Arithmetic is Necessary

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Abstract

Goodsell [12] establishes the noncontingency of sentences of first-order arithmetic, in a plausible higher-order modal logic. Here, the same result is derived using significantly weaker assumptions. Most notably, the assumption of rigid comprehension—that every property is coextensive with a modally rigid one—is weakened to the assumption that the Boolean algebra of properties under necessitation is countably complete. The results are generalized to extensions of the language of arithmetic, and are applied to answer a question posed by Bacon and Dorr [5].

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2 Introduction

Recent literature[1] has challenged the orthodox conception of pure logic and pure mathematics as modally fixed.[2] Goodsell [12] establishes a limitative result on speculation in this direction: given plausible connections between plural and modal reasoning, there is no contingency in pure arithmetic. For example, it is necessary one way or the other whether the Goldbach conjecture holds, whether there are infinitely many twin primes, and whether ZFC is consistent. More precisely, where I is the claim that it is possible that zero is not a successor and that the successor function is injective, Goodsell proves each instance of the following schema, where $\mathbf{A}$ ranges over sentences of the language of Peano arithmetic:

$$\Box(I \to \mathbf{A}) \vee \Box(I \to \neg\mathbf{A}) \tag{1}$$

Since I is a necessary precondition for even the possible truth of the axioms of arithmetic, this schema shows that the only possibilities where an arithmetical sentence gets a different truth-value to normal are those where the usual axioms of arithmetic are necessarily false. (With the plausible assumption that I is necessarily true we

[1] Linnebo and Shapiro [19], Scambler [22], Brauer [7], Scambler [23], Builes and Wilson [8], Hamkins and Linnebo [15], Linnebo and Shapiro [20], Bacon and Dorr [5], Bacon [3, 4]

[2] For countervailing considerations see [17, 24, 13].

get the unqualified necessity schema with instances of the form $\Box\mathbf{A} \vee \Box\neg\mathbf{A}$).

This paper improves on Goodsell's results in the following two respects. First, Goodsell's principle of *Rigidity*—here called *Rigid Comprehension* in accordance with more recent literature [11, 5]—is weakened to a special case of *Boolean Completeness*. Rigid Comprehension asserts that standard principles of modal plural logic (see, e.g., Linnebo [18]) hold of a special class of properties known as *rigid* properties. By contrast, Boolean Completeness asserts only that every collection of properties has a *least upper bound*, a property which is an *upper bound* in the sense of being necessitated by every member of that collection, but is also *least* in that it necessitates every other upper bound. Boolean Completeness is known to be a consequence of Rigid Comprehension, and is conjectured to be strictly weaker. Moreover, for this paper it is possible to restrict Boolean Completeness to *Countable* Boolean Completeness, which asserts only the existence of least upper bounds for countable collections of properties.

Second, the background modal higher-order logic is weakened. Goodsell employs, alternatively, a second-order logic with a Quantified KT modality, with extensionality for second-order variables, and where the axioms of arithmetic are assumed to be necessary, and then the higher-order logic of Classicism of [5], which identifies provably equivalent formulae, and in which a defined notion of necessity has a Quantified $\mathsf{S4}$ logic (see [2]), but where the necessity of the axioms of arithmetic may now be derived rather than assumed. It is Goodsell's second result, using higher-order logic, which is improved upon in this paper. Here, we use the logic HKC, which includes the following components (see Table 1):

- H, a minimal higher-order logic including classical propositional logic, classical quantificational logic at all types, and the standard $\beta\eta$ rules for λ-abstracts.[3]

- A K logic for a primitive modality $\Box$, which in addition to a rule of necessitation asserts exactly that $\Box$ distributes over material implication (the K axiom for $\Box$). (Since necessitation applies to provable formulae of H, the resulting logic includes the converse Barcan formula and the necessity of identity, but not the Barcan formula or the necessity of distinctness). $\mathsf{H}+\mathsf{K}$, or HK, is strictly weaker than Classicism if H is consistent.[4]

- The axiom of Countable Boolean Completeness (CBC).

The results are also modestly extended beyond the language of arithmetic, to include, e.g., that every sentence that can be stated in the language of arithmetic plus a truth-predicate for the language of arithmetic is either necessarily true or necessarily false.

The necessity of arithmetic puts paid to the hypothesis that the necessarily true sentences in a fundamental language that includes arithmetic are all consequences of a consistent recursively axiomatizable theory.[5] In light of this observation, the results are applied to give a negative answer to an outstanding question of Bacon

[3] H is first axiomatized by Church [9, p. 61], with Axioms 1-6^{α} (Church's final system in that paper includes axioms of infinity, function extensionality, and choice). It is also the system $\mathcal{J}$ of Andrews [1] and is the theory of models of class $\mathfrak{M}_{\beta\eta}$ as isolated by Benzmüller et al. [6]. Dorr [10] and Bacon [2] give H its current name and have raised it to prominence in recent philosophical literature.

[4] A notational variant of Classicism may be axiomatized by adding to HK the following schema:

Intensionalism Necessarily equivalent properties are identical.

$$\forall XY^{\vec{\sigma}t} \mathbin{.} (\Box\forall\vec{z} \mathbin{.} Xz \leftrightarrow Xy) \rightarrow X = Y \tag{2}$$

[5] A view of this sort is propounded by Ramsey [21, §12].

PC	Every propositional tautology
$\beta\eta$	$\mathbf{A} \to \mathbf{B}$, where $\mathbf{A} \sim_{\beta\eta} \mathbf{B}$
UI	$\forall \mathbf{A} \to \mathbf{A}\mathbf{b}$
MP	From $\Gamma \vdash \mathbf{A} \to \mathbf{B}$ and $\Gamma \vdash \mathbf{A}$ infer $\Gamma \vdash \mathbf{B}$
Gen	From $\Gamma \vdash \mathbf{A}$ infer $\Gamma \vdash \forall \mathbf{x} \,.\, \mathbf{A}$, when $\mathbf{x} \notin \mathrm{FV}(\Gamma)$
K	$\forall pq \,.\, \Box(p \to q) \to \Box p \to \Box q$
Nec	From $\vdash \mathbf{A}$ infer $\vdash \Box \mathbf{A}$
CBC	$\forall X^{\langle \sigma t \rangle t} \,.\, (\mathrm{Ctbl}\, X \to \exists (\mathrm{LUB}\, X))$ (see Definitions 1 and 6)

Table 1: Axiom schemata and rules (Footnote 6 lists some notational conventions). H adds $\beta\eta$ and UI axiom schemata to PC and closes under MP and Gen. HK adds K to H and closes under MP, Gen, and Nec. HKC adds CBC to HK and closes under MP and Gen (but not Nec, hence CBC is assumed without assuming it to be necessarily true).

and Dorr [5]: that of whether Boolean Completeness is consistent with a theory they call *maximalized* Classicism.

3 Boolean Completeness of Properties

3.1 Motivation and Statement

Table 1 includes the axioms and rules of the system HKC.[6] The system is unremarkable except for the axiom schema of Countable Boolean Completeness, CBC, which, by a theorem of Bacon and Dorr [5], is essential in deriving the necessity of arithmetic (see Section 6). Countable Boolean Completeness is motivated by viewing the properties modulo necessary equivalence as forming a Boolean algebra under necessitation. The ordering of this algebra is necessitation, i.e., the relation

$$\lambda XY^{\sigma t} \mathbin{.} \Box\forall z^{\sigma} \mathbin{.} Xz \to Yz. \tag{3}$$

Necessitation is straightforwardly transitive and reflexive in HK. There is a necessitation-maximal equivalence class of properties, namely those which necessarily everything has, such as $\lambda z^{\sigma} \mathbin{.} z = z$. Moreover, the union, intersection, and complements of

[6]Some notational conventions:

(a) Bold symbols are metavariables.

(b) Variable binders bind the variables immediately before $.$, and take greatest possible scope.

(c) $\forall \mathbf{v} \in \mathbf{A} \mathbin{.} \mathbf{B}$ abbreviates $\mathbf{A}$-restricted universal quantification, i.e., $\forall \mathbf{v} \mathbin{.} \mathbf{A}\mathbf{v} \to \mathbf{B}$. Analogously for existential quantification.

(d) Boolean connectives may be written in infix notation, and take greater scope than prefix and postfix applications, but lesser scope than variable binders. Moreover, $\to$ associates to the right (so $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ is $\mathbf{A} \to (\mathbf{B} \to \mathbf{C})$).

(e) Type decorations are omitted where they can be inferred. $\langle \sigma\tau \rangle$ is the simple functional type from σ to τ, and omitted angle brackets associate to the right, so that eet is $\langle e\langle et \rangle\rangle$.

properties, defined by lifting Boolean propositional operators as follows,

$$\cup := \lambda X Y^{\sigma t} z^{\sigma} \,.\, Xz \vee Yz \tag{4}$$

$$\cap := \lambda X Y^{\sigma t} z^{\sigma} \,.\, Xz \wedge Yz \tag{5}$$

$$- := \lambda X^{\sigma t} z^{\sigma} \,.\, \neg Xz, \tag{6}$$

yield the least upper bound, greatest lower bound, and complement respectively in the necessitation preordering. Therefore, modulo necessary equivalence, the properties form a Boolean algebra under necessitation.

It is natural to suppose that this Boolean algebra is *complete*: that for any collection of properties, there is a least upper bound for that entire collection.

Definition 1 (Upper bound; UB/Least upper bound; LUB)**.** For a property Y to be an *upper bound* of a property of properties X is for Y to be necessitated by every member of X.

$$\mathrm{UB}_{\sigma} := \lambda X^{\langle \sigma t \rangle t} Y^{\sigma t} \,.\, \forall Z \in X \,.\, \Box \forall x^{\sigma} \,.\, Zx \to Yx \tag{7}$$

For Y to be a *least* upper bound of X is for it to be an upper bound of X that necessitates every other upper bound thereof.

$$\mathrm{LUB}_{\sigma} := \lambda XY \,.\, \mathrm{UB}_{\sigma}\, XY \wedge \forall Y' \,.\, \mathrm{UB}\, XY' \to \Box \forall z \,.\, Yz \to Y'z \tag{8}$$

Least upper bounds need not be unique in HK, but are unique modulo necessary equivalence.

The principle of *Boolean Completeness* is as follows:

Boolean Completeness Every property of properties has a least upper bound.

$$\forall X^{\langle \sigma t \rangle t} \,.\, \exists (\mathrm{LUB}\, X) \tag{9}$$

For present purposes, we may restrict Boolean Completeness to countable properties of properties, corresponding to the hypothesis that the properties form a *countably* complete Boolean algebra modulo necessary equivalence:

Countable Boolean Completeness Every countable property of properties has a least upper bound.

$$\forall X^{\langle \sigma t \rangle t} \,.\, \mathrm{Ctbl}\, X \to \exists (\mathrm{LUB}\, X) \tag{10}$$

where countability, or Ctbl, is defined in Section 4 as being injectible into the natural numbers.

3.2 Comparison with Rigid Comprehension

Goodsell [12] uses the principle of *Rigid Comprehension* instead of Countable Boolean Completeness.

Definition 2 (Rigid property). A *rigid* property is one such that quantification restricted by that property obeys the Barcan and converse Barcan formulae:

$$\begin{aligned} \mathrm{Rigid}_\sigma := \lambda X^{\sigma t} \,.\, \forall Y^{\sigma t} \,.\, (\Box(\forall z \in X \,.\, Yz) \to \forall z \in X \,.\, \Box Yz) \\ \wedge ((\forall z \in X \,.\, \Box Yz) \to \Box \forall z \in X \,.\, Yz) \end{aligned} \tag{11}$$

Rigid Comprehension Every property is coextensive with some rigid property.

$$\forall X^{\sigma t} \mathbin{.} \exists Y^{\sigma t} \mathbin{.} \mathrm{Rigid}\, Y \wedge \forall z \mathbin{.} Xz \leftrightarrow Yz \tag{12}$$

As Goodsell points out, the Barcan and Converse Barcan Formualae are intuitively very tempting principles for regimenting plural quantification. For example, it may be a contingent matter who was a bandmate of Miles Davis, but for those very people that were in fact bandmates of Miles Davis, it is not a contingent matter who was one of *them*. Hence Rigid Comprehension is a tempting way of cashing out the idea that for every property X, there is some plurality of things which are all and only those which instantiate X.

Boolean Completeness is implied by Rigid Comprehension, by an argument of Bacon and Dorr [5]. The intuitive idea is as follows. For a given property of properties X, take the rigid property X' with which it is coextensive; then the least upper bound of X will be

$$\lambda z \mathbin{.} \exists Y \in X' \mathbin{.} Yz.^{7} \tag{13}$$

Restated using the analogy between rigid properties and pluralities: to find the least upper bound of X, take those very properties X' which are all and only the X

[7] Let U be the property

$$\lambda z \mathbin{.} \exists Y \in X' \mathbin{.} Yz. \tag{14}$$

To show that U is an upper bound of X, take Y in X. Then by the converse Barcan formula for X'-restricted quantification it is necessary that Y is in X', so it is necessary that, if Yz, then Uz.

To show that U is least, let Z be an arbitrary upper bound of X, which is to say

$$\forall Y \in X \mathbin{.} \Box \forall z \mathbin{.} Yz \to Zz. \tag{15}$$

properties. Then the least upper bound of X is the property of being some z that instantiates at least one of *them*, (i.e., at least one property that instantiates X').

Boolean Completeness is conjectured to be strictly weaker than Rigid Comprehension. This is because in HK, the least upper bound of X might have instances which are not instances of any instance of X, if there is no rigid property X' coextensive with X. (What is not known is whether this situation is compossible with *every* property of properties having a least upper bound.)

Countable Boolean Completeness is used here to show that, as a result of the inductive property of natural numbers, natural numberhood is a rigid property (Lemma 8). This turns out to be sufficient in HK for the necessity of arithmetic.

4 Formalization of Arithmetic

In H, basic arithmetical concepts can be defined in terms of the concepts of zero, 0, and the successor function, $\cdot^+$ (written postfix superscript), of respectively of some types of the forms ν and $\nu\nu$.

Since X' is coextensive with X this is equivalent to

$$\forall Y \in X' \,\mathbf{.}\, \Box \forall z \,\mathbf{.}\, Yz \rightarrow Zz, \tag{16}$$

which is equivalent by the Barcan formula for X'-restricted quantification to

$$\Box \forall Y \in X' \,\mathbf{.}\, \forall z \,\mathbf{.}\, Yz \rightarrow Zz \tag{17}$$

which implies

$$\Box \forall z \,\mathbf{.}\, (\exists Y \in X' \,\mathbf{.}\, Yz) \rightarrow Zz \tag{18}$$

which is the definition of U being less than Z on the necessitation preordering. So U is a least upper bound of X as required.

Definition 3 (Natural numberhood, $\mathbb{N}$)**.** To be a natural number is to be an instance of every property that contains 0 and that is closed under successor.

$$\mathbb{N} := \lambda m^{\nu} \centerdot \forall X^{\nu t} \centerdot X0 \to (\forall n \centerdot Xn \to Xn^{+}) \to Xm \tag{19}$$

Three-place relations of numbers Sum and Prod, which relate three numbers when the first two add or multiply to make the third, may be given standard recursive definitions as follows:[8]

Definition 4.

$$\mathrm{Sum} := \lambda mno^{\nu} \centerdot \forall R^{\nu\nu t} \centerdot R0m \to (\forall ij \in \mathbb{N} \centerdot Rij \to Ri^{+}j^{+}) \to Rno \tag{20}$$

$$\begin{aligned}&\mathrm{Prod} := \\ &\lambda mno^{\nu} \centerdot \forall R^{\nu\nu t} \centerdot R00 \to (\forall ijk \in \mathbb{N} \centerdot Rij \to \mathrm{Sum}\, mjk \to Ri^{+}k) \to Rno\end{aligned} \tag{21}$$

In usual pure number theory the specific choice of 0 and $\cdot^{+}$ is immaterial, so long as the expected structural constraints hold. In HK, the only important structural constraint can be stated as follows:

Definition 5 (I)**.** I is the claim that possibly, zero is not a successor and successor is injective on numbers.

$$I := \Diamond \forall mn \in \mathbb{N} \centerdot 0 \neq m^{+} \wedge (m^{+} = n^{+} \to m = n) \tag{22}$$

Whence we may derive Peano arithmetic, understood so that all quantification is

[8] HK does not include the resources for recursive definition of functions of numbers (as opposed to relations).

explicitly restricted to $\mathbb{N}$, and so as to use relational predicates Sum and Prod rather than the usual function symbols $+$ and $\times$:[9]

Lemma 1. *Where* $\mathbf{A}$ *is a theorem of Peano arithmetic with all quantification restricted to* $\mathbb{N}$*,*

$$I \to \mathbf{A} \tag{23}$$

is a theorem of HK *(with the basic arithmetical concepts defined as in Definitions 3 and 4).*

Proof. Routine given Lemma 5 (p. 15). $\square$

A particularly important way of understanding 0 and $\cdot^+$ is as relevant to *how many* things there can be. That is, 0 can be understood as *being zero in number*, and $\cdot^+$ as *that many and then one more.*

$$0 := \lambda X^{et} \,\mathbf{.}\, \neg\exists X \tag{24}$$

$$\cdot^+ := \lambda n^{\langle et \rangle t} X^{et} \,\mathbf{.}\, \exists y \in X \,\mathbf{.}\, n(\lambda z \,\mathbf{.}\, y \neq z \wedge Xz) \tag{25}$$

On this identification, I is strictly entailed by Goodsell's hypothesis of *Unboundedness*, which says that every number is possibly instantiated; $\forall n \,\mathbf{.}\, \mathbb{N}n \to \Diamond\exists n$ (I and Unboundedness are equivalent in Goodsell's stronger logic).

[9]Typical set theories such as ZFC do not have complex function expressions, so Peano arithmetic is generally understood in a similar way in reductions of Peano arithmetic to set theories. However, this subtlety is usually ignored because typical set theories are extensional, so adding a function symbol where there was provably functional relation does not affect anything important. HKC is not extensional, or even intensional, so the replacement of a functional relation symbol by a function symbol can make a difference in some contexts.

At any rate, this identification will not be assumed. As Goodsell shows, establishing the necessity of arithmetic on any way of understanding 0 and $\cdot^+$ for which I is possibly true is as good as any other, since by necessitating Dedekind's categoricity theorem for arithmetic one can show that on any two such interpretations, it is necessary that if I holds on both interpretations then the two understandings have an isomorphic structure so make the same arithmetical sentences true.

Ctbl is defined as follows:

Definition 6 (Ctbl)**.** To be (at most) countable is to stand in an injective relation with $\mathbb{N}$.

$$\mathrm{Ctbl}_\sigma := \lambda X^{\sigma t} \mathbin{.} \exists R^{\sigma\nu t} \mathbin{.} \forall yz \in X \mathbin{.} (\exists n \in \mathbb{N} \mathbin{.} Ryn \wedge Rzn) \leftrightarrow y = z \tag{26}$$

If I is not assumed it is consistent that $\cdot^+$ is identity, in which case Ctbl only applies to properties with one instance and Countable Boolean Completeness will not have the intuitively intended effect of asserting the existence of least upper bounds for some properties with infinitely many instances. Be this as it may, Countable Boolean Completeness will still suffice to establish the necessity schema. If I fails then it necessarily fails,[10] in which case the necessity schema is trivially true anyway.

[10]By necessitating Lemma 5 we have $\Box(\Diamond I^* \to I^*)$, where $\Diamond I^* = I$, hence $\Diamond\Diamond I^* \to \Diamond I^*$, which is equivalent to $\neg I \to \Box\neg I$.

5 Proof of Necessity of Arithmetic

5.1 Modal Rigidity of Natural Number

Lemma 2 (HK). *Any property that zero has and which is closed under successor is had by every number.*

$$\forall X^{\nu t} \,\textbf{.}\, X0 \to (\forall n \,\textbf{.}\, Xn \to Xn^+) \to \forall n \in \mathbb{N} \,\textbf{.}\, Xn \tag{27}$$

Proof. Immediate from the definition. □

Lemma 3 (HK). *Every number is necessarily a number.*

$$\forall n \in \mathbb{N} \,\textbf{.}\, \Box \mathbb{N} n \tag{28}$$

Proof. By induction (Lemma 2). We have $\Box\mathbb{N}0$ by the rule of necessitation. For the inductive step, first necessitate $\forall n \,\textbf{.}\, \mathbb{N}n \to \mathbb{N}n^+$, then apply the converse Barcan formula and K to get

$$\forall n \,\textbf{.}\, \Box\mathbb{N}n \to \Box\mathbb{N}n^+ \tag{29}$$

which completes the induction. □

Lemmas 4 to 7 will make use of the following abbreviation.

Definition 7 (I^*). I^* is I without the preceding $\Diamond$.

$$I^* := \forall mn \in \mathbb{N} \,\textbf{.}\, 0 \neq m^+ \wedge (m^+ = n^+ \to m = n) \tag{30}$$

(so $I = \Diamond I^*$).

Lemma 4 (HK)**.** *For any two distinct numbers, it is necessary that, if I^*, then they are distinct.*

$$\forall mn \in \mathbb{N} \,\blacksquare\, m \neq n \to \Box(I^* \to m \neq n) \tag{31}$$

Proof. By induction on m and n. The case where $m = n = 0$ is trivial. Now consider the case where $m = 0$ and $n = i^+$ for some number i. In that case the result follows from Lemma 3 and the definition of I^*.

Now suppose the result holds for all values of n when m takes the value k, and consider the case where $m = k^+$. We establish this case by induction on n. The case with $n = 0$ is established in the previous paragraph (since m and n are symmetric). So suppose $n = j^+$ for some number j. Now suppose $i^+ \neq j^+$. Then $i \neq j$, and by the inductive hypothesis we have

$$i \neq j \to \Box(I^* \to i \neq j) \tag{32}$$

and by the definition of I^* we have

$$\Box(I^* \to i \neq j \to i^+ \neq j^+) \tag{33}$$

hence $\Box(I^* \to i^+ \neq j^+)$, so the result holds by induction. $\Box$

We are now in a position to prove the non-routine part of Lemma 1

Lemma 5 (HK)**.** *If it is possible that zero is not a successor and successor is injective,* *then it is in fact the case.*

$$I \to I^* \tag{34}$$

Proof. Since $I = \Diamond I^*$, the converse of the desired conclusion is

$$\neg I^* \to \Box \neg I^*. \tag{35}$$

which is

$$\begin{aligned} &\exists mn \in \mathbb{N} \centerdot (m^+ = 0 \lor (m^+ = n^+ \land m \neq n)) \\ &\qquad \to \Box \exists mn \in \mathbb{N} \centerdot m^+ = 0 \lor (m^+ = n^+ \land m \neq n). \end{aligned} \tag{36}$$

Suppose that $m^+ = 0$ for some $m \in \mathbb{N}$. Then by Lemma 3 (the necessity of natural numberhood), we have the result.

Now suppose $m^+ = n^+ \land m \neq n$ for some $m, n \in \mathbb{N}$. Then by the necessity of identity and Lemma 4, we have

$$\Box(I^* \to (m^+ = n^+ \land m \neq n)), \tag{37}$$

hence by Lemma 3 we have

$$\Box(I^* \to \exists mn \in \mathbb{N} \centerdot m^+ = n^+ \land m \neq n) \tag{38}$$

which implies $\Box \neg I^*$. $\square$

The next two lemmas establish the necessity of distinctness, Sum, and Prod, conditional on I.

Lemma 6 (HK). *For any two distinct numbers, it is necessary that, if I, then they are distinct.*

$$\forall mn \in \mathbb{N} \,.\, m \neq n \rightarrow \Box(I \rightarrow m \neq n) \tag{39}$$

Proof. By necessitating Lemma 5 we have $\Box(I \rightarrow I^*)$, so the result is immediate from Lemma 4. $\Box$

Lemma 7 (HK). *For any three numbers, either I necessitates that the first two add/multiply to make the third, or I necessitates this is not so.*

$$\forall mno \in \mathbb{N} \,.\, \Box(I \rightarrow \mathrm{Sum}\, mno) \vee \Box(I \rightarrow \neg\, \mathrm{Sum}\, mno) \tag{40}$$

$$\forall mno \in \mathbb{N} \,.\, \Box(I \rightarrow \mathrm{Prod}\, mno) \vee \Box(I \rightarrow \neg\, \mathrm{Prod}\, mno) \tag{41}$$

Proof. It suffices to show

$$\forall mno \in \mathbb{N} \,.\, \mathrm{Sum}\, mno \rightarrow \Box\, \mathrm{Sum}\, mno \tag{42}$$

$$\forall mno \in \mathbb{N} \,.\, \mathrm{Prod}\, mno \rightarrow \Box\, \mathrm{Prod}\, mno \tag{43}$$

$$\forall mno \in \mathbb{N} \,.\, \neg\, \mathrm{Sum}\, mno \rightarrow \Box(I \rightarrow \neg\, \mathrm{Sum}\, mno) \tag{44}$$

$$\forall mno \in \mathbb{N} \,.\, \neg\, \mathrm{Prod}\, mno \rightarrow \Box(I \rightarrow \neg\, \mathrm{Prod}\, mno) \tag{45}$$

Eqs. (42) and (43) are established by a straightforward but tedious induction unpacking the definition of Sum and Prod. Eqs. (44) and (45) then follow by Lemmas 1, 5 and 6. $\square$

Countable Boolean Completeness is used essentially, and only, in the following lemma (which in conjunction with Lemma 3 implies natural numberhood is a rigid property in the sense of Definition 2):

Lemma 8 (HKC). *Every property that every number has necessarily, is such that necessarily, every number has it.*

$$\forall X^{\nu t} \,\mathbf{.}\, (\forall n \in \mathbb{N} \,\mathbf{.}\, \Box X n) \to \Box \forall n \in \mathbb{N} \,\mathbf{.}\, X n \tag{46}$$

Proof. Define

$$\mathbb{N}^{\{\cdot\}} := \lambda X^{\nu t} \,\mathbf{.}\, \exists n \in \mathbb{N} \,\mathbf{.}\, X = \{n\}, \tag{47}$$

where $\{n\} := \lambda m \,\mathbf{.}\, n = m$.

Let B be a least upper bound of $\mathbb{N}^{\{\cdot\}}$. Since B is an upper bound, we have $\forall n \in \mathbb{N} \,\mathbf{.}\, \Box \forall m \,\mathbf{.}\, \{n\} m \to B m$, hence also $\forall n \in \mathbb{N} \,\mathbf{.}\, \Box B n$ by the necessity of identity. It immediately follows that $\forall n \in \mathbb{N} \,\mathbf{.}\, \Box B n^{+}$. Thus $\lambda n \,\mathbf{.}\, B n^{+}$ is also an upper bound of $\mathbb{N}^{\{\cdot\}}$.

Since B is a least upper bound it necessitates $\lambda n \,\mathbf{.}\, B n^{+}$, which is to say

$$\Box \forall n \,\mathbf{.}\, B n \to B n^{+}, \tag{48}$$

and we also have $\Box B0$, since $\{0\}$ is an instance of $\mathbb{N}^{\{\cdot\}}$. Thus by Lemma 2 we have

$$\Box\forall n \,\text{.}\, \mathbb{N}n \to Bn. \tag{49}$$

Now, for an arbitrary X with $\forall n \in \mathbb{N} \,\text{.}\, \Box Xn$, X is an upper bound of $\mathbb{N}^{\{\cdot\}}$, so $\Box\forall n \,\text{.}\, Bn \to Xn$, and thus $\Box\forall n \in \mathbb{N} \,\text{.}\, Xn$ as required. $\Box$

5.2 Proof of Necessity Schema

Definition 8 (Arithmetical formula/sentence)**.** The *numerical terms* consist of variables of type ν, the symbol 0, and terms of the form $\mathbf{n}^+$, where $\mathbf{n}$ is a numerical term.

The *arithmetical formulae* are either *atomic* formulae of the forms

$$\mathbb{N}\mathbf{n} \quad \mathrm{Sum}\,\mathbf{mno} \quad \mathrm{Prod}\,\mathbf{mno} \quad (\mathbf{m} = \mathbf{n}) \tag{50}$$

where $\mathbf{m}$, $\mathbf{n}$, $\mathbf{o}$ are numerical terms, or *complex* formulae of the forms

$$(\neg\mathbf{A}) \quad (\mathbf{A} \vee \mathbf{B}) \quad (\mathbf{A} \wedge \mathbf{B}) \quad (\forall \mathbf{v} \in \mathbb{N} \,\text{.}\, \mathbf{A}) \tag{51}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arithmetical formulae and $\mathbf{v}$ is a variable of type ν.

An *arithmetical sentence* is an arithmetical formula with no free variables.[11]

Theorem 9. *Where* $\mathbf{A}$ *is an arithmetical formula and* $\vec{\mathbf{n}}$ *is a sequence of variables*

[11]Goodsell uses a definition of arithmetical formula where the only primitive relation is the ordering of numbers. This definition is too restrictive, since arithmetical formulae so-defined will not include anything about addition or multiplication, so will not include even the theorems of Peano arithmetic. Goodsell's more restrictive definition also does not immediately yield Theorem 11 in Section 6 of this paper, since Presburger arithmetic, which is axiomatized by the axioms of Peano arithmetic besides those concerning multiplication, is decidable.

of type ν that includes the free variables of $\mathbf{A}$*, the sentence*

$$\forall \vec{\mathbf{n}} \in \mathbb{N} \centerdot \Box(I \to \mathbf{A}) \vee \Box(I \to \neg\mathbf{A}) \tag{52}$$

is a theorem of HKC.

Proof. By induction on the complexity of $\mathbf{A}$. The atomic formulae are predications of $\mathbb{N}$, Sum, Prod, and $=$. For these, we use use, respectively, Lemma 3, then Lemma 7, then either the necessity of identity or Lemma 6.

Suppose formulae $\mathbf{A}$ and $\mathbf{B}$ satisfy Eq. (52), then the same goes for $(\neg\mathbf{A})$, $(\mathbf{A}\vee\mathbf{B})$, and $(\mathbf{A}\wedge\mathbf{B})$ by the inclusion of K for $\Box$.

Finally, suppose $\mathbf{A}$ satisfies Eq. (52), and let $\mathbf{m}$ be a variable of type ν. Then we have

$$\forall \vec{\mathbf{n}} \in \mathbb{N} \centerdot \forall \mathbf{m} \in \mathbb{N} \centerdot \Box(I \to \mathbf{A}) \vee \Box(I \to \neg\mathbf{A}) \tag{53}$$

which implies

$$\forall \vec{\mathbf{n}} \in \mathbb{N} \centerdot (\forall \mathbf{m} \in \mathbb{N} \centerdot \Box(I \to \mathbf{A})) \vee \exists \mathbf{m} \in \mathbb{N} \centerdot \Box(I \to \neg\mathbf{A}) \tag{54}$$

By Lemma 8 we have

$$(\forall \mathbf{m} \in \mathbb{N} \centerdot \Box(I \to \mathbf{A})) \to \Box(I \to \forall \mathbf{m} \in \mathbb{N} \centerdot \mathbf{A}), \tag{55}$$

And Lemma 3 yields

$$(\exists \mathbf{m} \in \mathbb{N} \,.\, \Box(I \to \neg \mathbf{A})) \to \Box(I \to \neg \forall \mathbf{m} \in \mathbb{N} \,.\, \mathbf{A}) \tag{56}$$

hence

$$\forall \vec{\mathbf{n}} \in \mathbb{N} \,.\, \Box(I \to \forall \mathbf{m} \in \mathbb{N} \,.\, \mathbf{A}) \vee \Box(I \to \neg \forall \mathbf{m} \in \mathbb{N} \,.\, \mathbf{A}) \tag{57}$$

as required. □

5.3 Extensions of the Language of Arithmetic

The necessity schema Eq. (1) on page 2 does not quite assert the necessity of arithmetic *tout court*, since the necessity of arithmetic is naturally thought of as a quantified semantic claim: that every arithmetical sentence is necessarily true, or necessarily false, given I. This would be formalized in the following form:

$$\forall x \in \mathrm{Sent} \,.\, \Box(I \to \mathrm{T}\, x) \vee \Box(I \to \neg\, \mathrm{T}\, x) \tag{58}$$

We may understand the syntactic functions Sent and T as applying to Gödel numbers of strings rather than of strings themselves, in which case Eq. (58) is in a broad sense itself a claim about numbers. However, if these constants are given standard definitions, Eq. (58) does not follow from the necessity schema in HK, because the definition of T is not given in the language of arithmetic as given in Definition 3 (due to Tarski's theorem on the undefinability of truth).

Nevertheless, HKC does imply Eq. (58), on standard recursive definitions of Sent and T.[12] The method of proof is an induction on formula-complexity exactly parallel to Theorem 9, only where the metavariables '**A**' and '**B**' are replaced with object-language variables of type ν. The proof is omitted because giving it precisely requires notational conventions for quotation that will take too much space to describe.

It is also worth remarking on another straightforward generalization of Theorem 9. Let an *arithmetical signature* be a finite list of terms of with the types of either numerals (ν), numerical predicates (νt, $\nu\nu t$, ...), or numerical functions ($\nu\nu$, $\nu\nu\nu$, ...). If the language of arithmetic is augmented by an arithmetical signature, one may wonder whether now it is possible to assert something contingent. We have the following result:

Definition 9 (Fixed)**.** When $\mathbf{a}$ is a term of type ν, we write Fixed($\mathbf{a}$) as an abbreviation for

$$\mathbb{N}\mathbf{a}. \tag{59}$$

When $\mathbf{f}$ is a term of the type of a k-adic function from numbers to numbers which does not contain any free occurrences of the variables $n_1, \dots, n_k$, we write Fixed($\mathbf{f}$) for

$$\forall n_1 \dots n_k \in \mathbb{N} \,\blacksquare\, \mathbb{N}(\mathbf{f}n_1 \dots n_k). \tag{60}$$

When $\mathbf{R}$ is a term of the type of a k-adic relation of numbers which does not contain

[12] See, e.g., Definition 8.1 of Halbach [14, p. 64].

any free occurrences of the variables $n_1, \ldots, n_k$, we write Fixed($\mathbf{R}$) for

$$\forall n_1 \ldots n_k \in \mathbb{N} \,\mathbf{.}\, \Box(I \to \mathbf{R}n_1 \ldots n_k) \vee \Box(I \to \neg\mathbf{R}n_1 \ldots n_k). \tag{61}$$

When Σ is a signature, i.e., a finite list of terms (not necessarily constants) of the above three categories of type, we write Fixed(Σ) for the conjunction of terms of the form

$$\mathrm{Fixed}(\mathbf{a}), \tag{62}$$

where $\mathbf{a}$ is a term in Σ.

The proof of Theorem 9 immediately generalizes as follows:

Theorem 10. *Let* $\mathbf{A}$ *be any sentence of the language of arithmetic plus the signature* Σ*, and let* $\vec{\mathbf{v}}$ *be the union of free variables of terms in* Σ *and let* $\vec{\mathbf{n}}$ *be a sequence of type-*ν *variables disjoint from* $\vec{\mathbf{v}}$*, including all those in* $\mathbf{A}$ *besides those in* $\vec{\mathbf{v}}$*. Then*

$$\forall \vec{\mathbf{v}} \,\mathbf{.}\, \mathrm{Fixed}(\Sigma) \to \forall \vec{\mathbf{n}} \in \mathbb{N} \,\mathbf{.}\, \Box(I \to \mathbf{A}) \vee \Box(I \to \neg\mathbf{A}) \tag{63}$$

is a theorem of HKC*.*

Proof. As Theorem 9 but with additional base cases for the new signature, which are all handled by the assumption of Fixed(Σ). $\Box$

It follows, for example, that everything that can be said in the language of arithmetic plus a truth predicate (given Eq. (58)) is either necessarily true if I or necessarily false if I.

6 Inconsistency of Boolean Completeness with Maximalized Systems

Various authors have been tempted by the idea that for some sufficiently rich axiomatic system of arithmetic, the theorems of that system are exactly the necessarily true sentences of arithmetic. The qualification to only a special class of true sentences, in this case the necessarily true ones, is needed, since by Gödel's first incompleteness theorem the truths of arithmetic are not recursively axiomatizable. Unsurprisingly, the necessity of arithmetic will spell trouble for this sort of picture.

Bacon and Dorr [5] make the picture more precise with the operation of *maximalizing* a theory.

Definition 10 (Modal maximalization; Max)**.** The (modal) *maximalization* of a set of sentences Γ, $\mathrm{Max}(\Gamma)$, is the set of sentences which includes Γ as well as every sentence of the form $\Diamond \mathbf{A}$, where $\mathbf{A}$ is a sentence such that $\neg\mathbf{A}$ is not included in Γ.[13]

They show that various interesting theories stated in the language of HK to have consistent maximalizations (supposing that they are consistent to begin with). Of particular interest to them is the system *Classicism*, which can be axiomatized by adding to HK every instance of the schema stating that relations are individuated by necessary coextensiveness:

$$\forall XY^{\sigma_1 \ldots \sigma_n t} \,\boldsymbol{.}\, (\Box \forall z_1 \ldots z_n \,\boldsymbol{.}\, Xz_1 \ldots z_n \leftrightarrow Yz_1 \ldots z_n) \to X = Y \tag{64}$$

Bacon and Dorr isolate Boolean Completeness as a plausible and interesting extension

[13] Bacon [3] calls the property of having a consistent maximalization *coherence*.

to Classicism, and pose the question of whether any extension of Classicism includes Boolean Completeness and has a consistent maximalization. The foregoing results answer this question in the negative, in light of the following general theorem about maximalizations of axiomatizable theories which are consistent with I.[14]

Theorem 11. *The maximalization of any recursively enumerable extension of* HK *that is consistent with* I *is inconsistent with some instance of Eq.* (1) *in* HK.

This theorem also shows that Rigid Comprehension, the principle Goodsell [12] used to establish the necessity schema, is inconsistent with the maximalization of any recursively enumerable extension of HK.

Proof of Theorem 11. Let Γ be a recursively enumerable set of sentences that does not include $\neg I$. If Γ is consistent in HK then the set of arithmetical sentences $\mathbf{A}$ for which $I \to \mathbf{A}$ is provable from Γ in HK, is a consistent extension of Peano arithmetic.

By Gödel's first incompleteness theorem it follows that if Γ is consistent in HK, then there is an arithmetical sentence $\mathbf{A}$ such that both

$$I \wedge \mathbf{A} \qquad I \wedge \neg \mathbf{A} \tag{65}$$

are consistent with Γ in HK. $\mathrm{Max}(\Gamma)$ therefore includes

$$\Diamond(I \wedge \mathbf{A}) \qquad \Diamond(I \wedge \neg \mathbf{A}) \tag{66}$$

[14] Assuming Classicism is consistent with I, i.e., that Classicism does not prove that there are necessarily only finitely many things of any type. Classicism plus I has equal consistency strength with higher-order logics which include axioms of infinity, e.g., those of Church [9] or Henkin [16], which in turn have much lower consistency strength than ZF set theory. This can be confirmed by inspecting Bacon and Dorr's model theory for Classicism.

which by basic modal logic yield

$$\neg(\Box(I \to \mathbf{A}) \wedge \Box(I \to \neg\mathbf{A})), \tag{67}$$

Which is the negation of an instance of the necessity schema.[15] $\Box$

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