalisable algebras [4, Section 3.7], the algebras validating the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$ [5], and the algebraic models of the tense logic of real time [7, Section 5.6].

A fundamental result in this context is that if a class K of structures is closed under ultraproducts, then the variety generated by $\mathsf{Cm}\,K$ is canonical [4, Theorem 3.6.7]. Of particular relevance here is the case that $K = \mathbb{S}\mathsf{tr}\,V$ for some variety V. Then for $\mathbb{S}\mathsf{tr}\,V$ to be closed under ultraproducts it suffices for it to be closed under ultrapowers. Indeed the condition $\mathbb{P}\mathsf{w}\mathbb{S}\mathsf{tr}\,V = \mathbb{S}\mathsf{tr}\,V$ is sufficient to ensure that $\mathbb{S}\mathsf{tr}\,V$ is an elementary class [4, Theorem 3.8.4]. In that case the variety generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$ is canonical. This generated variety is equal to V precisely when V is complete, so altogether it follows that

> *if* $\mathbb{S}\mathsf{tr}\,V$ *is elementary, then the variety* V *is canonical if and only if it is complete.*

We will use this observation several times in what follows.

3 Elementary Classes from Complex Operations

For any structure $\mathfrak{S}$, define $\mathsf{Sn}\,\mathfrak{S}$, the *singleton algebra* of $\mathfrak{S}$, to be the subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ generated by the set $\{\{s\} : s \in X\}$ of singleton subsets of the underlying set X of $\mathfrak{S}$. Let

$$\mathbb{W}\mathsf{st}\,V = \{\mathfrak{S} : \mathsf{Sn}\,\mathfrak{S} \in V\}.$$

$\mathbb{W}\mathsf{st}\,V$ is called the class of all *weak structures* for V [14], and is closed under isomorphism because $\mathfrak{S}_1 \cong \mathfrak{S}_2$ implies $\mathsf{Sn}\,\mathfrak{S}_1 \cong \mathsf{Sn}\,\mathfrak{S}_2$. Since V is closed under subalgebras, $\mathsf{Cm}\,\mathfrak{S} \in V$ always implies $\mathsf{Sn}\,\mathfrak{S} \in V$, so $\mathbb{S}\mathsf{tr}\,V \subseteq \mathbb{W}\mathsf{st}\,V$. We say that V is *singleton-persistent*, or more briefly Sn*-persistent*, if $\mathsf{Sn}\,\mathfrak{S} \in V$ implies $\mathsf{Cm}\,\mathfrak{S} \in V$, i.e. any weak structure for V is a structure for V. Thus V is Sn-persistent iff $\mathbb{W}\mathsf{st}\,V = \mathbb{S}\mathsf{tr}\,V$.[1]

There are many Sn-persistent varieties, as indicated early in Section 4 below. On the other hand some significant varieties are not Sn-persistent, including the variety **RRA** of representable relation algebras and the variety $\mathbf{RCA}_n$ of representable cylindric algebras of dimension n for all finite $n \geq 3$ [9]. Another example, which is explained immediately after Theorem 4.1 below, is the variety of algebras for the modal logic K4M.

It was shown in [14] that $\mathbb{W}\mathsf{st}\,V$ is an elementary class of structures whenever V is a variety. The proof was model-theoretic, and it was noted that it can be adapted to the case of V being a universal class. A structural proof of this fact will now be given in a generalised setting that applies to other class operations than Sn.

By a *complex operation* we will mean any operation Φ which assigns to each structure $\mathfrak{S}$ a subalgebra $\Phi\mathfrak{S}$ of $\mathsf{Cm}\,\mathfrak{S}$ and which preserves isomorphisms. Thus $\Phi\mathfrak{S} \rightarrowtail \mathsf{Cm}\,\mathfrak{S}$, and if $\mathfrak{S}_1 \cong \mathfrak{S}_2$ then $\Phi\mathfrak{S}_1 \cong \Phi\mathfrak{S}_2$. Let

$$\Phi^* V = \{\mathfrak{S} : \Phi\mathfrak{S} \in V\}.$$

Then since $\mathsf{S}\,V = V$, we always have $\mathbb{S}\mathsf{tr}\,V \subseteq \Phi^* V$. Also $\Phi^* V$ is always closed under isomorphism because Φ preserves isomorphism and V is closed under isomorphism.

[1] Singleton persistence corresponds in modal logic to the notion of a formula being "di-persistent"[1, Section 5.6], meaning that it is valid in a Kripke frame $\mathfrak{F}$ whenever it is valid in some general frame $(\mathfrak{F}, P)$ that is *discrete*, i.e. P contains all singleton subsets of $\mathfrak{F}$.

A class V will be called Φ-*persistent* if $\Phi^*V \subseteq \mathbb{S}\mathsf{tr}\,V$, i.e. $\Phi\mathfrak{S} \in V$ implies $\mathsf{Cm}\,\mathfrak{S} \in V$, and hence $\Phi^*V = \mathbb{S}\mathsf{tr}\,V$. Note that $\mathbb{W}\mathsf{st}\,V = \mathsf{Sn}^*V$.

A complex operation Φ will be said to *subcommute with ultraproducts* if for any collection $\{\mathfrak{S}_j : j \in J\}$ of structures and any ultrafilter U on J there is an injective BAO-homomorphism

$$\Phi(\textstyle\prod_J \mathfrak{S}_j/U) \rightarrowtail \textstyle\prod_J \Phi\mathfrak{S}_j/U.$$

We symbolise this property by $\Phi\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\Phi$. Similarly Φ *preserves ultrapower embeddings*, symbolised $\Phi \rightarrowtail \Phi\mathbb{P}\mathsf{w}$, if for any structure $\mathfrak{S}$ there is an injective homomorphism

$$\Phi\mathfrak{S} \rightarrowtail \Phi(\mathfrak{S}^J/U).$$

If both $\Phi\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\Phi$ and $\Phi \rightarrowtail \Phi\mathbb{P}\mathsf{w}$, then Φ will be called an *elementary* operation. The justification for this name is that an elementary operation creates elementary classes, as will now be shown.

Theorem 3.1

(1) *If Φ subcommutes with ultraproducts and V is closed under ultraproducts, then Φ^*V is closed under ultraproducts.*

(2) *If Φ preserves ultrapower embeddings, then Φ^*V is closed under ultraroots.*

Proof.

(1) Let $\{\mathfrak{S}_j : j \in J\} \subseteq \Phi^*V$. Then each $\Phi\mathfrak{S}_j$ is in V, so if $\mathsf{Pu}\,V \subseteq V$, $\prod_J \Phi\mathfrak{S}_j/U \in V$. Thus if $\Phi\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\Phi$, since $\mathsf{S}\,V \subseteq V$ we get $\Phi(\prod_J \mathfrak{S}_j/U) \in V$, and hence $\prod_J \mathfrak{S}_j/U \in \Phi^*V$.

(2) Let $\mathfrak{S}^J/U \in \Phi^*V$, i.e. $\Phi(\mathfrak{S}^J/U) \in V$. From $\Phi \rightarrowtail \Phi\mathbb{P}\mathsf{w}$ and $\mathsf{S}\,V \subseteq V$ we immediately get $\Phi\mathfrak{S} \in V$, and so $\mathfrak{S} \in \Phi^*V$ as desired.

□

Corollary 3.2 *Let Φ be an elementary complex operation. Then for any universal class V of BAO's, Φ^*V is an elementary class of structures. Consequently, if V is a Φ-persistent universal class, $\mathbb{S}\mathsf{tr}\,V$ is elementary.*

Proof. If V is universal, then $\mathsf{Pu}\,V = \mathsf{S}\,V = V$, so if $\Phi\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\Phi$ and $\Phi \rightarrowtail \Phi\mathbb{P}\mathsf{w}$, Theorem 3.1 yields that Φ^*V is closed under ultrapowers and ultraroots. But Φ^*V is closed under isomorphism, so altogether it is an elementary class. The last statement of the Theorem follows because Φ-persistence means that $\mathbb{S}\mathsf{tr}\,V = \Phi^*V$. □

Theorem 3.3 Sn *is an elementary complex operation. Hence for any universal class V, $\mathbb{W}\mathsf{st}\,V$ is elementary.*

Proof. For any collection $\{\mathfrak{S}_j : j \in J\}$ of structures there is an injective homomorphism

$$\theta : \prod_J \mathsf{Cm}\,\mathfrak{S}_j/U \rightarrowtail \mathsf{Cm}\,(\prod_J \mathfrak{S}_j/U)$$

given by the relationship

$$g/U \in \theta(G/U) \text{ iff } \{j \in J : g(j) \in G(j)\} \in U$$

for all $g \in \prod_J \mathfrak{S}_j$ and $G \in \prod_J \mathsf{Cm}\,\mathfrak{S}_j$ (see [4, Lemma 3.6.5]). Also, the inclusions $\mathsf{Sn}\,\mathfrak{S}_j \rightarrowtail \mathsf{Cm}\,\mathfrak{S}_j$ give rise to an injective homomorphism

$$\rho : \prod_J \mathsf{Sn}\,\mathfrak{S}_j/U \rightarrowtail \prod_J \mathsf{Cm}\,\mathfrak{S}_j/U.$$

Now each singleton $\{g/U\}$ in $\mathsf{Cm}\,(\prod_J \mathfrak{S}_j/U)$ is the image under $\theta \circ \rho$ of the element $\langle\{g(j)\} : j \in J\rangle/U$ of $\prod_J \mathsf{Sn}\,\mathfrak{S}_j/U$. Hence the image of $\theta \circ \rho$ is a subalgebra of $\mathsf{Cm}\,(\prod_J \mathfrak{S}_j/U)$ that contains all the singletons, and therefore includes the algebra $\mathsf{Sn}\,(\prod_J \mathfrak{S}_j/U)$. Consequently the inverse of $\theta \circ \rho$ is an injective homomorphism mapping $\mathsf{Sn}\,(\prod_J \mathfrak{S}_j/U)$ into $\prod_J \mathsf{Sn}\,\mathfrak{S}_j/U$, establishing that $\mathsf{Sn}\,\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\mathsf{Sn}$.

Next, to show that $\mathsf{Sn} \rightarrowtail \mathsf{Sn}\,\mathbb{P}\mathsf{w}$, observe that the injection θ above specializes, for a single structure $\mathfrak{S}$, to an injection $(\mathsf{Cm}\,\mathfrak{S})^J/U \rightarrowtail \mathsf{Cm}\,(\mathfrak{S}^J/U)$. Composing this with the standard elementary embedding $\mathsf{Cm}\,\mathfrak{S} \rightarrowtail (\mathsf{Cm}\,\mathfrak{S})^J/U$ gives an injective homomorphism $\tau : \mathsf{Cm}\,\mathfrak{S} \rightarrowtail \mathsf{Cm}\,(\mathfrak{S}^J/U)$. Now τ maps a singleton $\{s\}$ to the singleton $\{g_s/U\}$, where $g_s(j) = s$ for all $j \in J$. Hence τ maps all the singletons from $\mathsf{Cm}\,\mathfrak{S}$ into $\mathsf{Sn}\,(\mathfrak{S}^J/U)$. Since $\mathsf{Sn}\,\mathfrak{S}$ is generated by these singletons, it follows that τ embeds $\mathsf{Sn}\,\mathfrak{S}$ into $\mathsf{Sn}\,(\mathfrak{S}^J/U)$, giving the desired result.

Thus if V is universal, Corollary 3.2 entails that Sn^*V is an elementary class. But $\mathsf{Sn}^*V = \mathbb{W}\mathsf{st}\,V$. □

Corollary 3.4 *A canonical singleton-persistent variety is elementarily generated.*

Proof. Elementary generation of a variety V means that V is generated by $\mathsf{Cm}\,K$ for some elementary class K. Here we can take $K = \mathbb{S}\mathsf{tr}\,V$: if V is singleton-persistent then $\mathbb{S}\mathsf{tr}\,V$ is elementary by Theorem 3.3, and if V is canonical then any $\mathfrak{A} \in V$ has $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A} \in V$, so $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\,V$. Hence $V = \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$ and $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$ generates V. □

For other examples of elementary complex operations, let $\Phi = \mathsf{Fo}$ where $\mathsf{Fo}\mathfrak{S}$ is the class of all subsets of $\mathfrak{S}$ that are definable by some first-order formula with parameters in $\mathfrak{S}$. Then $\mathsf{Fo}\mathfrak{S}$ is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ that includes $\mathsf{Sn}\,\mathfrak{S}$, since the singleton $\{s\}$ is definable by the formula $(v = s)$. With the help of Łoś's Theorem and reasoning as in the proof of 3.3, it is readily seen that $\mathsf{Fo}\mathbb{P}\mathsf{u} \rightarrowtail \mathsf{Pu}\,\mathsf{Fo}$ and $\mathsf{Fo} \rightarrowtail \mathsf{Fo}\mathbb{P}\mathsf{w}$. The same conclusion holds for $\Phi = \mathsf{Fo}_n$, where $\mathsf{Fo}_n\mathfrak{S}$ is the algebra of subsets definable by first-order formulas with at most n free variables. Thus Fo^*V and Fo_n^*V are elementary classes whenever V is universal.

The paper [8] gives a remarkable proof that $\mathbb{S}\mathsf{tr}\,\mathbf{RRA}$ is not an elementary class. Hence

$$\mathbb{S}\mathsf{tr}\,\mathbf{RRA} \subsetneqq \mathsf{Fo}^*\mathbf{RRA},$$

and $\mathbf{RRA}$ is not Fo-persistent: there exists a structure whose algebra of first-order definable sets is a representable relational algebra but whose complex algebra is not.

4 Singleton-Persistence and Atom-Canonicity

If a BAO $\mathfrak{A} = (\mathfrak{B}, \{f_i : i \in I\})$ is *atomic*, i.e. $\mathfrak{B}$ is an atomic Boolean algebra, then the *atom structure* of $\mathfrak{A}$ is the relational structure

$$\mathbb{A}\mathsf{t}\,\mathfrak{A} = (\mathbb{A}\mathsf{t}\,\mathfrak{B}, \{R_{f_i} : i \in I\}),$$

where $\mathbb{A}\mathsf{t}\,\mathfrak{B}$ is the set of atoms of $\mathfrak{B}$, and

$$R_{f_i}(a, b_1, \dots, b_n) \text{ if and only if } a \leq f_i(b_1, \dots, b_n).$$

Define

$$\mathbb{A}\mathsf{t}\,V = \{\mathfrak{S} : \mathfrak{S} \cong \mathbb{A}\mathsf{t}\,\mathfrak{A} \text{ for some atomic } \mathfrak{A} \in V\}.$$

Now the atoms of a complex algebra $\mathsf{Cm}\,\mathfrak{S}$ are just the singleton subsets $\{s\}$ of $\mathfrak{S}$. $\mathsf{Cm}\,\mathfrak{S}$ is always atomic with $\mathbb{A}\mathsf{t}\,\mathsf{Cm}\,\mathfrak{S} \cong \mathfrak{S}$, the isomorphism being given by the correspondence $\{s\} \leftrightarrow s$. Since $\mathsf{Sn}\,\mathfrak{S}$ contains all the singletons, it is itself atomic, with $\mathbb{A}\mathsf{t}\,\mathsf{Sn}\,\mathfrak{S} = \mathbb{A}\mathsf{t}\,\mathsf{Cm}\,\mathfrak{S} \cong \mathfrak{S}$. Hence $\mathsf{Sn}\,\mathfrak{S} \in V$ implies $\mathfrak{S} \in \mathbb{A}\mathsf{t}\,V$. Therefore we have

$$\mathbb{S}\mathsf{tr}\,V \subseteq \mathbb{W}\mathsf{st}\,V \subseteq \mathbb{A}\mathsf{t}\,V.$$

V is called *atom-canonical* [13] if $\mathbb{A}\mathsf{t}\,V \subseteq \mathbb{S}\mathsf{tr}\,V$, or equivalently $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,V \subseteq V$. Obviously atom-canonicity implies Sn-persistence (i.e. $\mathbb{W}\mathsf{st}\,V = \mathbb{S}\mathsf{tr}\,V$). Any variety of conjugated BAO's defined by Sahlqvist equations is atom-canonical [13, Theorem 2] and therefore Sn-persistent.

It was shown in [6, Theorem 5.8] that if V is an atom-canonical variety then $\mathbb{A}\mathsf{t}\,V = \mathbb{S}\mathsf{tr}\,V$ is elementary. Venema's introduction of the class $\mathbb{W}\mathsf{st}\,V$ gave a deeper understanding of this situation: $\mathbb{W}\mathsf{st}\,V$ is always elementary, and is equal to $\mathbb{S}\mathsf{tr}\,V$ under atom-canonicity. But it is enough to to know that V is Sn-persistent to conclude that $\mathbb{S}\mathsf{tr}\,V$ is elementary, and hence that V is canonical if and only if it is complete. Moreover this can happen in the absence of atom-canonicity:

Theorem 4.1 *There exists a variety that is singleton-persistent but not atom canonical.*

Proof. Let η be the equation $f_1 x \leq f_2 f_1 x$, where f_1 and f_2 are unary operator symbols, and let V_η be the variety of all algebras satisfying η. Then results of [13] and [3] show that V_η fulfills the claim of the Theorem.

In [13, Theorem 3] an atomic normal BAO $\mathfrak{A}$ is constructed such that $\mathfrak{A} \models \eta$ and $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A} \not\models \eta$. Thus $\mathbb{A}\mathsf{t}\,\mathfrak{A} \in \mathbb{A}\mathsf{t}\,V_\eta$ but $\mathbb{A}\mathsf{t}\,\mathfrak{A} \notin \mathbb{S}\mathsf{tr}\,V_\eta$, so V_η is not atom-canonical.

On the other hand V_η is Sn-persistent. This follows from a stronger property of closure of V_η under "completions" analysed in [3], as will be explained below (see Theorem 5.1(1) and the discussion following that Theorem). But it is instructive to see a direct proof of the Sn-persistence of V_η that displays the role of the singletons.

If a structure $\mathfrak{S} = (X, R_1, R_2)$ has $\mathsf{Sn}\,\mathfrak{S} \in V_\eta$, then for each $b \in X$, since $\{b\} \in \mathsf{Sn}\,\mathfrak{S}$ we have $f_{R_1}(\{b\}) \subseteq f_{R_2}(f_{R_1}(\{b\}))$. This ensures that $\mathfrak{S}$ satisfies the first-order condition

$$aR_1b \text{ implies } \exists c(aR_2c \text{ and } cR_1b), \tag{i}$$

which is enough to yield $f_{R_1}(Y) \subseteq f_{R_2}(f_{R_1}(Y))$ for all $Y \subseteq X$. Hence $\mathsf{Cm}\,\mathfrak{S} \in V_\eta$ as desired.

Note that this argument shows directly that $\mathbb{W}\mathsf{st}\,V = \mathbb{S}\mathsf{tr}\,V =$ the elementary class defined by (i). □

There are varieties that are canonical but not atom-canonical. An example was described in [6, p. 592]: it is the variety V_{4M} of algebras for the modal logic K4M, the smallest normal logic having the transitivity axiom $\Box\alpha \to \Box\Box\alpha$ and the McKinsey axiom $\Box\Diamond\alpha \to \Diamond\Box\alpha$. In fact this variety is not even Sn-persistent, since it contains $\mathsf{Sn}\,(\omega, <)$ but not $\mathsf{Cm}\,(\omega, <)$. But it is elementarily generated: $\mathbb{S}\mathsf{tr}\,V_{4M}$ is an elementary class and V_{4M} is complete, hence canonical and generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V_{4M}$.

In the converse direction, the question was raised in [14, p. 304] as to whether every atom-canonical variety must be canonical. We will now give a counter-example. The point, as with Sn-persistence above, is that if a variety V is atom-canonical, then since its class $\mathbb{S}\mathsf{tr}\,V$ is elementary, V will be canonical iff complete. So our proposed counter-example must be atom-canonical but incomplete. Let V_0 be the variety of all BAO's with two normal operators, f and f', that satisfy

$$(\eta_0): \quad fx \leq f^d f'(x \cdot -f'x),$$

where f^d is the dual of f, i.e. $f^d y = -f-y$.

Theorem 4.2 *The variety V_0 is atom-canonical but incomplete, and therefore not canonical.*

Proof. If $\mathfrak{A}$ is any atomic member of V_0, then the atom structure $\mathbb{A}\mathsf{t}\,\mathfrak{A}$ has the form (X, R, R'), where X is the set of atoms of $\mathfrak{A}$, xRy iff $x \leq fy$, and $xR'y$ iff $x \leq f'y$. But for such a structure we will show that $R = \emptyset$, so that in the complex algebra $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ the operator f_R satisfies the equation $fx = 0$. This implies that $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A} \models \eta_0$, making $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ a member of V_0 and establishing that V_0 is atom-canonical.

To see that $R = \emptyset$, suppose on the contrary that there exist $x, y \in X$ with xRy, i.e. $x \leq fy$. Then we must also have $y \leq f'(y \cdot -f'y)$. For if not, then $y \leq -f'(y \cdot -f'y)$ as y is an atom, so as f is monotonic

$$x \leq fy \leq f{-}f'(y \cdot -f'y) = -f^d f'(y \cdot -f'y),$$

so $x \not\leq f^d f'(y \cdot -f'y)$ as $x \neq 0$. But since $\mathfrak{A} \models \eta_0$ we have

$$x \leq fy \leq f^d f'(y \cdot -f'y),$$

so this is a contradiction. Thus we must indeed have $y \leq f'(y \cdot -f'y)$, and so $y \leq f'y$. But now $f'(y \cdot -f'y) \neq 0$, as $y \neq 0$, so $y \cdot -f'y \neq 0$ as f' is normal. Since y is an atom, this implies $y \cdot -f'y = y$ and therefore $y \leq -f'y$. Since we already concluded $y \leq f'y$, this is an outright contradiction with $y \neq 0$. Thus the assumption $R \neq \emptyset$ is false, and the proof that V_0 is atom-canonical is finished.

Now we show that V_0 is incomplete, and therefore cannot be canonical. If $\mathfrak{S} \in \mathbb{S}\mathsf{tr}V_0$, then $\mathsf{Cm}\,\mathfrak{S} \in V_0$ and so putting $\mathfrak{A} = \mathsf{Cm}\,\mathfrak{S}$ in the proof just given shows that $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathsf{Cm}\,\mathfrak{S} \cong \mathsf{Cm}\,\mathfrak{S}$ satisfies $fx = 0$. Thus every member of $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}V_0$ satisfies $fx = 0$. If V_0 were complete it would be generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}V_0$, so every member of V_0 would satisfy $fx = 0$. But this is not so: consider the structure $\mathfrak{S}_0 = (X, R, R')$ with

$$\begin{aligned} X &= \{0, \dots, \omega + 1\}, \\ R &= \{(\omega + 1, \omega)\}, \\ R' &= \{(p, q) : p, q \leq \omega \text{ and } p > q\}. \end{aligned}$$

Let $\mathfrak{A}_0$ be the collection of all finite subsets of X that exclude ω together with the complements of such subsets. $\mathfrak{A}_0$ is closed under the operators f_R and $f_{R'}$, as well as the Boolean set operations, so is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}_0$. In that algebra we have

$$f_R(Y) = \begin{cases} \{\omega + 1\} & \text{if } \omega \in Y, \\ \emptyset & \text{if } \omega \notin Y. \end{cases}$$

In particular $\omega + 1 \in f_R(X)$ so $fx = 0$ fails in $\mathfrak{A}_0$. On the other hand $\mathfrak{A}_0$ is in V_0. For if $\omega \in Y \in \mathfrak{A}_0$, then Y must be cofinite and have a least element $p \in \omega$. Then $\omega R' p \in (Y \cap -f_{R'}Y)$, yielding $\omega \in f_{R'}(Y \cap -f_{R'}Y)$. This implies that $\omega + 1 \in {f_R}^d f_{R'}(Y \cap -f_{R'}Y)$, and so $f_R(Y) \subseteq f^d_R f_{R'}(Y \cap -f_{R'}Y)$. But this last inclusion also holds if $\omega \notin Y$, since then $f_R(Y) = \emptyset$, so $\mathfrak{A}_0$ satisfies η_0. □

Another question raised in [14] is whether every atom-canonical variety generated by its finite members must be canonical. This time the answer is positive, and requires only the hypothesis of Sn-persistence in place of atom-canonicity. This is because a finite algebra $\mathfrak{A}$ is isomorphic to the full complex algebra $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ (see below), so if V is generated by its finite members then it is generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}V$, i.e. is complete. But Sn-persistence of V entails that $\mathbb{S}\mathsf{tr}V$ is elementary, which together with completeness implies canonicity of V. Hence

> *a variety that is singleton-persistent and generated by its finite members is canonical.*

5 Complete Additivity and Atomic Generation

An operator on a Boolean algebra is *completely additive* if, in each argument, it preserves all existing joins of sets of elements. We say that a BAO is completely additive if all of its operators are, and a class of BAO's is completely additive if all of its members are.

For instance, in any complex algebra $\mathsf{Cm}\,\mathfrak{S}$ the operators f_R are completely additive. Conversely if $\mathfrak{A}$ is an atomic and complete BAO whose operators are completely additive, then $\mathfrak{A} \cong \mathsf{Cm}\,\mathbb{A}\mathsf{t}\mathfrak{A}$. This was shown by Jónsson and Tarski [11, Theorem 3.9], who established a bijective correspondence between completely additive normal n-ary operators on the powerset algebra of a set X and $n+1$-ary relations on X. More strongly, if $\mathfrak{A}$ is atomic then the standard embedding

$$x \mapsto \{a \in \mathbb{A}\mathsf{t}\mathfrak{A} : a \leq x\}$$

of the Boolean part of $\mathfrak{A}$ into the powerset of its set of atoms will preserve the operators of $\mathfrak{A}$ *if these operators are completely additive* (see [13, Lemma 1] or [14, Proposition 5.1] for details). In other words, when $\mathfrak{A}$ is atomic and completely additive there is an injective BAO-homomorphism from $\mathfrak{A}$ into $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\mathfrak{A}$. The image of this injection contains all the singleton subsets of $\mathbb{A}\mathsf{t}\mathfrak{A}$ (since $a \mapsto \{a\}$), and hence includes $\mathsf{Sn}\,\mathbb{A}\mathsf{t}\mathfrak{A}$. By inverting the injection we obtain an embedding of $\mathsf{Sn}\,\mathbb{A}\mathsf{t}\mathfrak{A}$ back into $\mathfrak{A}$, and altogether we have

$$\mathsf{Sn}\,\mathbb{A}\mathsf{t}\mathfrak{A} \rightarrowtail \mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{A}\mathsf{t}\mathfrak{A}$$

[14, Proposition 5.1]. But then if $\mathfrak{A}$ belongs to a class V, so does $\mathsf{Sn}\,\mathbb{A}\mathsf{t}\mathfrak{A}$ (remember V is S-closed), making $\mathbb{A}\mathsf{t}\mathfrak{A}$ a weak structure for V. This shows that

if V is completely additive, then $\mathbb{A}\mathsf{t}\,V = \mathbb{W}\mathsf{st}\,V$

[14, Theorem 1.4]. It follows immediately that

if V is completely additive, then V is singleton-persistent if and only if it is atom-canonical.

Any completely additive BAO has a *completion* $\mathfrak{A}^+$. The Boolean part of $\mathfrak{A}^+$ is complete and atomic and has $\mathfrak{A}$ as a dense subalgebra, i.e. each element of $\mathfrak{A}^+$ is the join of the elements of $\mathfrak{A}$ below it. Each n-ary operator f of $\mathfrak{A}$ lifts to a completely additive f^+ on $\mathfrak{A}^+$ with $f^+(x) = \sum\{f(y) : x \geq y \in \mathfrak{A}^n\}$. The existence and uniqueness (up to isomorphism) of $\mathfrak{A}^+$ was established in [12]. For further discussion of the construction see [3].

A class V of BAO's will be called *completion-closed* if $\mathfrak{A}^+ \in V$ whenever $\mathfrak{A}$ is a completely additive member of V. Note that this does not require all members of V to be completely additive.

Theorem 5.1

(1) *If V is completion-closed, then it is singleton-persistent.*

(2) *If V is completion-closed and completely additive, then it is atom-canonical.*

(3) *A completion-closed variety is canonical if and only if it is complete.*

Proof.

(1) For any structure $\mathfrak{S}$, $\mathsf{Cm}\,\mathfrak{S}$ is a completion of $\mathsf{Sn}\,\mathfrak{S}$, so if V is completion closed then $\mathsf{Sn}\,\mathfrak{S} \in V$ only if $\mathsf{Cm}\,\mathfrak{S} \in V$, i.e. V is Sn-persistent.

(2) This follows from (1), since Sn-persistence implies atom-canonicity in the completely additive setting (see above).

(3) If V is a completion closed variety then by (1) and Corollary 3.2, $\mathbb{S}\mathsf{tr}\,V = \mathbb{W}\mathsf{st}\,V$ is elementary, so V is canonical iff complete.

□

The conclusion of 5.1(1) cannot be strengthened to "V is atom-canonical", and so the hypothesis of complete additivity in 5.1(2) is essential. This is exemplified by the variety V_η, where η is $(f_1 x \leq f_2 f_1 x)$. V_η was shown not to be atom-canonical in the proof of Theorem 4.1. That V_η is closed under completions is an instance of the general fact that any inequality $\sigma \leq \sigma'$ is preserved under completions if σ and σ' are strictly positive terms [3, Corollary 31(iii)]. In the case of η the proof is quite straightforward. If $\mathfrak{A} \models \eta$ and $\mathfrak{A}^+$ is a completion of $\mathfrak{A}$, then using the fact that $f_i z \leq f_i^+ z$ for any $z \in \mathfrak{A}$, as well as the monotonicity of f_i^+, we calculate that if $y \in \mathfrak{A}$ and $y \leq x \in \mathfrak{A}^+$, then

$$f_1 y \leq f_2 f_1 y \leq f_2^+ f_1 y \leq f_2^+ f_1^+ y \leq f_2^+ f_1^+ x.$$

Thus $f_1^+ x = \sum\{f_1 y : x \geq y \in \mathfrak{A}\} \leq f_2^+ f_1^+ x$, showing that $\mathfrak{A}^+ \models \eta$.

In relation to 5.1(3), it is presently unknown whether there is a completion-closed variety that is incomplete. If not, then every completion-closed variety must be canonical. Either conclusion would be of interest in understanding the nature of varieties of BAO's. Note that the variety of algebras for the modal logic K4M (see Section 4) is canonical but not Sn-persistent, and therefore not completion-closed.

A further question raised in [14] is whether every variety of BAO's is *atomically generated*, i.e. generated by its atomic members. The following result bears on this issue.

Theorem 5.2 *Let V be a completely additive variety that is singleton-persistent. Then the following are equivalent.*

(1) *V is atomically generated.*

(2) *V is complete.*

(3) *V is canonical.*

Proof. We have already observed that (2) and (3) are equivalent when V is Sn-persistent. It is always true that (2) implies (1), since a complete variety is generated by complex algebras $\mathsf{Cm}\,\mathfrak{S}$ which are atomic. Thus our task is to show that (1) implies (2), and this is where we require completely additivity.

Let V_{at} be the class of all atomic members of V, and take $\mathfrak{A} \in V_{at}$. Now the hypotheses on V imply that it is atom-canonical, so then $\mathbb{A}\mathsf{t}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\,V$. But since $\mathfrak{A}$ is completely additive, $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ as explained above, so altogether

$$\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A} \in \mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V,$$

showing that $\mathfrak{A} \in \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$. Thus

$$V_{at} \subseteq \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V \subseteq V,$$

so the subvariety of V generated by $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$ must include V_{at}, and therefore must include the variety generated by V_{at}. Hence if V_{at} generates V itself, then so does $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\,V$, i.e. (1) implies (2). □

The assumption that V is completely additive is indispensible in Theorem 5.2, as may be seen by refining the example of the variety V_0 discussed in Section 4. Let V_1 be the subvariety generated by the atomic members of V_0. The algebra $\mathfrak{A}_0$ belongs to V_1 because it is atomic: its atoms are all the singleton subsets of $\mathfrak{S}_0$ except $\{\omega\}$ and every member of $\mathfrak{A}_0$ includes such a singleton. The operator f_R is not completely additive in $\mathfrak{A}_0$ since $\sum_{p\in\omega} f_R(\{p\}) = \emptyset$ but

$$f_R(\textstyle\sum_{p\in\omega}\{p\}) = f_R(\{0,\dots\omega\}) = \{\omega+1\}.$$

By definition, V_1 is atomically generated. Also V_1 is Sn-persistent, and indeed is atom-canonical, for if $\mathfrak{A}$ in V_1 is atomic then $\mathsf{Cm}\,\mathbb{A}\mathsf{t}\,\mathfrak{A}$ belongs to V_0 by atom-canonicity of V_0, and hence belongs to V_1 because it is atomic. However V_1 is not complete: its complex algebras all belong to V_0 and so satisfy $fx = 0$, but $\mathfrak{A}_0 \in V_1$ and $\mathfrak{A}_0 \not\models fx = 0$, so V_1 is not generated by its complex algebras. Thus V_1 is Sn-persistent but violates the conclusion of the Theorem.

It follows from Theorem 5.2 that to exhibit a variety that is not atomically generated, it suffices to construct a *completely additive* one that is Sn-persistent and incomplete. The variety V_0 is Sn-persistent and incomplete, but it lacks complete additivity. Now one way to impose complete additivity on a BAO $\mathfrak{A}$ with unary operators is to require each operator f of $\mathfrak{A}$ to have a *conjugate* f^c. Here f^c is a function on $\mathfrak{A}$ satisfying

$$f(x)\cdot y = 0 \text{ if and only if } x \cdot f^c(y) = 0.$$

This condition can be expressed equationally [11, Theorem 1.15], and if a function f has a conjugate in $\mathfrak{A}$ then f must be normal and completely additive [11, Theorem 1.14]. This all suggests that a candidate for a completely additive Sn-persistent and incomplete variety might be the variety of conjugated BAO's that satisfy the equation η_0. However that prospect is nullified by the following result, which implies that this is just the variety of conjugated BAO's satisfying $fx = 0$, a variety which is complete.

Theorem 5.3 *If* $\mathfrak{A} \models \eta_0$ *and* f *is conjugated in* $\mathfrak{A}$*, then* $\mathfrak{A} \models fx = 0$.

Proof. Let f^c be the conjugate of f in $\mathfrak{A}$. To prove that $fx = 0$ it is enough to show that $f^c 1 = 0$, for then $x \cdot f^c(1) = 0$ and so by conjugacy $f(x) \cdot 1 = 0$. We will use the fact that $f^c f^d z \leq z$ in general, which follows by conjugacy from $f^d z \cdot f(-z) = 0$.

Now if $fx \leq f^d y$ then

$$1 = -fx + f^d y = f^d{-x} + f^d y \leq f^d(-x + y).$$

Putting $y = f'(x \cdot -f'x)$ and invoking $\mathfrak{A} \models \eta_0$ here then yields

$$1 = f^d(-x + f'(x \cdot -f'x)).$$

Hence

$$f^c 1 = f^c f^d(-x + f'(x \cdot -f'x)) \leq (-x + f'(x \cdot -f'x)).$$

Now taking $x = f^c 1$ at this point gives

$$f^c 1 \leq (-f^c 1 + f'(f^c 1 \cdot -f' f^c 1)),$$

which by Boolean algebra and the monotonicity of f' yields

$$f^c 1 \leq f'(f^c 1 \cdot -f' f^c 1) \leq f' f^c 1.$$

Thus $f^c 1 \cdot -f' f^c 1 = 0$, and so

$$f^c 1 \leq f'(f^c 1 \cdot -f' f^c 1) = f' 0 = 0.$$

□

This result has an interesting interpretation in the field of modal logic. Consider the propositional language of two modalities $\Diamond_1$ and $\Diamond_2$, with duals $\Box_1$ and $\Box_2$. Let L be the smallest normal logic in this language that includes the schema

$$\Diamond_1 p \to \Box_1 \Diamond_2 (p \wedge \neg \Diamond_2 p).$$

The algebraic models for L are the members of the variety V_0. L is incomplete with respect to its Kripke semantics, which is based on structures $\mathfrak{S} = (X, R_1, R_2)$ in which R_i interprets the modality $\Diamond_i$. If such a structure validates L then it validates the formula $\neg \Diamond_1 \top$. This may be seen by adapting the argument that V_0 is atom-canonical. However $\neg \Diamond_1 \top$ is not a theorem of L, since the algebra $\mathfrak{A}_0$ validates L but not $\neg \Diamond_1 \top$.

Now in $\mathsf{Cm}\,\mathfrak{S}$, the operators f_{R_i} are conjugated. Their conjugates are the operators $f_{R_i^{-1}}$, where R_i^{-1} is the inverse relation to R_i [11, Theorem 3.6]. (The algebra $\mathfrak{A}_0$ is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}_0$ that is not closed under these conjugates, as Theorem 5.3 implies.) We may expand the language for L by adding new modalities $\Diamond_i^{-1}$ interpreted in a structure by the relations R_i^{-1}. Let L^+, the *minimal conjugated extension of* L, be the smallest normal logic in this language that includes L and the *conjugation schemata*

$$\Diamond_i^{-1}\Box_i p \to p, \qquad \Diamond_i \Box_i^{-1} p \to p.$$

The argument in the proof of Theorem 5.3 can be adapted proof-theoretically to show that $\neg\Diamond_1\top$ is a theorem of L^+, and in fact that L^+ is the smallest normal logic in the extended language that includes $\neg\Diamond_1\top$ and the conjugation schemata. By well-known methods it is readily inferred that L^+ is complete, and characterised by the class of all structures $(X, R_i, R_i^{-1})_{i=1,2}$ in which $R_1 = \emptyset$. Indeed the canonical L^+-frame satisfies this description and characterises L^+.

We thus see that L is an example of a logic that is incomplete for Kripke semantics, but whose minimal conjugated extension is complete and canonical. Moreover it is only necessary to add the conjugate modality for $\Diamond_1$ to achieve this canonicity.

In conclusion, let us record the open problems that have been identified above:

(1) Does there exist a completion-closed variety that is not canonical (or equivalently, incomplete)?

(2) Does there exist a completely additive variety that is singleton-persistent but incomplete, and therefore not generated by its atomic members?

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic.* Cambridge University Press, 2001.

[2] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra.* Springer-Verlag, 1981.

[3] S. Givant and Y. Venema. The Preservation of Sahlqvist Equations in Completions of Boolean Algebras With Operators. *Algebra Universalis*, 41:47–84, 1999.

[4] Robert Goldblatt. Varieties of Complex Algebras. *Annals of Pure and Applied Logic*, 44:173–242, 1989.

[5] Robert Goldblatt. The McKinsey Axiom is Not Canonical. *The Journal of Symbolic Logic*, 56:554–562, 1991.

[6] Robert Goldblatt. Elementary Generation and Canonicity for Varieties of Boolean Algebras with Operators. *Algebra Universalis*, 34:551–607, 1995.

[7] Robert Goldblatt. Algebraic Polymodal Logic: A Survey. *Logic Journal of the IGPL,* Special Issue on Algebraic Logic edited by István Németi and Ildikó Sain, 8(4):393–450, July 2000. Electronically available at: `http://www3.oup.co.uk/igpl/Volume_08/Issue_04/`.

[8] Robin Hirsch and Ian Hodkinson. Strongly Representable Atom Structures. Electronically available at `http://www.doc.ic.ac.uk/~imh`, 2000.

[9] Ian Hodkinson. Atom Structures of Cylindric Algebras and Relation Algebras. *Annals of Pure and Applied Logic*, 89:117–148, 1997.

[10] Bjarni Jónsson. A Survey of Boolean Algebras with Operators. In *Algebras and Orders*, volume 389 of *NATO ASI Series*, pages 239–286. Kluwer Academic Publishers, 1993.

[11] Bjarni Jónsson and Alfred Tarski. Boolean Algebras with Operators, Part I. *American Journal of Mathematics*, 73:891–939, 1951.

[12] J. D. Monk. Completions of Boolean Algebras With Operators. *Mathematische Nachrichten*, 46:47–55, 1970.

[13] Yde Venema. Atom Structures and Sahlqvist Equations. *Algebra Universalis*, 38:185–199, 1997.

[14] Yde Venema. Atom Structures. In Marcus Kracht, Maarten de Rijke, Heinrich Wansung, and Michael Zakharyaschev, editors, *Advances in Modal Logic Volume 1*, pages 291–305. CSLI Publications, 1998.