

ROBERT GOLDBLATT  
IAN HODKINSON

# The Finite Model Property for Logics with the Tangle Modality

**Abstract.** The tangle modality is a propositional connective that extends basic modal logic to a language that is expressively equivalent over certain classes of finite frames to the bisimulation-invariant fragments of both first-order and monadic second-order logic. This paper axiomatises several logics with tangle, including some that have the universal modality, and shows that they have the finite model property for Kripke frame semantics. The logics are specified by a variety of conditions on their validating frames, including local and global connectedness properties. Some of the results have been used to obtain completeness theorems for interpretations of tangled modal logics in topological spaces.

*Keywords:* Tangle modality, Finite model property, Kripke frame, Filtration, Connected, Universal modality.

## 1. Introduction

The *tangle modality*, which we denote by  $\langle t \rangle$ , is a polyadic propositional connective that creates a new formula  $\langle t \rangle \Gamma$  out of any finite non-empty set  $\Gamma$  of formulas.  $\langle t \rangle \Gamma$  has the following semantics in a model on Kripke frame  $(W, R)$ :

$\langle t \rangle \Gamma$  is true at  $x$  iff there is an endless  $R$ -path  $xRx_1 \cdots x_n Rx_{n+1} \cdots \cdots$   
in  $W$  with each member of  $\Gamma$  being true at  $w_n$  for infinitely many  $n$ .

This connective was introduced by Dawar and Otto [2] in a study of language fragments that are bisimulation-invariant over finite frames. It is well known that over the class of all frames, the bisimulation-invariant fragment of first-order logic is expressively equivalent to the basic modal language  $\mathcal{L}_\square$  of a single modality  $\square$  (van Benthem's Theorem [19,20]). This equivalence also holds over any elementary class of frames, such as the class of all transitive ones [2, Thm. 2.12], and over the class of all finite frames [13]. By contrast, the bisimulation-invariant fragment of monadic second-order logic is equivalent over all frames to the much more powerful modal mu-calculus

---

Presented by Yde Venema; Received June 16, 2016

[10]. But [2] proved the striking result that over the class of *finite transitive* frames, and a number of its subclasses, the bisimulation-invariant fragment of monadic second-order logic and the mu-calculus are both equivalent to the bisimulation-invariant fragment of first-order logic, and all three are equivalent, not to  $\mathcal{L}_\square$ , but to the language  $\mathcal{L}_\square^{\langle t \rangle}$  that expands  $\mathcal{L}_\square$  by the addition of the tangle modality  $\langle t \rangle$ .

Subsequently, Fernández-Duque [4, 5] studied the logic of  $\mathcal{L}_\square^{\langle t \rangle}$ -formulas valid in S4 frames, i.e. reflexive transitive frames, axiomatising it as an extension of S4, and showing that it has the finite model property. We call this logic S4t. Its essential axioms involving  $\langle t \rangle$  are

**Fix:**  $\langle t \rangle \Gamma \rightarrow \diamond(\gamma \wedge \langle t \rangle \Gamma)$ , all  $\gamma \in \Gamma$ ,

**Ind:**  $\square(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \rightarrow (\varphi \rightarrow \langle t \rangle \Gamma)$ ,

where  $\diamond$  is the dual modality to  $\square$ . These axioms encapsulate the fact that  $\langle t \rangle \Gamma$  has the same meaning as the mu-calculus formula

$$\nu p \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge p),$$

which is interpreted (loosely speaking) as the greatest fixed point of the function  $\varphi \mapsto \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)$ . Fix expresses that  $\langle t \rangle \Gamma$  is a (post)fixed point of this function, while Ind expresses that it is the greatest.

To explain this further, denote by  $\llbracket \varphi \rrbracket$  the set of points at which a formula  $\varphi$  is true in a model on a frame  $(W, R)$ , and let  $f_\Gamma$  be the function on subsets  $S$  of  $W$  defined by

$$f_\Gamma(S) = \bigcap_{\gamma \in \Gamma} R^{-1}(\llbracket \gamma \rrbracket \cap S),$$

where in general  $R^{-1}(V)$  is the  $R$ -preimage  $\{x \in W : \exists y(xRy \in V)\}$  of  $V$ . Since  $\llbracket \diamond \varphi \rrbracket = R^{-1}\llbracket \varphi \rrbracket$ , we see that

$$f_\Gamma(\llbracket \varphi \rrbracket) = \llbracket \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi) \rrbracket.$$

A set  $S$  is a *postfixed point* of  $f_\Gamma$  if  $S \subseteq f_\Gamma(S)$ . It can be shown that  $\llbracket \langle t \rangle \Gamma \rrbracket$  is equal to the union

$$\bigcup \{S \subseteq W : S \subseteq f_\Gamma(S)\} \tag{1.1}$$

of all postfixed points of  $f_\Gamma$ , which implies by the Knaster-Tarski Theorem [18] that  $\llbracket \langle t \rangle \Gamma \rrbracket$  is a fixed point of  $f_\Gamma$ , i.e.  $f_\Gamma \llbracket \langle t \rangle \Gamma \rrbracket = \llbracket \langle t \rangle \Gamma \rrbracket$ , that is larger than all others (see the introduction to [9] for more on the mu-calculus reading of  $\langle t \rangle$ ).

Fernández-Duque also provided the name ‘tangle’ for  $\langle t \rangle$ , motivated by a topological semantics for it studied further in [3, 6]. That interprets  $\square$  as the

interior operator in a topological space, and hence  $\diamond$  as the closure operator, while  $\langle t \rangle$  is interpreted as an operation of *tangled closure* assigning to any collection of sets the largest subset in which each member of the collection is dense. In an S4 frame,  $R^{-1}(V)$  is the topological closure of  $V$  under the Alexandroff topology generated by the *successor sets*  $R(x) = \{y \in W : xRy\}$  of all points  $x \in W$ . In this topology,  $\llbracket \langle t \rangle \Gamma \rrbracket$  is the tangled closure of  $\{\llbracket \gamma \rrbracket : \gamma \in \Gamma\}$ .

The purpose of the present paper is to axiomatise several other  $\mathcal{L}_{\square}^{\langle t \rangle}$ -logics whose  $\mathcal{L}_{\square}$  fragment is weaker than S4, and show that they have the finite model property for Kripke semantics. First we deal with the logics K4t and KD4t, characterised by validity in frames that are transitive, and serial and transitive, respectively. Then we study a sequence of axioms  $G_n$  in the variables  $p_0, \dots, p_n$ , introduced by Shehtman [15]. Putting  $Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j$  for each  $i \leq n$ , then  $G_n$  can be defined as the formula

$$\bigwedge_{i \leq n} \diamond Q_i \rightarrow \diamond \left( \bigwedge_{i \leq n} \diamond^* \neg Q_i \right),$$

where in general  $\diamond^* \varphi$  is  $\varphi \vee \diamond \varphi$ .  $G_n$  expresses a certain graph-theoretic *local n-connectedness* property of a frame as a directed graph, namely that the successor set  $R(x)$  of each point  $x$  has at most  $n$  path-connected components. We prove the finite model property over such frames for a logic K4G<sub>n</sub>t, and for a number of extensions of it. These include expanding the language by including the universal modality  $\forall$  and its dual  $\exists$ , and adding the axiom

$$\exists \varphi \wedge \exists \neg \varphi \rightarrow \exists (\diamond^* \varphi \wedge \diamond^* \neg \varphi),$$

which expresses global connectedness (any two points have a connecting path between them). We show that any weak<sup>1</sup> canonical frame of an extension of K4G<sub>n</sub>t is locally  $n$ -connected, using Fix and Ind to refine an analysis of the  $\mathcal{L}_{\square}$ -logic KD4G<sub>1</sub> given in [15].

Our initial motivation for this work involves a different topological semantics in which  $\diamond$  is interpreted as the derivative (i.e. set of limit points) operator of a topological space, and the interpretation of  $\langle t \rangle$  is modified to use the derivative in place of topological closure. In [8, 9] we have obtained completeness theorems for the resulting logics of a range of spaces. For instance, the 'tangle logic' KD4G<sub>1</sub>t is the logic of the Euclidean space  $\mathbb{R}^n$  for all  $n \geq 2$ , and includes the logic of every dense-in-itself metric space; KD4G<sub>2</sub>t is the logic of the real line  $\mathbb{R}$ ; and KD4t is the logic of any zero-dimensional dense-in-itself metric space (examples include the space of rationals  $\mathbb{Q}$ , the Cantor space,

---

<sup>1</sup>'Weak' means built from a language with finitely many variables.

and the Baire space  $\omega^\omega$ ). The technique used to prove these results, and others, is to construct validity preserving maps from the space in question onto finite frames for the logic, and to appeal to the finite model property to ensure that there are sufficiently many such frames available to yield completeness. Thus the work of this paper is an essential prerequisite to these completeness theorems. At the same time we consider that the paper has its own interest as a contribution to modal Kripke semantics that goes beyond, and is independent of, the topological applications.

Our approach to the finite model property for languages with  $\langle t \rangle$  differs from that of [5]. It follows a well known procedure of building a canonical Henkin model and then collapsing it to a finite one by the filtration process. But there are some stumbling blocks in the presence of  $\langle t \rangle$ . The first is that a canonical model, whose points are maximally consistent sets of formulas, may fail to satisfy the ‘Truth Lemma’ that a formula is true at point  $x$  iff it belongs to  $x$ . We show below that there is an endless  $R$ -path  $xRx_1R\cdots$  in the canonical model for  $K4t$  along which a variable  $q$  and its negation  $\neg q$  are each true infinitely often, so  $\langle t \rangle\{q, \neg q\}$  is true at  $x$ , but  $\langle t \rangle\{q, \neg q\} \notin x$ . Consequently, we are obliged to work with the membership relation  $\varphi \in x$  of a canonical model, rather than its truth relation.

The second problem is that the filtration process may reproduce the first problem. There may be endless  $R$ -paths in a finite collapsed model  $\mathcal{M}_\Phi$  that contradict the falsity of formulas of the form  $\langle t \rangle\Gamma$ . To overcome this we ‘untangle’ the binary relation of the frame of  $\mathcal{M}_\Phi$ , refining it to a subrelation that gives a new model  $\mathcal{M}_t$ , in such a way that such ‘bad paths’ do not occur in  $\mathcal{M}_t$ . This construction is the heart of the paper, and is carried out in Section 6 by making vital use of the tangle axioms Fix and Ind (with the latter modified slightly for the sub-S4 context).

Each result about the finite model property that we prove is stated as a formal Proposition, typically at the end of a section. In the final section there is a summary table listing all of the logics that we analyse, and giving for each of them of a class of frames over which it has the finite model property.

## 2. Syntax and Semantics

We assume familiarity with Kripke semantics for modal logic, but include some review of basics as we establish notation and terminology. Let  $\mathbf{Var}$  be a set of propositional variables, which may be finite or infinite. Formulas of the language  $\mathcal{L}_\square$  are constructed from these variables by the standard Boolean connectives  $\top$ ,  $\neg$ ,  $\wedge$  and the unary modality  $\square$ . The other Boolean

connectives  $\perp$ ,  $\vee$ ,  $\rightarrow$ ,  $\leftrightarrow$  are introduced as the usual abbreviations, and the dual modality  $\diamond$  is defined to be  $\neg\Box\neg$ .

The language  $\mathcal{L}_{\Box}^{(t)}$  is defined as for  $\mathcal{L}_{\Box}$  but with the additional formation of a formula  $\langle t \rangle \Gamma$  for each finite non-empty set  $\Gamma$  of formulas. Later we will add the universal modality  $\forall$  and its dual  $\exists$ .

A (*Kripke*) *frame* is a pair  $\mathcal{F} = (W, R)$  with  $R$  a binary relation on set  $W$ . For each  $x \in W$ , the set  $R(x) = \{y \in W : xRy\}$  is the set of *R-successors* or *R-alternatives* of  $x$ .

A *model*  $\mathcal{M} = (W, R, h)$  on a frame is given by a valuation function  $h : \text{Var} \rightarrow \wp W$ . The relation  $\mathcal{M}, x \models \varphi$  of a formula  $\varphi$  of  $\mathcal{L}_{\Box}^{(t)}$  being *true at  $x$  in  $\mathcal{M}$*  is defined by an induction on the formation of  $\varphi$  as follows:

1.  $\mathcal{M}, x \models p$  iff  $x \in h(p)$ , for  $p \in \text{Var}$ .
2.  $\mathcal{M}, x \models \top$ .
3.  $\mathcal{M}, x \models \neg\varphi$  iff  $\mathcal{M}, x \not\models \varphi$ .
4.  $\mathcal{M}, x \models \varphi \wedge \psi$  iff  $\mathcal{M}, x \models \varphi$  and  $\mathcal{M}, x \models \psi$ .
5.  $\mathcal{M}, x \models \Box\varphi$  iff  $\mathcal{M}, y \models \varphi$  for every  $y \in R(x)$ .
6.  $\mathcal{M}, x \models \langle t \rangle \Gamma$  iff there is a sequence  $x = x_0, x_1, \dots$  in  $W$  with  $x_n R x_{n+1}$  for each  $n < \omega$  and such that for each  $\gamma \in \Gamma$  there are infinitely many  $n < \omega$  with  $\mathcal{M}, x_n \models \gamma$ .

Consequently we have

7.  $\mathcal{M}, x \models \diamond\varphi$  iff  $\mathcal{M}, y \models \varphi$  for some  $y \in R(x)$ .

A formula  $\varphi$  is *true in model  $\mathcal{M}$*  if it is true at all points in  $\mathcal{M}$ , and *valid in frame  $\mathcal{F}$*  if it is true in all models on  $\mathcal{F}$ .

A *subframe* of a frame  $\mathcal{F}$  is any frame  $\mathcal{F}' = (W', R')$  for which  $W' \subseteq W$  and  $R'$  is the restriction of  $R$  to  $W'$ . Then  $\mathcal{F}'$  is an *inner subframe* of  $\mathcal{F}$  if it is closed under  $R$  in the sense that  $R(x) \subseteq W'$  for all  $x \in W'$ .

We say that a frame  $(W, R)$ , or any model on that frame, is *finite* if  $W$  is finite, and is *reflexive* if  $R$  is reflexive, *transitive* if  $R$  is transitive, etc.

### 3. Clusters in Transitive Frames

From now on we will work throughout with models on *transitive* frames  $(W, R)$ . If  $xRy$ , we may say that the  $R$ -successor  $y$  comes *R-after*  $x$ , or is *R-later than*  $x$ . We write  $xR^\bullet y$  when  $xRy$  but not  $yRx$ : then  $y$  is *strictly after/after*, or is a *proper R-successor*. A point  $x$  is *reflexive* if  $xRx$ , and

*irreflexive* otherwise.  $R$  is (ir)reflexive on a set  $X \subseteq W$  if every member of  $X$  is (ir)reflexive.

An  $R$ -cluster is a subset  $C$  of  $W$  that is an equivalence class under the equivalence relation

$$\{(x, y) : x = y \text{ or } xRyRx\}.$$

A cluster is *degenerate* if it is a singleton  $\{x\}$  with  $x$  irreflexive. Note that a cluster  $C$  can only contain an irreflexive point if it is a singleton. For, if  $C$  has more than one element, then for each  $x \in C$  there is some  $y \in C$  with  $x \neq y$ , so  $xRyRx$  and thus  $xRx$  by transitivity. On a non-degenerate cluster the relation  $R$  is universal. For  $C$  to be non-degenerate it suffices that there exist  $x, y \in C$  with  $xRy$ , regardless of whether  $x = y$  or not.

Write  $C_x$  for the  $R$ -cluster containing  $x$ . Thus  $C_x = \{x\} \cup \{y : xRyRx\}$ . The relation  $R$  lifts to a well-defined *partial* ordering of clusters by putting  $C_xRC_y$  iff  $xRy$ . A cluster  $C$  is  $R$ -maximal when there is no cluster that comes strictly  $R$ -after it, i.e. when  $CRC'$  implies  $C = C'$ . A point  $x \in W$  is  $R$ -maximal, or just *maximal* if  $R$  is understood, if  $C_x$  is a maximal cluster, or equivalently if  $xRy$  implies  $yRx$ . This means that  $R^\bullet(x) = \emptyset$ , where  $R^\bullet(x) = \{y \in W : xR^\bullet y\}$ .

An  $R$ -chain is a sequence  $C_1, C_2, \dots$  of pairwise distinct clusters with  $C_1RC_2R \dots$ . In a finite frame, such a chain is of finite length. Hence we can define a notion of *rank* in a finite frame by declaring the rank of a cluster  $C$  to be the number of clusters in the longest chain of clusters starting with  $C$ . So the rank is always  $\geq 1$ , and a rank-1 cluster is maximal. The rank of a point  $x$  is defined to be the rank of  $C_x$ . The key property of this notion is that if  $xR^\bullet y$ , equivalently if  $C_y$  comes strictly  $R$ -after  $C_x$ , then  $y$  has smaller rank than  $x$ .

An *endless  $R$ -path* is a sequence  $\{x_n : n < \omega\}$  such that  $x_nRx_{n+1}$  for all  $n$ , as in the semantic clause (6) for the truth of  $\langle t \rangle \Gamma$ . Such a path *starts at/from*  $x_0$ . The terms of the sequence need not be distinct: for instance, any reflexive point  $x$  gives rise to the endless  $R$ -path  $xRxRxR \dots$ . In a finite frame, an endless path must eventually enter some *non-degenerate* cluster  $C$  and stay there, i.e. there is some  $n$  such that  $x_m \in C$  for all  $m \geq n$ .

If  $(W', R')$  is an *inner* subframe of  $(W, R)$ , then every  $R'$ -cluster is an  $R$ -cluster, and every  $R$ -cluster that intersects  $W'$  is a subset of  $W'$  and is an  $R'$ -cluster.

In a model  $\mathcal{M}$ , a set  $\Gamma$  of formulas is *satisfied by the cluster  $C$*  if each member of  $\Gamma$  is true in  $\mathcal{M}$  at some point of  $C$ . So  $\Gamma$  fails to be satisfied by  $C$  if some member of  $\Gamma$  is false at every point of  $C$ . In a *finite* model, an endless path must eventually enter some non-degenerate cluster and stay there, so

we get that

$$x \models \langle t \rangle \Gamma \text{ iff there is a } y \text{ with } xRy \text{ and } \mathbf{yRy} \text{ and } \Gamma \text{ is satisfied by } C_y. \quad (3.1)$$

To put this another way,  $x \models \langle t \rangle \Gamma$  iff  $\Gamma$  is satisfied by some *non-degenerate* cluster following  $C_x$ .

Write  $\langle t \rangle \varphi$  for the formula  $\langle t \rangle \{\varphi\}$ . Then  $\langle t \rangle \varphi$  is true at  $x$  iff there is an endless path starting at  $x$  along which  $\varphi$  is true infinitely often. For finite models we have

$$x \models \langle t \rangle \varphi \text{ iff there is a } y \text{ with } xRy \text{ and } yRy \text{ and } y \models \varphi,$$

i.e. the meaning of  $\langle t \rangle \varphi$  is that there is a *reflexive* alternative at which  $\varphi$  is true. Thus for finite *reflexive* models (i.e. finite S4 models) this reduces to the standard Kripkean interpretation (7) of  $\diamond$ . More strongly, it is evident that while  $\langle t \rangle \varphi \rightarrow \diamond \varphi$  is valid in all transitive frames, reflexive transitive frames validate  $\langle t \rangle \varphi \leftrightarrow \diamond \varphi$ .

Observe further that in a finite model that is partially ordered (i.e.  $R$  is reflexive, transitive and anti-symmetric),  $\langle t \rangle \Gamma$  is equivalent to  $\diamond \bigwedge \Gamma$  since each cluster is a non-degenerate singleton  $\{y\}$  which satisfies  $\Gamma$  iff  $\bigwedge \Gamma$  is true at  $y$ . On the other hand, in an irreflexive finite model no formula  $\langle t \rangle \Gamma$  can be true anywhere, since all clusters are degenerate.

Write  $\diamond^* \varphi$  for the formula  $\varphi \vee \diamond \varphi$ , and  $\square^* \varphi$  for  $\varphi \wedge \square \varphi$ . In any transitive frame, define  $R^* = R \cup \{(x, x) : x \in W\}$ . Then  $R^*$  is the reflexive-transitive closure of  $R$ , and in any model  $\mathcal{M}$  on the frame we have

$$\mathcal{M}, x \models \square^* \varphi \text{ iff for all } y, \text{ if } xR^*y \text{ then } \mathcal{M}, y \models \varphi.$$

and

$$\mathcal{M}, x \models \diamond^* \varphi \text{ iff for some } y, xR^*y \text{ and } \mathcal{M}, y \models \varphi.$$

Note that if  $C_x = C_y$ , then  $xR^*y$ . For each  $x$  let  $R^*(x) = \{y \in W : xR^*y\}$ . Then  $R^*(x) = \{x\} \cup R(x)$ .

## 4. Tangle Logics

A *tangle logic*, in any language including  $\mathcal{L}_{\square}^{\langle t \rangle}$ , is a set of formulas that includes all tautologies and all instances of the schemes

$$\mathbf{K}: \square(\varphi \rightarrow \psi) \rightarrow (\square\varphi \rightarrow \square\psi)$$

$$\mathbf{4}: \diamond\diamond\varphi \rightarrow \diamond\varphi$$

$$\mathbf{Fix}: \langle t \rangle \Gamma \rightarrow \diamond(\gamma \wedge \langle t \rangle \Gamma), \quad \text{all } \gamma \in \Gamma,$$

$$\mathbf{Ind}: \square^*(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \rightarrow (\varphi \rightarrow \langle t \rangle \Gamma),$$

and whose rules include modus ponens and  $\Box$ -generalisation (from  $\varphi$  infer  $\Box\varphi$ ). These schemes are all true in any transitive model  $\mathcal{M}$ , and so  $\{\varphi : \varphi \text{ is true in } \mathcal{M}\}$  is a tangle logic. So too is  $\{\varphi : \varphi \text{ is valid in } \mathcal{F}\}$  for any transitive frame  $\mathcal{F}$ .

Our naming convention will be that if  $N$  is the name of some logic in a language without  $\langle t \rangle$ , then  $Nt$  denotes the smallest tangle logic that contains all instances of members of  $N$ . Thus the smallest tangle logic will be denoted  $K4t$ , since  $K4$  is the smallest normal  $\mathcal{L}_{\Box}$ -logic to contain the scheme 4.

Any tangle logic includes the following schemes:

$$\langle t \rangle\varphi \rightarrow \Diamond\varphi$$

$$4_*: \Diamond\Diamond^*\varphi \rightarrow \Diamond\varphi$$

$$4_t: \Diamond\langle t \rangle\Gamma \rightarrow \langle t \rangle\Gamma$$

$\langle t \rangle\varphi \rightarrow \Diamond\varphi$  is derived from Fix, and  $4_*$  from 4, using the modal principles  $\Diamond(\varphi \wedge \psi) \rightarrow \Diamond\varphi$  and  $\Diamond(\varphi \vee \psi) \rightarrow \Diamond\varphi \vee \Diamond\psi$  and Boolean reasoning.  $4_t$  will be explicitly needed in Section 9, to establish a condition called (r4). Here is a derivation of  $4_t$ , in which the justification ‘‘Bool’’ means by principles of Boolean logic, ‘‘Reg’’ is the Regularity rule *from*  $\varphi \rightarrow \psi$  *infer*  $\Diamond\varphi \rightarrow \Diamond\psi$ , and ‘‘Nec’’ is the Necessitation rule *from*  $\varphi$  *infer*  $\Box^*\varphi$ .

For each  $\gamma \in \Gamma$  we derive

- |  |                           |
|--|---------------------------|
| 1. $\langle t \rangle\Gamma \rightarrow \Diamond(\gamma \wedge \langle t \rangle\Gamma)$                                 | Fix                       |
| 2. $\Diamond(\gamma \wedge \langle t \rangle\Gamma) \rightarrow \Diamond\langle t \rangle\Gamma$                         | K-theorem (Bool + Reg)    |
| 3. $\langle t \rangle\Gamma \rightarrow \Diamond\langle t \rangle\Gamma$   | 1, 2 Bool                 |
| 4. $\gamma \wedge \langle t \rangle\Gamma \rightarrow \gamma \wedge \Diamond\langle t \rangle\Gamma$                     | 3, Bool                   |
| 5. $\Diamond(\gamma \wedge \langle t \rangle\Gamma) \rightarrow \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)$ | 4, Reg                    |
| 6. $\langle t \rangle\Gamma \rightarrow \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)$                         | 1, 5 Bool                 |
| 7. $\Diamond\langle t \rangle\Gamma \rightarrow \Diamond\Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)$         | 6, Reg                    |
| 8. $\Diamond\langle t \rangle\Gamma \rightarrow \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)$                 | 7, <b>Axiom 4</b> , Bool. |

Since this holds for every  $\gamma \in \Gamma$  we can continue with

- |   |   |
|---|---|
| 9. $\Diamond\langle t \rangle\Gamma \rightarrow \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)$  | 8 for all $\gamma$<br>in $\Gamma$ , Bool.                 |
| 10. $\Box^*(\Diamond\langle t \rangle\Gamma \rightarrow \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma))$   | 9, Nec.   |
| 11. $\Box^*(\Diamond\langle t \rangle\Gamma \rightarrow \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge \Diamond\langle t \rangle\Gamma)) \rightarrow (\Diamond\langle t \rangle\Gamma \rightarrow \langle t \rangle\Gamma)$ | Ind with<br>$\varphi = \Diamond\langle t \rangle\Gamma$ . |
| 12. $\Diamond\langle t \rangle\Gamma \rightarrow \langle t \rangle\Gamma$   | 10, 11, Bool.   |

The members of a logic  $L$  may be referred to as the *L-theorems*. A formula  $\varphi$  is *L-consistent* if  $\neg\varphi$  is not an *L-theorem*. If  $\mathcal{K}$  is a class of frames, then we will say that  $L$  has the *finite model property over*  $\mathcal{K}$  if it is validated by



all finite members of  $\mathcal{K}$ , and each  $L$ -consistent formula is true at some point in some model on some finite member of  $\mathcal{K}$ . Equivalently, this means that  $L$  is *sound and complete* over the class of finite members of  $\mathcal{K}$ , i.e. a formula is an  $L$ -theorem iff it is valid in all finite members of  $\mathcal{K}$ .<sup>2</sup> We may say that  $L$  has the *finite model property*, simpliciter, if it has the finite model property over some class of frames. This implies that  $L$  has the finite model property over the class of all frames that validate  $L$ .

## 5. Canonical Frame

For a tangle logic  $L$ , the canonical frame is  $\mathcal{F}_L = (W_L, R_L)$ , with  $W_L$  the set of maximally  $L$ -consistent sets of formulas, and  $xR_Ly$  iff  $\{\diamond\varphi : \varphi \in y\} \subseteq x$  iff  $\{\varphi : \Box\varphi \in x\} \subseteq y$ . The relation  $R_L$  is transitive, by axiom 4.

Suppose  $\mathcal{F} = (W, R)$  is an inner subframe of  $\mathcal{F}_L$ , i.e.  $W$  is an  $R_L$ -closed subset of  $W_L$ , and  $R$  is the restriction of  $R_L$  to  $W$ .

By standard canonical frame theory (e.g. [1, Chapter 4] or [7, Chapter 3]), we have that for all formulas  $\varphi$  and all  $x \in W$ :

$$\diamond\varphi \in x \text{ iff for some } y \in W, xRy \text{ and } \varphi \in y. \quad (5.1)$$

$$\diamond^*\varphi \in x \text{ iff for some } y \in W, xR^*y \text{ and } \varphi \in y. \quad (5.2)$$

$$\Box\varphi \in x \text{ iff for all } y \in W, xRy \text{ implies } \varphi \in y. \quad (5.3)$$

$$\Box^*\varphi \in x \text{ iff for all } y \in W, xR^*y \text{ implies } \varphi \in y. \quad (5.4)$$

We will say that a sequence  $\{x_n : n < \omega\}$  in  $\mathcal{F}$  *fulfils* the formula  $\langle t \rangle \Gamma$  if each member of  $\Gamma$  belongs to  $x_n$  for infinitely many  $n$ . The role of the axiom Fix is to provide such sequences:

LEMMA 5.1. *In  $\mathcal{F}$ , if  $\langle t \rangle \Gamma \in x$  then there is an endless  $R$ -path starting from  $x$  that fulfils  $\langle t \rangle \Gamma$ . Moreover,  $\langle t \rangle \Gamma$  belongs to every member of this path.*

PROOF. Let  $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ . Put  $x_0 = x$ . From  $\langle t \rangle \Gamma \in x_0$  by axiom Fix we get  $\diamond(\gamma_1 \wedge \langle t \rangle \Gamma) \in x_0$ , so by (5.1) there exists  $x_1 \in W$  with  $x_0R x_1$  and  $\gamma_1, \langle t \rangle \Gamma \in x_1$ . Since  $\langle t \rangle \Gamma \in x_1$ , by Fix again there exists  $x_2 \in W$  with  $x_1R x_2$  and  $\gamma_2, \langle t \rangle \Gamma \in x_2$ . Continuing in this way ad infinitum cycling through the list  $\gamma_1, \dots, \gamma_k$  we generate a sequence fulfilling  $\langle t \rangle \Gamma$ , with  $\gamma_i \in x_n$  whenever  $n \equiv i \pmod k$ , and  $\langle t \rangle \Gamma \in x_n$  for all  $n < \omega$ . ■

---

<sup>2</sup>This does not imply that  $L$  is sound over the class of *all* members of  $\mathcal{K}$ . For example, the well-known Gödel-Löb provability logic has the finite model property over the class of all finite irreflexive transitive frames, but is invalidated by some infinite irreflexive transitive frames.

The canonical model  $\mathcal{M}_L$  on  $\mathcal{F}_L$  has  $h(p) = \{x \in W_L : p \in x\}$  for all  $p \in \text{Var}$ , and has  $\mathcal{M}_L, x \models \varphi$  iff  $\varphi \in x$ , *provided that  $\varphi$  is  $\langle t \rangle$ -free*. But this ‘Truth Lemma’ can fail for formulas containing the tangle connective, even though all instances of the tangle axioms belong to every member of  $W_L$ . For this reason we will work directly with the structure of  $\mathcal{F}_L$  and the membership relation  $\varphi \in x$ , rather than with truth in  $\mathcal{M}_L$ .

For an example of failure of the Truth Lemma, consider the set

$$\Sigma = \{\diamond p_0\} \cup \{\square(p_{2n} \rightarrow \diamond(p_{2n+1} \wedge \neg q)), \square(p_{2n+1} \rightarrow \diamond(p_{2n+2} \wedge q)) : n < \omega\},$$

where  $q$  and the  $p_n$ ’s are distinct variables. Each finite subset of  $\Sigma \cup \{\neg \langle t \rangle \{q, \neg q\}\}$  is satisfiable in a transitive frame, and so is  $\text{K4}t$ -consistent. Explanation: if  $\Gamma$  is a finite subset,  $\mathcal{M}$  a model with transitive frame, and  $\mathcal{M}, x \models \Gamma$ , then  $\{\varphi : \varphi \text{ is true in } \mathcal{M}\}$  is a tangle logic that excludes  $\neg \bigwedge \Gamma$ , so  $\neg \bigwedge \Gamma \notin \text{K4}t$ .

Since the proof theory of  $\text{K4}$  is finitary, it follows that  $\Sigma \cup \{\neg \langle t \rangle \{q, \neg q\}\}$  is  $\text{K4}t$ -consistent, so is included in some member  $x$  of  $W_{\text{K4}t}$ . Using the fact that  $\Sigma \subseteq x$ , together with (5.1) and (5.3), we can construct an endless  $R_{\text{K4}t}$ -path starting from  $x$  that fulfills  $\{q, \neg q\}$ , hence satisfies each of  $q$  and  $\neg q$  infinitely often in  $\mathcal{M}_{\text{K4}t}$ . Thus  $\mathcal{M}_{\text{K4}t}, x \models \langle t \rangle \{q, \neg q\}$ . But  $\langle t \rangle \{q, \neg q\} \notin x$ , since  $\neg \langle t \rangle \{q, \neg q\} \in x$  and  $x$  is  $\text{K4}t$ -consistent.

## 6. Definable Reductions

Fix a finite set  $\Phi$  of formulas closed under subformulas. We now develop a refinement of the filtration method of reducing a model to a finite one that is equivalent in terms of satisfaction of members of  $\Phi$ . Let  $\Phi^t$  be the set of all formulas in  $\Phi$  of the form  $\langle t \rangle \Gamma$ , and  $\Phi^\diamond$  be the set of all formulas in  $\Phi$  of the form  $\diamond \varphi$ .

Let  $\mathcal{F} = (W, R)$  be an inner subframe of  $\mathcal{F}_L$ . Then by a *definable reduction of  $\mathcal{F}$  via  $\Phi$*  we mean a pair  $(\mathcal{M}_\Phi, f)$ , where  $\mathcal{M}_\Phi = (W_\Phi, R_\Phi, h_\Phi)$  is a model on a finite transitive frame,<sup>3</sup> and  $f : W \rightarrow W_\Phi$  is a surjective function, such that the following hold for all  $x, y \in W$ :

- (r1):  $p \in x$  iff  $f(x) \in h_\Phi(p)$ , for all  $p \in \text{Var} \cap \Phi$ .
- (r2):  $f(x) = f(y)$  implies  $x \cap \Phi = y \cap \Phi$ .
- (r3):  $xRy$  implies  $f(x)R_\Phi f(y)$ .
- (r4):  $f(x)R_\Phi f(y)$  implies  $y \cap \Phi^t \subseteq x \cap \Phi^t$  and  $\{\diamond \varphi \in \Phi : \diamond^* \varphi \in y\} \subseteq x$ .

---

<sup>3</sup> $\mathcal{M}_\Phi$  is not uniquely determined by  $\Phi$ .

(r5): For each subset  $C$  of  $W_\Phi$  there is a formula  $\varphi$  that defines  $f^{-1}(C)$  in  $W$ , i.e.  $\varphi \in y$  iff  $f(y) \in C$ .

The existence of definable reductions will be shown later in Section 9. We will be making crucial use of the following consequence of their definition.

LEMMA 6.1. *If  $f(x)$  and  $f(y)$  belong to the same  $R_\Phi$ -cluster, then  $x \cap \Phi^t = y \cap \Phi^t$  and  $x \cap \Phi^\diamond = y \cap \Phi^\diamond$ .*

PROOF. If  $f(x) = f(y)$ , then  $x \cap \Phi = y \cap \Phi$  by (r2) and so  $x \cap \Phi^t = y \cap \Phi^t$  and  $x \cap \Phi^\diamond = y \cap \Phi^\diamond$ . But if  $f(x) \neq f(y)$ , then  $f(x)R_\Phi f(y)R_\Phi f(x)$ , and so  $y \cap \Phi^t \subseteq x \cap \Phi^t \subseteq y \cap \Phi^t$  by (r4). Also if  $\diamond\varphi \in y \cap \Phi$  then  $\diamond^*\varphi = \varphi \vee \diamond\varphi \in y$ , and so  $\diamond\varphi \in x$  by (r4), and likewise  $\diamond\varphi \in x \cap \Phi$  implies  $\diamond\varphi \in y$ . ■

Note that the second conclusion of (r4) is a concise way of expressing that both

$$\{\diamond\varphi \in \Phi : \varphi \in y\} \subseteq x \quad \text{and} \quad \{\diamond\varphi \in \Phi : \diamond\varphi \in y\} \subseteq x.$$

Given a definable reduction  $(\mathcal{M}_\Phi, f)$  of  $\mathcal{F}$ , we will replace  $R_\Phi$  by a weaker relation  $R_t$ , producing a new model  $\mathcal{M}_t = (W_\Phi, R_t, h_\Phi)$ , the *untangling* of  $\mathcal{M}_\Phi$ , with the property that satisfaction in  $\mathcal{M}_t$  of any formula  $\varphi \in \Phi$  corresponds exactly via  $f$  to membership of  $\varphi$  in points of  $\mathcal{F}$ . In other words,

$$\varphi \in x \text{ iff } \mathcal{M}_t, f(x) \models \varphi,$$

a result we refer to as the *Reduction Lemma*. This result could fail if  $\mathcal{M}_\Phi$  is put in place of  $\mathcal{M}_t$ : there may be a formula  $\langle t \rangle \Gamma \in \Phi$  with  $\langle t \rangle \Gamma \notin x$  but  $\Gamma$  is satisfied in  $\mathcal{M}_\Phi$  by some  $R_\Phi$ -cluster coming  $R_\Phi$ -after  $f(x)$ , so that  $\mathcal{M}_\Phi, f(x) \models \langle t \rangle \Gamma$ . The definition of  $R_t$  will cause each  $R_\Phi$ -cluster to be decomposed into a partially ordered set of smaller  $R_t$ -clusters, in such a way that this obstruction is removed.

In what follows we will write  $|x|$  for  $f(x)$ . Then as  $f$  is surjective, each member of  $W_\Phi$  is equal to  $|x|$  for some  $x \in W$ . In later applications the set  $W_\Phi$  will be a set of equivalence classes  $|x|$  of points  $x \in W$ , under a suitable equivalence relation, and  $f$  will be the natural map  $x \mapsto |x|$ .

Our first step makes the key use of the axiom Ind:

LEMMA 6.2. *Let  $\langle t \rangle \Gamma \in \Phi$ . Suppose that  $\langle t \rangle \Gamma \notin x$ , where  $x \in W$ , and let  $|x| \in C \subseteq W_\Phi$ . Then there is a formula  $\gamma \in \Gamma$  and some  $y \in W$  such that  $xR^*y$ ,  $|y| \in C$  and*

$$\text{if } yRz \text{ and } |z| \in C, \text{ then } \gamma \notin z. \quad (6.1)$$

PROOF. By (r5) there is a formula  $\varphi$  that defines  $\{y \in W : |y| \in C\}$ , i.e.  $\varphi \in y$  iff  $|y| \in C$ . Then  $\varphi \in x$  and  $\langle t \rangle \Gamma \notin x$ , so by the axiom Ind,

$\square^*(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \notin x$ . Hence by (5.4) there is a  $y$  with  $xR^*y$  and  $(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \notin y$ . Then  $\varphi \in y$ , so  $|y| \in C$ , and for some  $\gamma \in \Gamma$  we have  $\diamond(\gamma \wedge \varphi) \notin y$ . Hence by (5.1), if  $yRz$  and  $|z| \in C$ , then  $\gamma \wedge \varphi \notin z$  and  $\varphi \in z$ , so  $\gamma \notin z$ , which gives (6.1). ■

LEMMA 6.3. *Let formulas  $\langle t \rangle \Gamma_1, \dots, \langle t \rangle \Gamma_k$  belong to  $\Phi$  but not to  $x$ . Suppose that  $|x| \in C \subseteq W_\Phi$ . Then there are formulas  $\gamma_1 \in \Gamma_1, \dots, \gamma_k \in \Gamma_k$  and some  $y_k \in W$  such that  $xR^*y_k$ ,  $|y_k| \in C$  and*

$$\text{if } y_kRz \text{ and } |z| \in C, \text{ then } \{\gamma_1, \dots, \gamma_k\} \cap z = \emptyset. \quad (6.2)$$

PROOF. By induction on  $k$ . If  $k = 1$ , by Lemma 6.2, there exists  $\gamma_1 \in \Gamma_1$  and  $y_1 \in W$  such that  $xR^*y_1$ ,  $|y_1| \in C$  and

$$\text{if } y_1Rz \text{ and } |z| \in C, \text{ then } \gamma_1 \notin z,$$

which gives (6.2) in this base case.

For the induction case, assume the result holds for  $k$ , and take formulas  $\langle t \rangle \Gamma_1, \dots, \langle t \rangle \Gamma_k, \langle t \rangle \Gamma_{k+1} \in \Phi - x$  with  $|x| \in C$ . Then by the induction hypothesis there are formulas  $\gamma_1 \in \Gamma_1, \dots, \gamma_k \in \Gamma_k$  and some  $y_k \in W$  such that  $xR^*y_k$ ,  $|y_k| \in C$  and (6.2) holds.

Now  $\langle t \rangle \Gamma_{k+1} \notin x$ , so  $\diamond \langle t \rangle \Gamma_{k+1} \notin x$  by scheme 4<sub>t</sub>. Hence  $\diamond^* \langle t \rangle \Gamma_{k+1} = \langle t \rangle \Gamma_{k+1} \vee \diamond \langle t \rangle \Gamma_{k+1} \notin x$ . As  $xR^*y_k$ , this implies  $\langle t \rangle \Gamma_{k+1} \notin y_k$  by (5.2). So by Lemma 6.2, with  $y_k$  in place of  $x$ , there exists  $\gamma_{k+1} \in \Gamma_{k+1}$  and  $y_{k+1} \in W$  such that  $y_kR^*y_{k+1}$ ,  $|y_{k+1}| \in C$  and

$$\text{if } y_{k+1}Rz \text{ and } |z| \in C, \text{ then } \gamma_{k+1} \notin z. \quad (6.3)$$

Now by transitivity of  $R^*$  we have  $xR^*y_{k+1}$ . We now show that (6.2) holds with  $k$  replaced by  $k+1$ . If  $y_{k+1}Rz$  and  $|z| \in C$ , then from  $y_kR^*y_{k+1}Rz$  we get  $y_kRz$ , and so  $\{\gamma_1, \dots, \gamma_k\} \cap z = \emptyset$  by (6.2). Together with (6.3) this shows that  $\{\gamma_1, \dots, \gamma_{k+1}\} \cap z = \emptyset$ . This establishes (6.2) with  $k$  replaced by  $k+1$ , proving that the result holds for  $k+1$  and completing the induction case. ■

Define a formula  $\varphi \in \Phi$  to be *realised* at a member  $|z|$  of  $W_\Phi$  iff  $\varphi \in z$ . Note that this definition does not depend on how the member is named, for if  $|z| = |z'|$ , then  $z \cap \Phi = z' \cap \Phi$  by (r2), and so  $\varphi \in z$  iff  $\varphi \in z'$ .

LEMMA 6.4. *Let  $C$  be any  $R_\Phi$ -cluster. Then there is some  $y \in W$  with  $|y| \in C$ , such that for any formula  $\langle t \rangle \Gamma \in \Phi^t - y$  there is a formula in  $\Gamma$  that is not realised at any  $|z| \in C$  such that  $yRz$ .*

PROOF. Take any  $|x| \in C$ . It  $\Phi^t - x$  is empty, then putting  $y = x$  immediately makes the statement of the Lemma true (vacuously).

Alternatively, if  $\Phi^t - x \neq \emptyset$ , put  $\Phi^t - x = \{\langle t \rangle \Gamma_1, \dots, \langle t \rangle \Gamma_k\}$ . By Lemma 6.3 there is some  $y$  with  $xR^*y$  and  $|y| \in C$ , and formulas  $\gamma_i \in \Gamma_i$  for  $1 \leq i \leq k$  such that if  $yRz$  and  $|z| \in C$ , then  $\gamma_i \notin z$ , hence  $\gamma_i$  is not realised at  $|z|$ .

Now  $|x|$  and  $|y|$  belong to the same  $R_\Phi$ -cluster  $C$ , so  $y \cap \Phi^t = x \cap \Phi^t$  by Lemma 6.1. Hence  $\Phi^t - y = \Phi^t - x$ . So if  $\langle t \rangle \Gamma \in \Phi^t - y$ , then  $\Gamma = \Gamma_i$  for some  $i$ , and then  $\gamma_i$  is a member of  $\Gamma$  not realised at any  $|z| \in C$  such that  $yRz$ . ■

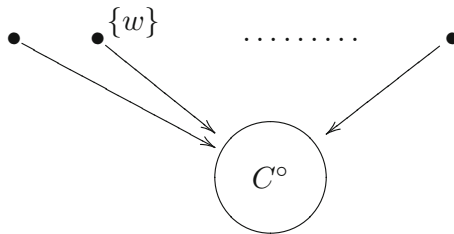
Now for each  $R_\Phi$ -cluster  $C$ , choose and fix a point  $y$  as given by Lemma 6.4. Call  $y$  the *critical point* for  $C$ , and put

$$C^\circ = \{|z| \in C : yRz\}.$$

Lemma 6.4 states that if  $\langle t \rangle \Gamma \in \Phi^t - y$ , then there is a formula in  $\Gamma$  that is not realised at any point of  $C^\circ$ .

We call  $C^\circ$  the *nucleus* of the cluster  $C$ . If  $yRy$  then  $|y| \in C^\circ$ , but in general  $|y|$  need not belong to  $C^\circ$ . Indeed the nucleus could be empty. For instance, it must be empty when  $C$  is a degenerate cluster. To show this, suppose that  $C^\circ \neq \emptyset$ . Then there is some  $|z| \in C$  with  $yRz$ , hence  $|y|R_\Phi|z|$  by (r3), so as  $|y| \in C$  this shows that  $C$  is *non-degenerate*. Consequently, if the nucleus is non-empty then the relation  $R_\Phi$  is universal on it.

We introduce the subrelation  $R_t$  of  $R_\Phi$  to refine the structure of  $C$  by decomposing it into the nucleus  $C^\circ$  as an  $R_t$ -cluster together with a singleton *degenerate*  $R_t$ -cluster  $\{w\}$  for each  $w \in C - C^\circ$ . These degenerate clusters all have  $C^\circ$  as an  $R_t$ -successor but are incomparable with each other. So the structure replacing  $C$  looks like the diagram below, with the black dots being the degenerate clusters determined by the points of  $C - C^\circ$ . Doing this to each cluster of  $(W_\Phi, R_\Phi)$  produces a new *transitive* frame  $\mathcal{F}_t = (W_\Phi, R_t)$  with  $R_t \subseteq R_\Phi$ .



$R_t$  can be more formally defined on  $W_\Phi$  simply by specifying, for each  $w, v \in W_\Phi$ , that  $wR_tv$  iff  $wR_\Phi v$  and either

- $w$  and  $v$  belong to different  $R_\Phi$ -clusters; or
- $w$  and  $v$  belong to the same  $R_\Phi$ -cluster  $C$ , and  $v \in C^\circ$ .

This ensures that each member of  $C$  is  $R_t$ -related to every member of the nucleus of  $C$ . The restriction of  $R_t$  to  $C$  is equal to  $C \times C^\circ$ , so we could also define  $R_t$  as the union of the relations  $C \times C^\circ$  for all  $R_\Phi$ -clusters  $C$ , plus all inter-cluster instances of  $R_\Phi$ .

If the nucleus is empty, then so is the relation  $R_t$  on  $C$ , and  $C$  decomposes into a set of pairwise incomparable degenerate clusters. If  $C = C^\circ$ , then  $R_t$  is universal on  $C$ , identical to the restriction of  $R_\Phi$  to  $C$ .

LEMMA 6.5. (*Reduction Lemma*) *Every formula in  $\Phi$  is true in  $\mathcal{M}_t = (W_\Phi, R_t, h_\Phi)$  precisely at the points at which it is realised, i.e. for all  $\varphi \in \Phi$  and all  $x \in W$ ,*

$$\mathcal{M}_t, |x| \models \varphi \text{ iff } \varphi \in x. \quad (6.4)$$

PROOF. This is by induction on the formation of formulas. For the base case of a variable  $p \in \Phi$ , we have  $\mathcal{M}_t, |x| \models p$  iff  $|x| \in h_\Phi(p)$ , which holds iff  $p \in x$  by (r1). The inductive cases of the Boolean connectives are standard.

Next, take the case of a formula  $\diamond\varphi \in \Phi$ , under the induction hypothesis that (6.4) holds for all  $x \in W$ . Suppose first that  $\mathcal{M}_t, |x| \models \diamond\varphi$ . Then there is some  $y \in W$  with  $|x|R_t|y|$  and  $\mathcal{M}_t, |y| \models \varphi$ , hence  $\varphi \in y$  by the induction hypothesis on  $\varphi$ . Then  $\diamond^*\varphi \in y$ . But  $R_t \subseteq R_\Phi$ , so  $|x|R_\Phi|y|$ , implying that  $\diamond\varphi \in x$ , as required, by (r4). Conversely, suppose that  $\diamond\varphi \in x$ . Let  $C$  be the  $R_\Phi$ -cluster of  $|x|$ , and  $y$  the critical point for  $C$ . Then  $\diamond\varphi \in y$  by Lemma 6.1, so there is some  $z$  with  $yRz$  and  $\varphi \in z$ , hence  $\mathcal{M}_t, |z| \models \varphi$  by induction hypothesis. Now if  $|z| \in C$ , then  $|z|$  belongs to the nucleus of  $C$  and hence  $|x|R_t|z|$ . But if  $|z| \notin C$ , then as  $|y|R_\Phi|z|$  by (r3), and hence  $|x|R_\Phi|z|$ , the  $R_\Phi$ -cluster of  $|z|$  is strictly  $R_\Phi$ -later than  $C$ , and again  $|x|R_t|z|$ . So in any case we have  $|x|R_t|z|$  and  $\mathcal{M}_t, |z| \models \varphi$ , giving  $\mathcal{M}_t, |x| \models \diamond\varphi$ . That completes this inductive case of  $\diamond\varphi$ .

Finally we have the most intricate case of a formula  $\langle t \rangle \Gamma \in \Phi$ , under the induction hypothesis that (6.4) holds for every member of  $\Gamma$  for all  $x \in W$ . Then we have to show that for all  $z \in W$ ,

$$\mathcal{M}_t, |z| \models \langle t \rangle \Gamma \text{ iff } \langle t \rangle \Gamma \in z. \quad (6.5)$$

The proof proceeds by strong induction on the *rank* of  $|z|$  in  $(W_\Phi, R_\Phi)$ , i.e. the number of  $R_\Phi$ -clusters in the longest chain of such clusters starting with the  $R_\Phi$ -cluster of  $|z|$ . Take  $x \in W$  and suppose that (6.5) holds for every  $z$  for which the rank of  $|z|$  is *less than* the rank of  $|x|$ . We show that  $\mathcal{M}_t, |x| \models \langle t \rangle \Gamma$  iff  $\langle t \rangle \Gamma \in x$ . Let  $C$  be the  $R_\Phi$ -cluster of  $|x|$ , and  $y$  the critical point for  $C$  (which does exist by Lemma 6.4, even if  $C$  is degenerate).

Assume first that  $\langle t \rangle \Gamma \in x$ . Then  $\langle t \rangle \Gamma \in y$  by Lemma 6.1. By Lemma 5.1, there is an endless  $R$ -path  $\{y_n : n < \omega\}$  starting from  $y = y_0$  that fulfills

$\langle t \rangle \Gamma$  and has  $\langle t \rangle \Gamma$  belonging to each point. Then by (r3) the sequence  $\{|y_n| : n < \omega\}$  is an endless  $R_\Phi$ -path in  $W_\Phi$  starting at  $|y| \in C$ . But to make  $\langle t \rangle \Gamma$  true at a point in  $\mathcal{M}_t$  we need an endless  $R_t$ -path.

Suppose that  $|y_n| \in C$  for all  $n$ . Then for all  $n > 0$ , since  $yRy_n$  we get  $|y_n| \in C^\circ$ . So there is the endless  $R_t$ -path  $\pi = |x|R_t|y_1|R_t|y_2|R_t \cdots$  starting at  $|x|$ . As  $\{y_n : n < \omega\}$  fulfills  $\langle t \rangle \Gamma$ , for each  $\gamma \in \Gamma$  there are infinitely many  $n$  for which  $\gamma \in y_n$  and so  $\mathcal{M}_t, |y_n| \models \gamma$  by the induction hypothesis on members of  $\Gamma$ . Thus each member of  $\Gamma$  is true infinitely often along  $\pi$ , implying that  $\mathcal{M}_t, |x| \models \langle t \rangle \Gamma$ .

If however there is an  $n > 0$  with  $|y_n| \notin C$ , then the  $R_\Phi$ -cluster of  $|y_n|$  is strictly  $R_\Phi$ -later than  $C$ , so  $|x|R_t|y_n|$  and  $|y_n|$  has smaller rank than  $|x|$ . Since  $\langle t \rangle \Gamma \in y_n$ , the induction hypothesis (6.5) on rank then implies that  $\mathcal{M}_t, |y_n| \models \langle t \rangle \Gamma$ . So there is an endless  $R_t$ -path  $\pi$  from  $|y_n|$  along which each member of  $\Gamma$  is true infinitely often. Since  $|x|R_t|y_n|$ , we can append  $|x|$  to the front of  $\pi$  to obtain such an  $R_t$ -path starting from  $|x|$ , showing that  $\mathcal{M}_t, |x| \models \langle t \rangle \Gamma$  (this last part is an argument for soundness of  $4_t$ ). So in both cases we get  $\mathcal{M}_t, |x| \models \langle t \rangle \Gamma$ . That proves the forward implication of (6.4) for  $\langle t \rangle \Gamma$ .

For the converse implication, suppose  $\mathcal{M}_t, |x| \models \langle t \rangle \Gamma$ . Since  $W_\Phi$  is finite, it follows by (3.1) that there exists a  $z \in W$  with  $|x|R_t|z|$  and  $|z|R_t|z|$  and the  $R_t$ -cluster of  $|z|$  satisfies  $\Gamma$ . By the induction hypothesis (6.4) on members of  $\Gamma$ , every formula in  $\Gamma$  is realised at some point of this cluster. Suppose first there is such a  $z$  for which the rank of  $|z|$  is less than that of  $|x|$ . Then as the  $R_t$ -cluster of  $|z|$  is non-degenerate and satisfies  $\Gamma$ , we have  $\mathcal{M}_t, |z| \models \langle t \rangle \Gamma$ . Induction hypothesis (6.5) then implies that  $\langle t \rangle \Gamma \in z$ . But  $|x|R_\Phi|z|$ , as  $|x|R_t|z|$ , so by (r4) we get the required conclusion that  $\langle t \rangle \Gamma \in x$ .

If however there is no such  $z$  with  $|z|$  of lower rank than  $|x|$ , then the  $|z|$  that does exist must have the same rank as  $|x|$ , so it belongs to  $C$ . Hence as  $|x|R_t|z|$ , the definition of  $R_t$  implies that  $|z| \in C^\circ$ . Thus the  $R_t$ -cluster of  $|z|$  is  $C^\circ$ . Therefore every formula in  $\Gamma$  is realised at some point of  $C^\circ$ , i.e. at some  $|z'| \in C$  with  $yRz'$ . But Lemma 6.4 states that if  $\langle t \rangle \Gamma \notin y$ , then some member of  $\Gamma$  is not realised in  $C^\circ$ . Therefore we must have  $\langle t \rangle \Gamma \in y$ . Then  $\langle t \rangle \Gamma \in x$  as required, by Lemma 6.1. That finishes the inductive proof that  $\mathcal{M}_t$  satisfies the Reduction Lemma. ■

## 7. Adding Seriality

If the logic  $L$  contains the D-axiom  $\diamond\top$ , then  $R_L$  is *serial*:  $\forall x \exists y (xR_L y)$ . This follows from (5.1), since each  $x \in W_L$  has  $\diamond\top \in x$ . The relation  $R$  of the inner subframe  $\mathcal{F}$  is then also serial. From this we can show that  $R_t$  is serial.

The key point is that any maximal  $R_\Phi$ -cluster  $C$  must have a *non-empty* nucleus. For, if  $y$  is the critical point for  $C$ , then there is a  $z$  with  $yRz$ , as  $R$  is serial. But then  $|y|R_\Phi|z|$  by (r3) and so  $|z| \in C$  as  $C$  is maximal. Hence  $|z| \in C^\circ$ , making the nucleus non-empty. Now every member of  $C$  is  $R_t$ -related to any member of  $C^\circ$  so altogether this implies that  $R_t$  is serial on the rank 1 cluster  $C$ . But any point of rank  $> 1$  will be  $R_t$ -related to points of lower rank, and indeed to points in the nucleus of some rank 1 cluster. Since  $R_t$  is reflexive on a nucleus, this shows that  $R_t$  satisfies the stronger condition that  $\forall w \exists v (wR_tvR_tv)$  — “every world sees a reflexive world”.

## 8. Adding Reflexivity

Suppose that  $L$  contains the scheme

$$\mathbf{T}: \varphi \rightarrow \diamond\varphi.$$

Then it contains

$$\mathbf{T}_t: \bigwedge \Gamma \rightarrow \langle t \rangle \Gamma.$$

To see this, let  $\varphi = \bigwedge \Gamma$ . Then  $\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} (\gamma \wedge \varphi)$  is a tautology, hence derivable. From that we derive

$$\square^*(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \tag{8.1}$$

using the instances  $(\gamma \wedge \varphi) \rightarrow \diamond(\gamma \wedge \varphi)$  of axiom T and K-principles. But (8.1) is an antecedent of axiom Ind, so we apply it to derive  $\varphi \rightarrow \langle t \rangle \Gamma$ , which is  $\mathbf{T}_t$  in this case.

Axiom T ensures that the canonical frame relation  $R_L$  is reflexive, and hence so is  $R_\Phi$  by (r3). Thus no  $R_\Phi$ -cluster is degenerate. We modify the definition of  $R_t$  to make it reflexive as well. The change occurs in the case of an  $R_\Phi$ -cluster  $C$  having  $C \neq C^\circ$ . Then instead of making the singletons  $\{w\}$  for  $w \in C - C^\circ$  be degenerate, we make them all into *non- $R_t$ -degenerate* clusters by requiring that  $wR_tw$ . Formally this is done by adding to the definition of  $wR_tv$  the third possibility that

- $w$  and  $v$  belong to the same  $R_\Phi$ -cluster  $C$ , and  $w = v \in C - C^\circ$ .

Equivalently, the restriction of  $R_t$  to  $C$  is equal to  $(C \times C^\circ) \cup \{(w, w) : w \in C - C^\circ\}$ .

The proof of the Reduction Lemma for the resulting reflexive and transitive model  $\mathcal{M}_t$  now requires an adjustment in one place, in its last paragraph, where  $|x|R_t|z| \in C$ . In the original proof above, this implied that



the  $R_t$ -cluster of  $|z|$  is  $C^\circ$ . But now we have the new possibility that  $|x| = |z| \in C - C^\circ$ . Then the  $R_t$ -cluster of  $|z|$  is  $\{|z|\}$ , so every formula of  $\Gamma$  is realised at  $|z|$ , implying  $\bigwedge \Gamma \in z$ . The scheme  $T_t$  now ensures that  $\langle t \rangle \Gamma \in z$ , so by Lemma 6.1 we still get the required result that  $\langle t \rangle \Gamma \in x$ , and the Reduction Lemma still holds for this modified reflexive version of  $\mathcal{M}_t$ .

### 9. Finite Model Property for $K4t$ , $KD4t$ and $S4t$

Given a tangle logic  $L$  and a finite set  $\Phi$  of formulas closed under subformulas, we can construct a definable reduction of any inner subframe  $\mathcal{F} = (W, R)$  of  $\mathcal{F}_L$  by filtration through  $\Phi$ . An equivalence relation  $\sim$  on  $W$  is given by putting  $x \sim y$  iff  $x \cap \Phi = y \cap \Phi$ . Then with  $|x| = \{y \in W : x \sim y\}$  we put  $W_\Phi = \{|x| : x \in W\}$ . The set  $W_\Phi$  is finite, because the map  $|x| \mapsto x \cap \Phi$  is a well-defined injection of  $W_\Phi$  into the finite powerset  $\wp\Phi$ . Thus  $W_\Phi$  has size at most  $2^{\text{size } \Phi}$ .

Letting  $R_\lambda = \{(|x|, |y|) : xRy\}$  (the least filtration of  $R$  through  $\Phi$ ), we define  $R_\Phi \subseteq W_\Phi \times W_\Phi$  to be the *transitive closure* of  $R_\lambda$ . Thus  $wR_\Phi v$  iff there exist  $w_1, \dots, w_n \in W_\Phi$ , for some  $n > 1$ , such that  $w = w_1 R_\lambda \dots R_\lambda w_n = v$ . The definition of  $\mathcal{M}_\Phi$  is completed by putting  $h_\Phi(p) = \{|x| : p \in x\}$  for  $p \in \Phi$ , and  $h_\Phi(p) = \emptyset$  (or anything) otherwise. We call  $\mathcal{M}_\Phi$  the *standard transitive filtration through  $\Phi$* .

The surjective function  $f : W \rightarrow W_\Phi$  is given by  $f(x) = |x|$ . The conditions (r1) and (r2) for a definable reduction are then immediate, and the definability condition (r5) is standard (e.g. [7, p. 36]). For (r3) observe that  $xRy$  implies  $|x|R_\lambda|y|$  and hence  $|x|R_\Phi|y|$ .

(r4) takes more work, but is also standard for the case of  $\diamond$ , and similar for  $\langle t \rangle$ . To prove it, let  $|x|R_\Phi|y|$ . Then by definition of  $R_\Phi$  as the transitive closure of  $R_\lambda$ , there are finitely many elements  $x_1, y_1, \dots, x_n, y_n$  of  $W$  (for some  $n \geq 1$ ) such that

$$x \sim x_1 R y_1 \sim x_2 R y_2 \sim \dots \sim x_n R y_n \sim y.$$

Then  $\langle t \rangle \Gamma \in y \cap \Phi^t$  implies  $\langle t \rangle \Gamma \in y_n$  as  $y_n \sim y$ , hence  $\diamond \langle t \rangle \Gamma \in x_n$  as  $x_n R y_n$ , which implies  $\langle t \rangle \Gamma \in x_n$  by the scheme  $4_t$ . If  $n = 1$  we then get  $\langle t \rangle \Gamma \in x$  because  $x \sim x_1$ . But if  $n > 1$ , we repeat this argument back along the above chain of relations, leading to  $\langle t \rangle \Gamma \in x_{n-1}, \dots, \langle t \rangle \Gamma \in x_1$ , and then  $\langle t \rangle \Gamma \in x$  as required to conclude that  $y \cap \Phi^t \subseteq x \cap \Phi^t$ .

To show that  $\{\diamond \varphi \in \Phi : \diamond^* \varphi \in y\} \subseteq x$ , note that if  $\diamond^* \varphi \in y$ , then either  $\varphi \in y$  or  $\diamond \varphi \in y$ . If  $\varphi \in y$ , then  $\varphi \in y_n$  as  $y_n \sim y$  and  $\varphi \in \Phi$ , hence  $\diamond \varphi \in x_n$  as  $x_n R y_n$ . But if  $\diamond \varphi \in y$  then  $\diamond \varphi \in y_n$ , hence  $\diamond \diamond \varphi \in x_n$ , and so again

$\diamond\varphi \in x_n$ , this time by scheme 4. Repeating this back along the chain leads to  $\diamond\varphi \in x$  as required.<sup>4</sup>

Thus  $(\mathcal{M}_\Phi, f)$  as constructed is a definable reduction of  $\mathcal{F}$ .

PROPOSITION 9.1. 1. *K4t, the smallest tangle logic, has the finite model property over transitive frames.*

2. *KD4t, the smallest tangle logic containing  $\diamond\top$ , has the finite model property over serial transitive frames.*

3. *S4t, the smallest tangle logic containing  $\top$ , has the finite model property over reflexive transitive (i.e. S4) frames.*

PROOF. 1. Let  $L = \text{K4t}$  and let  $\mathcal{F}$  be its canonical frame  $\mathcal{F}_{\text{K4t}}$ . If  $\varphi$  is a K4t-consistent formula then  $\varphi \in x$  for some point  $x$  of  $\mathcal{F}$ . Let  $\Phi$  be the set of subformulas of  $\varphi$ , and  $\mathcal{M}_t$  the untangling of the standard transitive filtration  $\mathcal{M}_\Phi$  through  $\Phi$  as just defined. Then  $\mathcal{M}_t, |x| \models \varphi$  by the Reduction Lemma. Since the finite frame  $\mathcal{F}_t = (W_\Phi, R_t)$  is transitive, this shows that  $\varphi$  is true at a point of some finite transitive model. But all transitive frames validate K4t.

2. If we replace K4t in (1) by the smallest tangle logic containing  $\diamond\top$ , then the frame  $\mathcal{F}_t$  is serial by Section 7, hence it validates  $\diamond\top$  and thus validates KD4t. Thus KD4t has the finite model property over serial transitive frames, which are precisely the frames that validate the  $\mathcal{L}_{\square}$ -logic KD4.

3. By Section 8 we get that if  $L$  contains the scheme  $\top$ , then the frame  $\mathcal{F}_t$  above is reflexive, so it validates  $\top$  and thus validates S4t. ■

## 10. Universal Modality

Extend the syntax of  $\mathcal{L}_{\square}^{(t)}$  to include the universal modality  $\forall$  with semantics

$$\mathcal{M}, x \models \forall\varphi \text{ iff for all } y \in W, \mathcal{M}, y \models \varphi.$$

Let  $\mathcal{L}_{\square\forall}^{(t)}$  be the resulting language, and K4t.U be the smallest tangle logic in this language that includes the S5 axioms and rules for  $\forall$ , and the scheme

$$\mathbf{U}: \forall\varphi \rightarrow \square\varphi,$$

---

<sup>4</sup>The arguments in the last two paragraphs could be made more formal by proving by induction over all  $k$  having  $0 \leq k < n$  that  $(t)\Gamma \in x_{n-k}$  and  $\diamond\varphi \in x_{n-k}$ .

equivalently  $\diamond\varphi \rightarrow \exists\varphi$ , where  $\exists = \neg\forall\neg$  is the dual modality to  $\forall$ . These axioms and rules involving  $\forall$  are sound in any model.

Let  $L$  be any tangle logic in  $\mathcal{L}_{\square\forall}^{(t)}$  that extends  $\text{K4t.U}$ . Define a relation  $S_L$  on  $W_L$  by:  $xS_Ly$  iff  $\{\varphi : \forall\varphi \in x\} \subseteq y$ . Then also  $xS_Ly$  iff  $\{\exists\varphi : \varphi \in y\} \subseteq x$ , and  $S_L$  is an equivalence relation with  $R_L \subseteq S_L$ . Moreover,

$$\forall\varphi \in x \text{ iff for all } y \in W_L, xS_Ly \text{ implies } \varphi \in y$$

(this is essentially the result (5.3) for the modality  $\forall$  in place of  $\square$ ). For any fixed  $x \in W_L$ , let  $W^x$  be the equivalence class  $S_L(x) = \{y \in W_L : xS_Ly\}$ . Then for  $z \in W^x$ ,

$$\forall\varphi \in z \text{ iff for all } y \in W^x, \varphi \in y. \tag{10.1}$$

Let  $R^x$  be the restriction of  $R_L$  to  $W^x$ . Since  $R_L \subseteq S_L$  it follows that  $\mathcal{F}^x = (W^x, R^x)$  is an inner subframe of  $(W_L, R_L)$ . If  $\mathcal{M}_\Phi$  is a definable reduction of  $\mathcal{F}^x$ , and  $\mathcal{M}_t$  its untangling, then using (10.1) it can be shown that if a formula  $\varphi \in \Phi$  satisfies the Reduction Lemma

$$\mathcal{M}_t, |z| \models \varphi \text{ iff } \varphi \in z$$

for all  $z$  in  $W^x$ , then so does  $\forall\varphi$ . So the Reduction Lemma holds for all members of  $\Phi$ .

**PROPOSITION 10.1.** *K4t.U has the finite model property over transitive frames; KD4t.U has the finite model property over serial transitive frames; and S4t.U has the finite model property over reflexive transitive frames.*

**PROOF.** The standard transitive filtration can be applied to  $\mathcal{F}^x$  to produce a definable reduction of it. Consequently, if  $L$  is a tangle logic in  $\mathcal{L}_{\square\forall}^{(t)}$  that extends  $\text{K4t.U}$  as above,  $\varphi$  is an  $L$ -consistent formula,  $x$  is a point of  $W_L$  with  $\varphi \in x$ , and  $\Phi$  is the set of all subformulas of  $\varphi$ , then  $\mathcal{M}_t, |x| \models \varphi$  where  $\mathcal{M}_t$  is the untangling of the standard transitive filtration of  $\mathcal{F}^x$  through  $\Phi$ . Since  $\text{K4t.U}$  is valid in any transitive frame this gives the finite model property for  $\text{K4t.U}$  over transitive frames.

This construction preserves seriality and reflexiveness in passing from  $R_L$  to  $R^x$  and then  $R_t$ . Consequently, the finite model property holds for the tangle systems  $\text{KD4t.U}$  and  $\text{S4t.U}$  over the  $\text{KD4}$  and  $\text{S4}$  frames, respectively. ■

## 11. Path Connectedness

A *connecting path* between  $w$  and  $v$  in a frame  $(W, R)$  is a finite sequence  $w = w_0, \dots, w_n = v$ , for some  $n \geq 0$ , such that for all  $i < n$ , either  $w_i R w_{i+1}$

or  $w_{i+1}Rw_i$ . We say that such a path has *length*  $n$ . The points  $w$  and  $v$  of  $W$  are *path connected* if there exists a connecting path between them of some finite length. Note that any point  $w$  is connected to itself by a path of length 0 (put  $n = 0$  and  $w = w_0$ ). The relation “ $w$  and  $v$  are path connected” is an equivalence relation whose equivalence classes are the *path components* of the frame. The frame is *path connected* if it has a single path component, i.e. any two points have a connecting path between them.

Later we will make use of the fact that a path component  $P$  is  $R$ -closed. For if  $x \in P$  and  $xRy$ , then  $x$  and  $y$  are path connected, so  $y \in P$ . It follows that any  $R$ -cluster  $C$  that intersects  $P$  must be included in  $P$ , for if  $x \in P \cap C$  and  $y \in C$ , then  $xR^*y$  and so  $y \in P$ , showing that  $C \subseteq P$ .

We now wish to show that in passing from the frame  $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$  to its untangling  $\mathcal{F}_t$ , as above, there is no loss of path connectivity. The two frames have the same path connectedness relation and so have the same path components. The idea is that the relations that are broken by the untangling only occur between elements of the same  $R_\Phi$ -cluster, so it suffices to show that such elements are still path connected in  $\mathcal{F}_t$ . *For this we need to make the assumption that  $\Phi$  contains the formula  $\diamond\top$ .* This is harmless as we can always add it and its subformula  $\top$ , preserving finiteness of  $\Phi$ .

LEMMA 11.1. *Let  $\diamond\top \in \Phi$ . If  $w, w'$  are points in  $W_\Phi$  with  $wR_\Phi w'$  or  $w'R_\Phi w$ , but neither  $wR_t w'$  or  $w'R_t w$ , then there exist a  $v$  with  $wR_t v$  and  $w'R_t v$ .*

PROOF. If  $wR_\Phi w'$ , then since not  $wR_t w'$  we must have  $w$  and  $w'$  in the same cluster. The same follows if  $w'R_\Phi w$ , since not  $w'R_t w$ .

Thus there is an  $R_\Phi$ -cluster  $C$  with  $w, w' \in C$ , so both  $wR_\Phi w'$  and  $w'R_\Phi w$ . If  $C$  is not  $R_\Phi$ -maximal, then there is an  $R_\Phi$ -cluster  $C'$  with  $CR_\Phi C'$  and  $C \neq C'$ . Taking any  $v \in C'$  we then get  $wR_t v$  and  $w'R_t v$ .

The alternative is that  $C$  is  $R_\Phi$ -maximal. Then we show that the nucleus  $C^\circ$  is non-empty. Let  $w = |u|$  and  $w' = |s|$ . Since  $|u|R_\Phi|s|$ ,  $\top \in s$ , and  $\diamond\top \in \Phi$ , property (r4) implies that  $\diamond\top \in u$ . Now if  $y$  is the critical point for  $C$ , then  $\diamond\top \in y$  by Lemma 6.1. Hence there is a  $z$  with  $yRz$ . So  $|y|R_\Phi|z|$  by (r3). Maximality of  $C$  then ensures that  $|z| \in C$ , so this implies that  $|z| \in C^\circ$ . Then by definition of  $R_t$ , since  $w, w' \in C$  we have  $wR_t|z|$  and  $w'R_t|z|$ . ■

LEMMA 11.2. *If  $\diamond\top \in \Phi$ , then two members of  $W_\Phi$  are path connected in  $\mathcal{F}_\Phi$  if, and only if, they are path connected in  $\mathcal{F}_t$ . Hence the two frames have the same path components.*

PROOF. Since  $R_t \subseteq R_\Phi$ , a connecting path in  $\mathcal{F}_t$  is a connecting path in  $\mathcal{F}_\Phi$ , so points that are path connected in  $\mathcal{F}_t$  are path connected in  $\mathcal{F}_\Phi$ .

Conversely, let  $\pi = w_0, \dots, w_n$  be a connecting path in  $\mathcal{F}_\Phi$ . If, for all  $i < n$ , either  $w_i R_t w_{i+1}$  or  $w_{i+1} R_t w_i$ , then  $\pi$  is a connecting path in  $\mathcal{F}_t$ . If not, then for each  $i$  for which this fails, by Lemma 11.1 there exists some  $v_i$  with  $w_i R_t v_i$  and  $w_{i+1} R_t v_i$ . Insert  $v_i$  between  $w_i$  and  $w_{i+1}$  in the path. Doing this for all “defective”  $i < n$ , creates a new sequence that is now a connecting path in  $\mathcal{F}_t$  between the same endpoints. ■

Now let  $\text{K4}t.\text{UC}$  be the smallest extension of system  $\text{K4}t.\text{U}$  in the language  $\mathcal{L}_{\Box\forall}^{(t)}$  that includes the scheme

$$\text{C: } \forall(\Box^* \varphi \vee \Box^* \neg \varphi) \rightarrow (\forall \varphi \vee \forall \neg \varphi),$$

or equivalently  $\exists \varphi \wedge \exists \neg \varphi \rightarrow \exists(\Diamond^* \varphi \wedge \Diamond^* \neg \varphi)$ . This scheme is valid in any path connected frame [16].

Let  $L$  be any  $\text{K4}t.\text{UC}$ -logic. Let  $\mathcal{F}^x$  be a point-generated subframe of  $(W_L, R_L)$  as above, and  $\mathcal{M}_\Phi$  its standard transitive filtration through  $\Phi$ . Then the frame  $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$  of  $\mathcal{M}_\Phi$  is path connected, as shown by Shehtman [16] as follows. If  $P$  is the path component of  $|x|$  in  $\mathcal{F}_\Phi$ , take a formula  $\varphi$  that defines  $f^{-1}(P)$  in  $W^x$ , i.e.  $\varphi \in y$  iff  $|y| \in P$ , for all  $y \in W^x$ . Suppose, for the sake of contradiction, that  $P \neq W_\Phi$ . Then there is some  $z \in W^x$  with  $|z| \notin P$ , hence  $\neg \varphi \in z$ . Since  $\varphi \in x$ , this gives  $\exists \varphi \wedge \exists \neg \varphi \in x$ . By the scheme C and (10.1) it follows that for some  $y \in W^x$ ,  $\Diamond^* \varphi \wedge \Diamond^* \neg \varphi \in y$ . Hence there are  $s, u \in W^x$  with  $y R^* s$ ,  $\varphi \in s$ ,  $y R^* u$  and  $\neg \varphi \in u$ .

From this we get  $|y| R_\Phi^* |s|$  and  $|y| R_\Phi^* |u|$  so the sequence  $|s|, |y|, |u|$  is a connecting path between  $|s|$  and  $|u|$  in  $\mathcal{F}_\Phi$ . But  $|s| \in P$  as  $\varphi \in s$ , so this implies  $|u| \in P$ . Hence  $\varphi \in u$ , contradicting the fact that  $\neg \varphi \in u$ . The contradiction forces us to conclude that  $P = W_\Phi$ , and hence that  $\mathcal{F}_\Phi$  is path connected.

From Lemma 11.2 it now follows that the untangling  $\mathcal{F}_t$  of  $\mathcal{F}_\Phi$  is also path connected when  $\Diamond \top \in \Phi$ . Thus if  $\varphi$  is an  $L$ -consistent formula, we take  $\Phi$  to be the finite set of all subformulas of  $\varphi$  or  $\Diamond \top$  and proceed as in the  $\text{K4}t.\text{U}$  case to obtain a model  $\mathcal{M}_t$  that has  $\varphi$  true at some point, and is based on a path connected frame by the argument just given, because  $L$  now includes scheme C and  $\Diamond \top \in \Phi$ . But path connected frames validate  $\text{K4}t.\text{UC}$ . Moreover, the arguments for the preservation of seriality and reflexivity by  $\mathcal{F}_t$  continue to hold here. So these observations establish the following.

**PROPOSITION 11.3.**  *$\text{K4}t.\text{UC}$  has the finite model property over the path-connected transitive frames;  $\text{KD4}t.\text{UC}$  has the finite model property over path-connected serial transitive frames; and  $\text{S4}t.\text{UC}$  has the finite model property over path-connected reflexive transitive frames.* ■

Note that for the  $\mathcal{L}_{\square\forall}$ -fragments of these logics (i.e. their restrictions to the language without  $\langle t \rangle$ ), our analysis reconstructs the finite model property proof of [16] by using  $\mathcal{M}_\Phi$  instead of  $\mathcal{M}_t$ . For, restricting to this language, if  $\mathcal{M}_\Phi$  is a standard transitive filtration of an inner subframe of  $\mathcal{F}_L$ , then any  $\langle t \rangle$ -free formula is true in  $\mathcal{M}_\Phi$  precisely at the points at which it is realised (for  $\mathcal{L}_\square$  this is a classical result first formulated and proved in [14]). Thus a finite satisfying model for a consistent  $\mathcal{L}_{\square\forall}$ -formula can be obtained as a model of this form  $\mathcal{M}_\Phi$ . Since seriality and reflexivity are preserved in passing from  $R_L$  to  $R_\Phi$ , and  $\mathcal{F}_\Phi$  is path connected in the presence of axiom C, this implies that the finite model property holds for each of the systems K4.UC, KD4.UC and S4.UC in the language  $\mathcal{L}_{\square\forall}$ .

## 12. The Schemes $G_n$

Fix  $n \geq 1$  and take  $n + 1$  variables  $p_0, \dots, p_n$ . For each  $i \leq n$ , define the formula

$$Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j. \quad (12.1)$$

$G_n$  is the scheme consisting of all uniform substitution instances of the  $\mathcal{L}_{\square}$ -formula

$$\bigwedge_{i \leq n} \diamond Q_i \rightarrow \diamond(\bigwedge_{i \leq n} \diamond^* \neg Q_i). \quad (12.2)$$

This is a theorem of S4, indeed of KT, and is true in any model at any reflexive point.

(12.2) is equivalent in any logic to

$$\square(\bigvee_{i \leq n} \square^* Q_i) \rightarrow \bigvee_{i \leq n} \square \neg Q_i,$$

the form in which the  $G_n$ 's were introduced in [15]. When  $n = 1$ , (12.2) is

$$\diamond(p_0 \wedge \neg p_1) \wedge \diamond(p_1 \wedge \neg p_0) \rightarrow \diamond(\diamond^* \neg(p_0 \wedge \neg p_1) \wedge \diamond^* \neg(p_1 \wedge \neg p_0)). \quad (12.3)$$

As an axiom, (12.3) is equivalent to

$$\diamond p \wedge \diamond \neg p \rightarrow \diamond(\diamond^* p \wedge \diamond^* \neg p), \quad (12.4)$$

or in dual form  $\square(\square^* p \vee \square^* \neg p) \rightarrow \square p \vee \square \neg p$ , which is the form in which  $G_1$  was first defined in [15]. To derive (12.4) from (12.3), substitute  $p$  for  $p_0$  and  $\neg p$  for  $p_1$  in (12.3). Conversely, substituting  $p_0 \wedge \neg p_1$  for  $p$  in (12.4) leads to a derivation of (12.3).

For the semantics of  $G_n$ , we use the set  $R(x) = \{y \in W : xRy\}$  of  $R$ -successors of  $x$  in a frame  $(W, R)$ . We can view  $R(x)$  as a frame in its own

right, under the restriction of  $R$  to  $R(x)$ , and consider whether it is path connected, or how many path components it has etc.  $(W, R)$  is called *locally  $n$ -connected* if, for all  $x \in W$ , the frame  $\mathcal{F}(x) = (R(x), R \upharpoonright R(x))$  has at most  $n$  path components. Note that path components in  $\mathcal{F}(x)$  are defined by connecting paths in  $(W, R)$  that lie entirely within  $R(x)$ . If  $x$  is reflexive, then  $R(x)$  has a single path component: any  $y, z \in R(x)$  have the connecting path  $y, x, z$  since  $x \in R(x)$ .

A K4 frame validates  $G_n$  iff it is locally  $n$ -connected. For a proof of this see [12, Theorem 3.7].

When  $\diamond\varphi$  is interpreted in a topological space as the set of limit points of the set interpreting  $\varphi$ , then the  $\mathcal{L}_\square$ -logic of  $\mathbb{R}^n$  is  $\text{KD4G}_1$  for  $n \geq 2$ , and is  $\text{KD4G}_2$  when  $n = 1$ . This was shown by Shehtman [15, 17], and was the motivation for introducing the  $G_n$ 's. The  $n = 1$  result was also proven by Lucero-Bryan [12].

### 13. Weak Models

We now assume that the set  $\text{Var}$  of variables is *finite*. The adjective “weak” is sometimes applied to languages with finitely many variables, as well as to models for weak languages and to canonical frames built from them. Weak models may enjoy special properties. For instance, a proof is given by Shehtman in [15, Lemma 8] that in a weak *distinguished*<sup>5</sup> model on a transitive frame, there are only finitely many maximal clusters. This was used to show that a weak canonical frame for the  $\mathcal{L}_\square$ -logic  $\text{KD4G}_1$  is locally 1-connected, giving a completeness theorem for  $\text{KD4G}_1$  over locally 1-connected frames, and then from this to obtain the finite model property for that logic by filtration. The corresponding versions of these results for  $\text{KD4G}_n$  with  $n \geq 2$  are worked out in [12].

We wish to lift these results to the language  $\mathcal{L}_\square^{(t)}$  with tangle. One issue is that the property of a canonical model being distinguished depends on it satisfying the Truth Lemma:  $\mathcal{M}_L, x \models \varphi$  iff  $\varphi \in x$ . As we have seen, this can fail for tangle logics. Therefore we must continue to work directly with the relation of membership of formulas in points of  $W_L$ , rather than with their truth in  $\mathcal{M}_L$ . We will see that it is still possible to recover Shehtman’s analysis of maximal clusters in  $\mathcal{F}_L$  with the help of the tangle axioms  $\text{Fix}$  and  $\text{Ind}$ .

---

<sup>5</sup>A model is distinguished if for any two of its distinct points there is a formula that is true in the model at one of the points and not the other.

Another issue is that we want to work over  $K4G_n$  without assuming the seriality axiom. This requires further adjustments, and care with the distinction between  $R$  and  $R^*$ .

Let  $L$  be any tangle logic in our weak language. Put  $\text{At} = \text{Var} \cup \{\diamond\top\}$ . For each  $s \subseteq \text{At}$  define the formula

$$\chi(s) = \bigwedge_{\varphi \in s} \varphi \wedge \bigwedge_{\varphi \in \text{At} \setminus s} \neg\varphi.$$

For each point  $x$  of  $W_L$  define  $\tau(x) = x \cap \text{At}$ . Think of  $\text{At}$  as a set of “atoms” and  $\tau(x)$  as the “atomic type” of  $x$ . It is evident that for any  $x \in W_L$  and  $s \subseteq \text{At}$  we have

$$\chi(s) \in x \text{ iff } s = \tau(x). \quad (13.1)$$

Writing  $\chi(x)$  for the formula  $\chi(\tau(x))$ , we see from (13.1) that  $\chi(x) \in x$ , and in general  $\chi(y) \in x$  iff  $\tau(y) = \tau(x)$ .

Now fix an inner subframe  $\mathcal{F} = (W, R)$  of  $\mathcal{F}_L$ . If  $C$  is an  $R$ -cluster in  $\mathcal{F}$ , let

$$\delta C = \{\tau(x) : x \in C\}$$

be the set of atomic types of members of  $C$ . We are going to show that maximal clusters in  $\mathcal{F}$  are determined by their atomic types. The key to this is:

**LEMMA 13.1.** *Let  $C$  and  $C'$  be maximal clusters in  $\mathcal{F}$  with  $\delta C = \delta C'$ . Then for all formulas  $\varphi$ , if  $x \in C$  and  $x' \in C'$  have  $\tau(x) = \tau(x')$ , then  $\varphi \in x$  iff  $\varphi \in x'$ . Thus,  $x = x'$ .*

**PROOF.** Suppose  $C$  and  $C'$  are maximal with  $\delta C = \delta C'$ . The key property of maximality that is used is that if  $x \in C$  and  $xRy$ , then  $y \in C$ , and likewise for  $C'$ .

The proof proceeds by induction on the formation of  $\varphi$ . The base case, when  $\varphi \in \text{Var}$ , is immediate from the fact that then  $\varphi \in x$  iff  $\varphi \in \tau(x)$ . The induction cases for the Boolean connectives are straightforward from properties of maximally consistent sets.

Now take the case of a formula  $\diamond\varphi$  under the induction hypothesis that the result holds for  $\varphi$ , i.e.  $\varphi \in x$  iff  $\varphi \in x'$  for any  $x \in C$  and  $x' \in C'$  such that  $\tau(x) = \tau(x')$ . Take such  $x$  and  $x'$ , and assume  $\diamond\varphi \in x$ . Then  $\varphi \in y$  for some  $y$  such that  $xRy$ . Then  $y \in C$  as  $C$  is maximal. Hence  $\tau(y) \in \delta C = \delta C'$ , so  $\tau(y) = \tau(y')$  for some  $y' \in C'$ . Therefore  $\varphi \in y'$  by the induction hypothesis on  $\varphi$ . But  $\diamond\top \in x$  (as  $xRy$ ), so  $\diamond\top \in \tau(x) = \tau(x')$ . This gives  $\diamond\top \in x'$  which ensures that  $x'Rz$  for some  $z$ , with  $z \in C'$  as  $C'$  is maximal, hence  $C'$



is a non-degenerate cluster.<sup>6</sup> It follows that  $x'Ry'$ , so  $\diamond\varphi \in x'$  as required. Likewise  $\diamond\varphi \in x'$  implies  $\diamond\varphi \in x$ , and the Lemma holds for  $\diamond\varphi$ .

Finally we have the case of a formula  $\langle t \rangle \Gamma$  under the induction hypothesis that the result holds for every  $\gamma \in \Gamma$ . Suppose  $x \in C$  and  $\tau(x) = \tau(x')$  for some  $x' \in C'$ . Let  $\langle t \rangle \Gamma \in x$ . Then by axiom Fix, for each  $\gamma \in \Gamma$  we have  $\diamond(\gamma \wedge \langle t \rangle \Gamma) \in x$ , implying that  $\diamond\gamma \in x$ . Then applying to  $\diamond\gamma$  the analysis of  $\diamond\varphi$  in the previous paragraph, we conclude that  $C'$  is non-degenerate and there is some  $y_\gamma \in C'$  with  $\gamma \in y_\gamma$ . Now if  $x'R^*z$ , then  $z \in C'$ , so for each  $\gamma \in \Gamma$  we have  $zRy_\gamma$ , implying that  $\diamond\gamma \in z$ . This proves that  $\Box^*(\bigwedge_{\gamma \in \Gamma} \diamond\gamma) \in x'$ . But putting  $\varphi = \top$  in axiom Ind shows that the formula

$$\Box^*(\top \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \top)) \rightarrow (\top \rightarrow \langle t \rangle \Gamma)$$

is an  $L$ -theorem. From this we can derive that  $\Box^*(\bigwedge_{\gamma \in \Gamma} \diamond\gamma) \rightarrow \langle t \rangle \Gamma$  is an  $L$ -theorem, and hence belongs to  $x'$ . Therefore  $\langle t \rangle \Gamma \in x'$  as required. Likewise  $\langle t \rangle \Gamma \in x'$  implies  $\langle t \rangle \Gamma \in x$ , and so the Lemma holds for  $\langle t \rangle \Gamma$ . ■

**COROLLARY 13.2.** *If  $C$  and  $C'$  are maximal clusters in  $\mathcal{F}$  with  $\delta C = \delta C'$ , then  $C = C'$ .*

**PROOF.** If  $x \in C$ , then  $\tau(x) \in \delta C = \delta C'$ , so there exists  $x' \in C'$  with  $\tau(x) = \tau(x')$ . Lemma 13.1 then implies that  $x = x' \in C'$ , showing  $C \subseteq C'$ . Likewise  $C' \subseteq C$ . ■

**COROLLARY 13.3.** *The set  $M$  of all maximal clusters of  $\mathcal{F}$  is finite.*

**PROOF.** The map  $C \mapsto \delta C$  is an injection of  $M$  into the double power set  $\wp\wp\text{At}$  of the finite set  $\text{At}$ . This gives an upper bound of  $2^{2^{n+1}}$  on the number of maximal clusters, where  $n$  is the size of  $\text{Var}$ . ■

Given subsets  $X, Y$  of  $W$  with  $X \subseteq Y$ , we say that  $X$  is *definable within  $Y$  in  $\mathcal{F}$*  if there is a formula  $\varphi$  such that for all  $y \in Y$ ,  $y \in X$  iff  $\varphi \in y$ . We now work towards showing that within each inner subframe  $R(x)$  in  $\mathcal{F}$ , each path component is definable. For each cluster  $C$ , define the formula

$$\alpha(C) = \bigwedge_{s \in \delta C} \diamond^* \chi(s) \wedge \bigwedge_{s \in \wp\text{At} \setminus \delta C} \neg \diamond^* \chi(s).$$

The next result shows that a maximal cluster is definable within the set of all maximal elements of  $\mathcal{F}$ .

**LEMMA 13.4.** *If  $C$  is a maximal cluster and  $x$  is any maximal element of  $\mathcal{F}$ , then  $x \in C$  iff  $\alpha(C) \in x$ .*

---

<sup>6</sup>That is the reason for including  $\diamond\top$  in  $\text{At}$ .

PROOF. Let  $x \in C$ . If  $s \in \delta C$ , then  $s = \tau(y)$  for some  $y$  such that  $y \in C$ , hence  $xR^*y$ , and  $\chi(s) = \chi(y) \in y$ , showing that  $\diamond^*\chi(s) \in x$ . The converse of this also holds: if  $\diamond^*\chi(s) \in x$ , then for some  $y$ ,  $xR^*y$  and  $\chi(s) \in y$ . Hence  $y \in C$  by maximality of  $C$ , and  $s = \tau(y)$  by (13.1), so  $s \in \delta C$ . Contrapositively then, if  $s \notin \delta C$ , then  $\diamond^*\chi(s) \notin x$ , so  $\neg\diamond^*\chi(s) \in x$ . Altogether this shows that all conjuncts of  $\alpha(C)$  are in  $x$ , so  $\alpha(C) \in x$ .

In the opposite direction, suppose  $\alpha(C) \in x$ . Let  $C'$  be the cluster of  $x$ . Then we want  $C = C'$  to conclude that  $x \in C$ . Since  $x$  is maximal, i.e.  $C'$  is maximal, it is enough by Corollary 13.2 to show that  $\delta C = \delta C'$ .

Now if  $s \in \delta C$ , then  $s = \tau(y)$  for some  $y \in C$ . But  $\diamond^*\chi(s)$  is a conjunct of  $\alpha(C) \in x$ , so  $\diamond^*\chi(s) \in x$ . Hence there exists  $z$  with  $xR^*z$  and  $\chi(s) \in z$ . Then  $z \in C'$  by maximality of  $C'$ , and by (13.1)  $s = \tau(z) \in \delta C'$ .

Conversely, if  $s \in \delta C'$ , with  $s = \tau(y)$  for some  $y \in C'$ , then  $xR^*y$  as  $x \in C'$ , and so  $\diamond^*\chi(s) \in x$  as  $\chi(s) = \chi(y) \in y$ . Hence  $\neg\diamond^*\chi(s) \notin x$ . But then we must have  $s \in \delta C$ , for otherwise  $\neg\diamond^*\chi(s)$  would be a conjunct of  $\alpha(C)$  and so would belong to  $x$ . ■

It is shown in [15] that any transitive canonical frame (weak or not) has the *Zorn property*:

$$\forall x \exists y (xR^*y \text{ and } y \text{ is } R\text{-maximal}).$$

Note the use of  $R^*$ : the statement is that either  $x$  is  $R$ -maximal, or it has an  $R$ -maximal successor. The essence of the proof is that the relation  $\{(x, y) : xR^*y \text{ or } x = y\}$  is a partial ordering for which every chain has an upper bound, so by Zorn's Lemma  $R(x)$  has a maximal element provided that it is non-empty.

The Zorn property is preserved under inner substructures, so it holds for our frame  $\mathcal{F}$ . One interesting consequence is:

LEMMA 13.5. *For each  $x \in W$ , the frame  $\mathcal{F}(x) = (R(x), R \upharpoonright R(x))$  has finitely many path components, as does  $\mathcal{F}$  itself.*

PROOF. The following argument works for both  $\mathcal{F}$  and  $\mathcal{F}(x)$ , noting that the  $R \upharpoonright R(x)$ -cluster of an element of  $\mathcal{F}(x)$  is the same as its  $R$ -cluster in  $\mathcal{F}$ , and that all maximal clusters of  $\mathcal{F}(x)$  are maximal in  $\mathcal{F}$ .

Let  $P$  be a path component and  $y \in P$ . By the Zorn property there is an  $R$ -maximal  $z$  with  $yR^*z$ . Then  $z \in P$  as  $P$  is  $R^*$ -closed. So the  $R$ -cluster of  $z$  is a subset of  $P$ . Since this cluster is maximal, that proves that every path component contains a maximal cluster.

Now distinct path components are disjoint and so cannot contain the same maximal cluster. Since there are finitely many maximal clusters (Corollary 13.3), there can only be finitely many path components. ■

LEMMA 13.6. *Let  $C$  be a maximal cluster in  $\mathcal{F}$ . Then for all  $x \in W$ :*

1.  $C \subseteq R(x)$  iff  $\diamond\Box^*\alpha(C) \in x$ .
2.  $C \subseteq R^*(x)$  iff  $\diamond^*\Box^*\alpha(C) \in x$ .

PROOF. For (1), first let  $C \subseteq R(x)$ . Take any  $y \in C$ . Then if  $yR^*z$  we have  $z \in C$  as  $C$  is maximal, therefore  $\alpha(C) \in z$  by Lemma 13.4. Thus  $\Box^*\alpha(C) \in y$ . But  $y \in R(x)$ , so then  $\diamond\Box^*\alpha(C) \in x$ .

Conversely, if  $\diamond\Box^*\alpha(C) \in x$  then for some  $y$ ,  $xRy$  and  $\Box^*\alpha(C) \in y$ . By the Zorn property, take a maximal  $z$  with  $yR^*z$ . Then  $\alpha(C) \in z$ , so  $z \in C$  by Lemma 13.4. From  $xRyR^*z$  we get  $xRz$ , so  $z \in R(x) \cap C$ . Since  $R(x)$  is  $R^*$ -closed, this is enough to force  $C \subseteq R(x)$ .

The proof of (2) is similar to (1), replacing  $R$  by  $R^*$  where required. ■

For a given  $x \in W$ , let  $P$  be a path component of the frame  $\mathcal{F}(x) = (R(x), R|R(x))$ . Let  $M(P)$  be the set of all maximal  $R$ -clusters  $C$  that have  $C \subseteq P$ . Then  $M(P) \subseteq M$ , where  $M$  is the set of all maximal clusters of  $\mathcal{F}$ , so  $M(P)$  is finite by Corollary 13.3. Define the formula

$$\alpha(P) = \bigvee \{ \diamond^*\Box^*\alpha(C) : C \in M(P) \}.$$

Then  $\alpha(P)$  defines  $P$  within  $R(x)$ :

LEMMA 13.7. *For all  $y \in R(x)$ ,  $y \in P$  iff  $\alpha(P) \in y$ .*

PROOF. Let  $y \in R(x)$ . If  $y \in P$ , take an  $R$ -maximal  $z$  with  $yR^*z$ , by the Zorn property. Then  $z \in R(x)$ , and  $z$  is path connected to  $y \in P$ , so  $z \in P$ . The cluster  $C_z$  of  $z$  is then included in  $P$  (if  $w \in C_z$  then  $zR^*w$  so  $w \in P$ ), and  $C_z$  is maximal, so  $C_z \in M(P)$ . The maximality of  $C_z$  together with Lemma 13.4 then ensure that  $\Box^*\alpha(C_z) \in z$ . Hence  $\diamond^*\Box^*\alpha(C_z) \in y$ . But  $\diamond^*\Box^*\alpha(C_z)$  is a disjunct of  $\alpha(P)$ , so  $\alpha(P) \in y$ .

Conversely, if  $\alpha(P) \in y$ , then  $\diamond^*\Box^*\alpha(C) \in y$  for some  $C \in M(P)$ . By Lemma 13.6(2),  $C \subseteq R^*(y)$ . Taking any  $z \in C$ , since also  $C \subseteq P$  we have  $yR^*z \in P$ , hence  $y \in P$ . ■

THEOREM 13.8. *Suppose that  $L$  includes the scheme  $G_n$ . Then every inner subframe  $\mathcal{F}$  of  $\mathcal{F}_L$  is locally  $n$ -connected.*

PROOF. Let  $x \in W$ . We have to show that  $R(x)$  has at most  $n$  path components. If it has fewer than  $n$  there is nothing to do, so suppose  $R(x)$  has at least  $n$  path components  $P_0, \dots, P_{n-1}$ . Put  $P_n = R(x) \setminus (P_0 \cup \dots \cup P_{n-1})$ . We will prove that  $P_n = \emptyset$ , confirming that there can be no more components.

For each  $i < n$ , let  $\varphi_i$  be the formula  $\alpha(P_i)$  that defines  $P_i$  within  $R(x)$  according to Lemma 13.7. Let  $\varphi_n = \neg \bigvee \{ \alpha(P_i) : 0 \leq i < n \}$ , so  $\varphi_n$  defines

$P_n$  within  $R(x)$ . Now for all  $i \leq n$  let  $\psi_i$  be the formula obtained by uniform substitution of  $\varphi_0, \dots, \varphi_n$  for  $p_0, \dots, p_n$  in the formula  $Q_i$  of (12.1). Observe that since the  $n + 1$  sets  $P_0, \dots, P_n$  form a partition of  $R(x)$ , each  $y \in R(x)$  contains  $\psi_i$  for exactly one  $i \leq n$ , and indeed  $\psi_i$  defines the same subset of  $R(x)$  as  $\varphi_i$ .

Now suppose, for the sake of contradiction, that  $P_n \neq \emptyset$ .<sup>7</sup> Then for each  $i \leq n$  we can choose an element  $y_i \in P_i$ . Then  $xRy_i$  and  $\psi_i \in y_i$ . It follows that  $\bigwedge_{i \leq n} \diamond \psi_i \in x$ . Since all instances of  $G_n$  are in  $x$ , we then get  $\diamond(\bigwedge_{i \leq n} \diamond^* \neg \psi_i) \in x$ . So there is some  $y \in R(x)$  such that for each  $i \leq n$  there exists a  $z_i \in R^*(y)$  such that  $\neg \psi_i \in z_i$ , hence  $\psi_i \notin z_i$ . Now let  $P$  be the path component of  $y$ . If  $P = P_i$  for some  $i < n$ , then as  $y \in P_i$  and  $yR^*z_i$ , we get  $z_i \in P_i$ , and so  $\psi_i \in z_i$  – which is false. Hence it must be that  $P$  is disjoint from  $P_i$  for all  $i < n$ , and so is a subset of  $P_n$ . But then as  $yR^*z_n$  we get  $z_n \in P \subseteq P_n$ , and so  $\psi_n \in z_n$ . That is also false, and shows that the assumption that  $P_n \neq \emptyset$  is false. ■

### 14. Completeness and Finite Model Property for $K4G_n t$

For the language  $\mathcal{L}_\square$  without  $\langle t \rangle$ , Theorem 13.8 provides a completeness theorem for any system extending  $K4G_n$  by showing that any consistent formula  $\varphi$  is satisfiable in a locally  $n$ -connected weak canonical model (take a finite  $\text{Var}$  that includes all variables of  $\varphi$  and enough variables to have  $G_n$  as a formula in the weak language). But the “satisfiable” part of this depends on the Truth Lemma, which is unavailable in the presence of  $\langle t \rangle$ . We will need to apply filtration/reduction to establish completeness itself for  $K4G_n t$ , by showing it has the finite model property.

Suppose that  $L$  is a weak tangle logic that includes  $G_n$ ,  $\mathcal{F} = (W, R)$  is an inner subframe of  $\mathcal{F}_L$ , and  $\Phi$  is a finite set of formulas that is closed under subformulas.

Recall that  $M$  is the set of all maximal clusters of  $\mathcal{F}$ , shown to be finite in Corollary 13.3. For each  $x \in W$ , define

$$M(x) = \{C \in M : C \subseteq R(x)\}.$$

Then  $M(x)$  is finite, being a subset of  $M$ .

Define an equivalence relation  $\approx$  on  $W$  by putting

$$x \approx y \text{ iff } x \cap \Phi = y \cap \Phi \text{ and } M(x) = M(y).$$

---

<sup>7</sup>In that case  $P_n$  is the union of finitely many path components, by Lemma 13.5, but we do not need that fact.

We then repeat the earlier standard transitive filtration construction, but using the finer relation  $\approx$  in place of  $\sim$ . Thus we put  $|x| = \{y \in W : x \approx y\}$  and  $W_\Phi = \{|x| : x \in W\}$ . The set  $W_\Phi$  is finite, because the map  $|x| \mapsto (x \cap \Phi, M(x))$  is a well-defined injection of  $W_\Phi$  into the finite set  $\wp\Phi \times \wp M$ , so  $W_\Phi$  has size at most  $2^{\text{size } \Phi} \cdot 2^{\text{size } M}$ . The surjective function  $f : W \rightarrow W_\Phi$  is given by  $f(x) = |x|$ .

Let  $\mathcal{M}_\Phi = (W_\Phi, R_\Phi, h_\Phi)$ , where  $R_\Phi \subseteq W_\Phi \times W_\Phi$  is the transitive closure of  $R_\lambda = \{(|x|, |y|) : xRy\}$ ,  $h_\Phi(p) = \{|x| : p \in x\}$  for  $p \in \Phi$ , and  $h_\Phi(p) = \emptyset$  otherwise.

We now verify that the pair  $(\mathcal{M}_\Phi, f)$  as just defined satisfies the axioms (r1)–(r5) of a definable reduction of  $\mathcal{F}$  via  $\Phi$ .

(r1):  $p \in x$  iff  $|x| \in h_\Phi(p)$ , for all  $p \in \text{Var} \cap \Phi$ .

By definition of  $h_\Phi$ .

(r2):  $|x| = |y|$  implies  $x \cap \Phi = y \cap \Phi$ .

If  $|x| = |y|$  then  $x \approx y$ , so  $x \cap \Phi = y \cap \Phi$  by definition of  $\approx$ .

(r3):  $xRy$  implies  $|x|R_\Phi|y|$ .

$xRy$  implies  $|x|R_\lambda|y|$  and  $R_\lambda \subseteq R_\Phi$ .

(r4):  $|x|R_\Phi|y|$  implies  $y \cap \Phi^t \subseteq x \cap \Phi^t$  and  $\{\diamond\varphi \in \Phi : \diamond^*\varphi \in y\} \subseteq x$ .

The proof is the same as the proof given earlier of (r4) for the standard transitive filtration, but using  $\approx$  in place of  $\sim$  and the fact that  $x \approx y$  implies  $x \cap \Phi = y \cap \Phi$ .

(r5): For each subset  $C$  of  $W_\Phi$  there is a formula  $\varphi$  that defines  $f^{-1}(C)$  in  $W$ , i.e.  $\varphi \in y$  iff  $|y| \in C$ .

To see this, for each  $x \in W$  let  $\gamma_x$  be the conjunction of  $(x \cap \Phi) \cup \{\neg\psi : \psi \in \Phi \setminus x\}$ . Then for any  $y \in W$ ,

$$\gamma_x \in y \quad \text{iff} \quad x \cap \Phi = y \cap \Phi.$$

Next, let  $\mu_x$  be the conjunction of the finite set of formulas

$$\{\diamond\Box^*\alpha(C) : C \in M(x)\} \cup \{\neg\diamond\Box^*\alpha(C) : C \in M \setminus M(x)\}.$$

Lemma 13.6 showed that each  $C \in M$  has  $C \in M(x)$  iff  $\diamond\Box^*\alpha(C) \in x$ . From this it follows readily that for any  $y \in W$ ,

$$\mu_x \in y \quad \text{iff} \quad M(x) = M(y).$$

So putting  $\varphi_x = \gamma_x \wedge \mu_x$ , we get that in general

$$\varphi_x \in y \quad \text{iff} \quad x \approx y \quad \text{iff} \quad |y| \in \{|x|\}.$$

Now if  $C = \emptyset$ , then  $\perp$  defines  $f^{-1}(C)$  in  $W$ . Otherwise if  $C = \{|x_1|, \dots, |x_n|\}$ , then the disjunction  $\varphi_{x_1} \vee \dots \vee \varphi_{x_n}$  defines  $f^{-1}(C)$  in  $W$ .<sup>8</sup>

Consequently, the reduction  $\mathcal{M}_t$  of  $\mathcal{M}_\Phi$  satisfies the Reduction Lemma. We will show that  $G_n$  is valid in the frame of  $\mathcal{M}_t$ . But first we show that it is valid in the frame of  $\mathcal{M}_\Phi$ . Both cases involve some preliminary analysis, involving linking points of  $R_\Phi(|y|)$  and  $R_t(|y|)$  back to points of  $R(y)$ . This requires further work with maximal elements and clusters.

LEMMA 14.1. *For all  $x, y \in W$ ,  $|x|R_\Phi^*|y|$  implies  $M(y) \subseteq M(x)$ .*

PROOF. If  $|x|R_\Phi^*|y|$  there is a finite sequence  $x = z_0, \dots, z_k = y$  for some  $k \geq 1$  such that for all  $i < k$ , either  $z_i \approx z_{i+1}$  or  $z_i R z_{i+1}$ . But  $z_i \approx z_{i+1}$  implies  $M(z_i) = M(z_{i+1})$ , and  $z_i R z_{i+1}$  implies  $M(z_{i+1}) \subseteq M(z_i)$  by transitivity of  $R$ . This yields  $M(z_k) \subseteq M(z_0)$  by induction on  $k$ . ■

LEMMA 14.2. *Suppose  $\text{At} \subseteq \Phi$  and  $a \in W$  is  $R$ -maximal. Then for all  $x \in W$ ,  $xRa$  iff  $|x|R_\Phi|a|$ .*

PROOF.  $xRa$  implies  $|x|R_\Phi|a|$  by (r3). For the converse, suppose  $|x|R_\Phi|a|$  and let  $K$  be the maximal  $R$ -cluster of  $a$ .

If  $K$  is non-degenerate then  $K \subseteq R(a)$ , so  $K \in M(a)$ . Then from  $|x|R_\Phi|a|$  we get  $K \in M(x)$  by Lemma 14.1, implying  $xRa$  as required.

But if  $K$  is degenerate, then  $K = \{a\}$  and  $R(a) = M(a) = \emptyset$ . Also  $\diamond\top \notin a$ . Since  $|x|R_\Phi|a|$ , by definition of  $R_\Phi$  there are  $z, w \in W$  with  $|x|R_\Phi^*|z|$  and  $zRw \approx a$ . As  $\text{At} \subseteq \Phi$ , from  $w \approx a$  we get  $w \cap \text{At} = a \cap \text{At}$ , i.e.  $\tau(w) = \tau(a)$ . In particular  $\diamond\top \notin w$ , hence  $w$  is also  $R$ -maximal. Therefore  $a$  and  $w$  are maximal elements with the same atomic type, so  $w = a$  by Lemma 13.1. Thus  $zRa$  and so  $K \in M(z)$ . Since  $|x|R_\Phi^*|z|$  this implies  $K \in M(x)$  by Lemma 14.1, giving the required  $xRa$  again. ■

LEMMA 14.3. *Suppose  $\text{At} \subseteq \Phi$  and for any  $y \in W$ , let  $A$  be the set of all  $R$ -maximal points in  $R(y)$ . Then each point  $v \in R_\Phi^*(|y|)$  has  $vR_\Phi^*|a|$  for some  $a \in A$ .*

PROOF. Let  $v = |z| \in R_\Phi^*(|y|)$ . By the Zorn property there exists an  $a$  with  $zR^*a$  and  $a$  is  $R$ -maximal. Then  $|z|R_\Phi^*|a|$  by (r3). Since  $|y|R_\Phi|z|$ , this implies  $|y|R_\Phi|a|$  by transitivity. Hence  $yRa$  by Lemma 14.2, and so  $a \in A$ . ■

---

<sup>8</sup>An alternative approach to this construction would be to enlarge  $\Phi$  by adding  $\{\diamond\Box^*\alpha(C) : C \in M\}$  and closing under subformulas. Then  $\Phi$  would still be finite and the relation  $\sim$  it induces would have  $x \sim y$  only if  $M(x) = M(y)$ . This would obviate the need to verify that  $\approx$  gives rise to a definable reduction, but would also multiply the upper bound on the size of  $W_\Phi$  by more than  $2^{\text{size } M}$ .

THEOREM 14.4. *If  $\text{At} \subseteq \Phi$ , the frame  $\mathcal{F}_\Phi = (W_\Phi, R_\Phi)$  is locally  $n$ -connected.*

PROOF. For any point  $|y| \in W_\Phi$ , we have to show that  $R_\Phi(|y|)$  has at most  $n$  path components. But if it had more than  $n$ , then by picking points from different components we would get a sequence of more than  $n$  points no two of which were path connected. We show that this is impossible, by taking an arbitrary sequence  $v_0, \dots, v_n$  of  $n+1$  points in  $R_\Phi(|y|)$ , and proving that there must exist distinct  $i$  and  $j$  such that  $v_i$  and  $v_j$  are path connected in  $R_\Phi(|y|)$ .

For each  $i \leq n$ , by Lemma 14.3 there is an  $R$ -maximal  $a_i \in R(y)$  with  $v_i R_\Phi^* |a_i|$ . This gives us a sequence  $a_0, \dots, a_n$  of members of  $R(y)$ . But  $R(y)$  has at most  $n$  path components, by Theorem 13.8. Hence there exist  $i \neq j \leq n$  such that there is a connecting  $R$ -path  $a_i = w_0, \dots, w_n = a_j$  between  $a_i$  and  $a_j$  that lies in  $R(y)$ . So for all  $i < n$  we have  $y R w_i$  and either  $w_i R w_{i+1}$  or  $w_{i+1} R w_i$ , hence  $|y| R_\Phi |w_i|$  and either  $|w_i| R_\Phi |w_{i+1}|$  or  $|w_{i+1}| R_\Phi |w_i|$ .

This shows that  $|a_i|$  and  $|a_j|$  are path connected in  $R_\Phi(|y|)$  by the sequence  $|w_0|, \dots, |w_n|$ . Since  $v_i R_\Phi^* |a_i|$  and  $v_j R_\Phi^* |a_j|$ , it follows that  $v_i$  and  $v_j$  are path connected in  $R_\Phi(|y|)$ , as required. ■

PROPOSITION 14.5. 1. *In the language  $\mathcal{L}_\square$ , for all  $n \geq 1$  the finite model property holds for  $\text{K4G}_n$  and  $\text{KD4G}_n$  over locally  $n$ -connected  $\text{K4}$  and  $\text{KD4}$  frames, respectively.*

2. *In the language  $\mathcal{L}_{\square\forall}$ , the finite model property holds for the four families of logics  $\text{K4G}_n.\text{U}$ ,  $\text{K4G}_n.\text{UC}$ ,  $\text{KD4G}_n.\text{U}$  and  $\text{KD4G}_n.\text{UC}$ .*

PROOF. For (1), take a consistent  $\mathcal{L}_\square$ -formula  $\varphi$  and let  $\Phi$  be the closure under  $\mathcal{L}_\square$ -subformulas of  $\text{At} \cup \{\varphi\}$ . Then  $\Phi$  is finite and  $\varphi$  is satisfiable in the model  $\mathcal{M}_\Phi$  (see the remarks about  $\mathcal{M}_\Phi$  at the end of Section 11). But the frame  $\mathcal{F}_\Phi$  of  $\mathcal{M}_\Phi$  is locally  $n$ -connected by the theorem just proved, so validates  $\text{G}_n$ . Together with the preservation of seriality by  $\mathcal{F}_\Phi$ , this implies the finite model property results for  $\text{K4G}_n$  and  $\text{KD4G}_n$ .

(2) follows correspondingly, using the results about  $\forall$  from Section 10 and the fact that  $\mathcal{F}_\Phi$  is path connected in the presence of axiom C. ■

The result for  $\text{KD4G}_n$  in part (1) of this Proposition was conjectured in general and proven for  $n = 1$  in [15]. The conjecture was proven in [21]. In part (2) the cases involving D were shown in [12].

We turn now to the corresponding results for the versions of these systems that include the tangle connective.

LEMMA 14.6. *If  $y \in W$  is the critical point for some  $R_\Phi$ -cluster, then  $z \in R(y)$  implies  $|z| \in R_t(|y|)$ .*

PROOF. Let  $y$  be critical for cluster  $C$ . If  $z \in R(y)$ , then  $|y|R_\Phi|z|$  (r3), so if  $|z| \notin C$  then immediately  $|y|R_t|z|$ . But if  $|z| \in C$ , then  $|z| \in C^\circ$  and again  $|y|R_t|z|$ . ■

LEMMA 14.7. *Suppose  $\diamond\top \in \Phi$ . Let  $y \in W$  be a critical point, and  $z, z' \in R(y)$ . If  $z$  and  $z'$  are path connected in  $R(y)$ , then  $|z|$  and  $|z'|$  are path connected in  $R_t(|y|)$ .*

PROOF. Let  $z = z_0, \dots, z_n = z'$  be a connecting path between  $z$  and  $z'$  within  $R(y)$ . The criticality of  $y$  ensures, by Lemma 14.6, that  $|z_0|, \dots, |z_n|$  are all in  $R_t(|y|)$ . We apply Lemma 11.1 to convert this sequence into a connecting  $R_t$ -path within  $R_t(|y|)$ .

For each  $i < n$  we have  $z_i R z_{i+1}$  or  $z_{i+1} R z_i$ , hence either  $|z_i|R_\Phi|z_{i+1}|$  or  $|z_{i+1}|R_\Phi|z_i|$  by (r3). So if there is such an  $i$  that is “defective” in the sense that neither  $|z_i|R_t|z_{i+1}|$  nor  $|z_{i+1}|R_t|z_i|$ , then by Lemma 11.1, which applies since  $\diamond\top \in \Phi$ , there exists a  $v_i$  with  $|z_i|R_t v_i$  and  $|z_{i+1}|R_t v_i$ . Then  $v_i \in R_t(|y|)$  by transitivity of  $R_t$ , as  $|z_i| \in R_t(|y|)$ . We insert  $v_i$  between  $|z_i|$  and  $|z_{i+1}|$  in the sequence. Doing this for all defective  $i < n$  turns the sequence into a connecting  $R_t$ -path in  $R_t(|y|)$  with unchanged endpoints  $|z|$  and  $|z'|$ . ■

LEMMA 14.8. *Suppose  $\text{At} \subseteq \Phi$  and  $a \in W$  is  $R$ -maximal. Then for all  $x \in W$ ,  $|x|R_t|a|$  iff  $|x|R_\Phi|a|$ .*

PROOF.  $|x|R_t|a|$  implies  $|x|R_\Phi|a|$  by definition of  $R_t$ . For the converse, suppose  $|x|R_\Phi|a|$ , and let  $C$  be the  $R_\Phi$ -cluster of  $|x|$ . If  $|a| \notin C$ , then since  $|x|R_\Phi|a|$  it is immediate that  $|x|R_t|a|$  as required. We are left with the case  $|a| \in C$ . Then since  $|x|R_\Phi|a|$ , we see that  $C$  is non-degenerate, so if  $y$  is the critical point for  $C$  then  $|y|R_\Phi|a|$ . Hence  $yRa$  by Lemma 14.2. But then  $|a| \in C^\circ$  and so again  $|x|R_t|a|$  as required. ■

THEOREM 14.9. *If  $\text{At} \subseteq \Phi$ , the frame  $\mathcal{F}_t = (W_\Phi, R_t)$  is locally  $n$ -connected.*

PROOF. This refines the proof of Theorem 14.4. If  $u \in W_\Phi$ , we have to show that  $R_t(u)$  has at most  $n$  path components. Now if  $C$  is the  $R_\Phi$ -cluster of  $u$ , then  $R_t(u)$  is the union of the nucleus  $C^\circ$  and all the  $R_\Phi$ -clusters coming strictly  $R_\Phi$ -after  $C$ . Hence  $R_t(u) = R_t(w)$  for all  $w \in C$ . In particular,  $R_t(u) = R_t(|y|)$  where  $y$  is the critical point of  $C$ . So we show that  $R_t(|y|)$  has at most  $n$  path components. We take an arbitrary sequence  $v_0, \dots, v_n$  of  $n+1$  points in  $R_t(|y|)$ , and prove that there must exist distinct  $i$  and  $j$  such that  $v_i$  and  $v_j$  are path connected in  $R_t(|y|)$ .

Let  $A$  be the set of all  $R$ -maximal points in  $R(y)$ . For each  $i \leq n$  we have  $v_i \in R_\Phi(|y|)$  and so by Lemma 14.3 there is an  $a_i \in A \subseteq R(y)$  such that  $v_i R_\Phi^* |a_i|$ . Hence  $v_i R_t^* |a_i|$  by Lemma 14.8. This gives us a sequence  $a_0, \dots, a_n$



of members of  $R(y)$ . But  $R(y)$  has at most  $n$  path components, by Theorem 13.8. Hence there exist  $i \neq j \leq n$  such that  $a_i$  and  $a_j$  are path connected in  $R(y)$ . Therefore by Lemma 14.7,  $|a_i|$  and  $|a_j|$  are path connected in  $R_t(|y|)$ . Since  $v_i R_t^* |a_i|$  and  $v_j R_t^* |a_j|$ , and  $v_i, v_j \in R_t(|y|)$ , it follows that  $v_i$  and  $v_j$  are path connected in  $R_t(|y|)$ . That shows that  $R_t(|y|)$  does not have more than  $n$  path components. ■

This result combines with the analysis as in other cases to give the following results.

PROPOSITION 14.10. *The finite model property holds for the tangle logics  $K4G_n t$ ,  $K4G_n t.U$ ,  $K4G_n t.UC$ ,  $KD4G_n t$ ,  $KD4G_n t.U$  and  $KD4G_n t.UC$ , for all  $n \geq 1$ .*

PROOF. The proof for  $K4G_n t$  is like that for  $K4G_n$  in Proposition 14.5, but using  $\mathcal{M}_t$  in place of  $\mathcal{M}_\Phi$  and observing that  $\mathcal{F}_t$  validates  $G_n$  by Theorem 14.9. This then combines with the analysis as in other cases to give the remaining results. ■

It is noteworthy that for languages without  $\langle t \rangle$  there is an alternative approach to Theorem 14.4 due to Kudinov and Shehtman [11, Lemma 7]. It expands the filtrating set  $\Phi$  to include formulas of the form  $\Box^* \varphi$ , where  $\varphi$  is a Boolean combination of the formulas from our Lemma 13.6(2). These formulas are used to define the path components of  $R_\Phi(|y|)$  in such a way that the axiom  $G_n$  can be applied more directly to prove local  $n$ -connectedness of  $\mathcal{F}_\Phi$  in the manner of Theorem 13.8, without having to first prove the latter. Moreover, this is done by taking  $\mathcal{F}_\Phi$  to be the classical filtration constructed from the equivalence relation  $\sim$  as in Section 9, rather than working with the stronger relation  $\approx$  of this Section which ensures that  $|x| = |y|$  implies  $M(x) = M(y)$ . However the method of [11] depends on  $R_\Phi$  satisfying (r3), a property that is lost when we untangle  $R_\Phi$  and replace it by  $R_t$ . It would be possible to apply the method to the stronger filtration based on  $\approx$  and with the relation  $R_t$ , using the partial restoration of (r3) provided here in Lemmas 14.2 and 14.8. But the use of  $\approx$  instead of  $\sim$  multiplies the upper bound on the size of  $W_\Phi$  by  $2^{\text{size } M}$  as we noted, and the inclusion of the formulas  $\Box^* \varphi$  in  $\Phi$  multiplies that upper bound by an even greater exponential factor. So this alternative method does not appear to provide an advantage from a complexity standpoint for languages that include the tangle connective.

## 15. Summing Up

The table below summarizes our results on the finite model property (fmp) for tangle logics in the languages  $\mathcal{L}_{\square}^{(t)}$  and  $\mathcal{L}_{\square\vee}^{(t)}$  over various classes  $\mathcal{K}$  of frames. The result for  $S4t$  is due to [5]. The others are new here. Several of them are essential to completeness theorems for certain spatial interpretations of tangle logics in [8, 9], as explained in the Introduction to this paper.

A natural direction for further study would be to obtain completeness theorems for the tangle extension of logics in other languages, for instance the logics of [11] with the difference modality  $[\neq]$  expressing “at all other points”, or more strongly, logics with graded modalities that can count the number of successors of a given point.

Conditions defining $\mathcal{K}$	Logics with the fmp over $\mathcal{K}$
Transitive	$K4t$ , $K4t.U$
Transitive and serial	$KD4t$ , $KD4t.U$
Transitive and reflexive	$S4t$ , $S4t.U$
Transitive and path connected	$K4t.UC$
Transitive, serial and path connected	$KD4t.UC$
Transitive, reflexive and path connected	$S4t.UC$
Transitive and locally $n$ -connected	$K4G_n t$ , $K4G_n t.U$
Transitive, serial and locally $n$ -connected	$KD4G_n t$ , $KD4G_n t.U$
Transitive, path connected and locally $n$ -connected	$K4G_n t.UC$
Transitive, serial, path connected and locally $n$ -connected	$KD4G_n t.UC$

Another direction would be to study the general relationship between logics and their tangle extensions, considering what properties are preserved in passing from  $L$  to  $Lt$ , such as conditions under which a Kripke-frame complete  $L$  would have a Kripke-frame complete  $Lt$ .

**Acknowledgements.** The authors would like to thank the referees for their very helpful comments and suggestions. The second author was supported by UK EPSRC overseas travel grant EP-L020750-1.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the

original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

## References

- [1] BLACKBURN, P., M. DE RIJKE, and Y. VENEMA, *Modal Logic*. Cambridge University Press, 2001.
- [2] DAWAR, A., and M. OTTO, Modal characterisation theorems over special classes of frames. *Annals of Pure and Applied Logic* 161:1–42, 2009.
- [3] FERNÁNDEZ-DUQUE, D., A sound and complete axiomatization for dynamic topological logic. *The Journal of Symbolic Logic* 77:947–969, 2012.
- [4] FERNÁNDEZ-DUQUE, D., On the modal definability of simulability by finite transitive models. *Studia Logica* 98(3):347–373, 2011.
- [5] FERNÁNDEZ-DUQUE, D., Tangled modal logic for spatial reasoning, in T. Walsh, (ed.), *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI)*, pp. 857–862. AAAI Press/IJCAI, 2011.
- [6] FERNÁNDEZ-DUQUE, D., Tangled modal logic for topological dynamics. *Annals of Pure and Applied Logic* 163:467–481, 2012.
- [7] GOLDBLATT, R., *Logics of Time and Computation*. CSLI Lecture Notes No. 7. CSLI Publications, Stanford University, second edition, 1992.
- [8] GOLDBLATT, R., and I. HODKINSON, Spatial logic of tangled closure operators and modal mu-calculus. *Annals of Pure and Applied Logic* 168:1032–1090, 2017. doi:[10.1016/j.apal.2016.11.006](https://doi.org/10.1016/j.apal.2016.11.006).
- [9] GOLDBLATT, R., and I. HODKINSON, The tangled derivative logic of the real line and zero-dimensional spaces, in L. Beklemishev, S. Demri, and A. Máté, (eds.), *Advances in Modal Logic, Volume 11*, pp. 342–361. College Publications, 2016. <http://www.aiml.net/volumes/volume11/>.
- [10] JANIN D., and I. WALUKIEWICZ, On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic, in U. Montanari and V. Sassone, (eds.), *CONCUR '96: Concurrency Theory*, volume 1119 of *Lecture Notes in Computer Science*, Springer, 1996, pp. 263–277.
- [11] KUDINOV, A., and V. SHEHTMAN, Derivational modal logics with the difference modality, in G. Bezhanishvili, (ed.), *Leo Esakia on Duality in Modal and Intuitionistic Logics*, volume 4 of *Outstanding Contributions to Logic*, Springer, 2014, pp. 291–334.
- [12] LUCERO-BRYAN, J. G., The d-logic of the real line. *Journal of Logic and Computation* 23(1):121–156, 2013. doi:[10.1093/logcom/exr054](https://doi.org/10.1093/logcom/exr054).
- [13] ROSEN, E., Modal logic over finite structures. *Journal of Logic and Computation* 6:427–439, 1997.
- [14] SEGERBERG, K., Decidability of S4.1. *Theoria* 34:7–20, 1968.
- [15] SHEHTMAN, V., Derived sets in Euclidean spaces and modal logic. Technical Report X-1990-05, University of Amsterdam, 1990. <http://www.illc.uva.nl/Research/Publications/Reports/X-1990-05.text.pdf>.
- [16] SHEHTMAN, V., «Everywhere» and «Here». *Journal of Applied Non-Classical Logics* 9(2–3):369–379, 1999.

- [17] SHEHTMAN, V., *Modal logic of Topological Spaces*. Habilitation thesis, Moscow, 2000. In Russian.
- [18] TARSKI, A., A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics* 5:285–309, 1955.
- [19] VAN BENTHEM, J. F. A. K., *Modal Correspondence Theory*. PhD thesis, University of Amsterdam, 1976.
- [20] VAN BENTHEM, J. F. A. K., *Modal Logic and Classical Logic*. Bibliopolis, Naples, 1983.
- [21] ZAKHARYASCHEV, M., A sufficient condition for the finite model property of modal logics above K4. *Bulletin of the IGPL* 1:13–21, 1993.

R. GOLDBLATT  
School of Mathematics and Statistics  
Victoria University of Wellington  
Wellington  
New Zealand  
rob.goldblatt@msor.vuw.ac.nz

I. HODKINSON  
Department of Computing  
Imperial College  
London SW7 2AZ  
UK  
i.hodkinson@imperial.ac.uk