**Universität Bielefeld/IMW**

**Working Papers**
**Institute of Mathematical Economics**

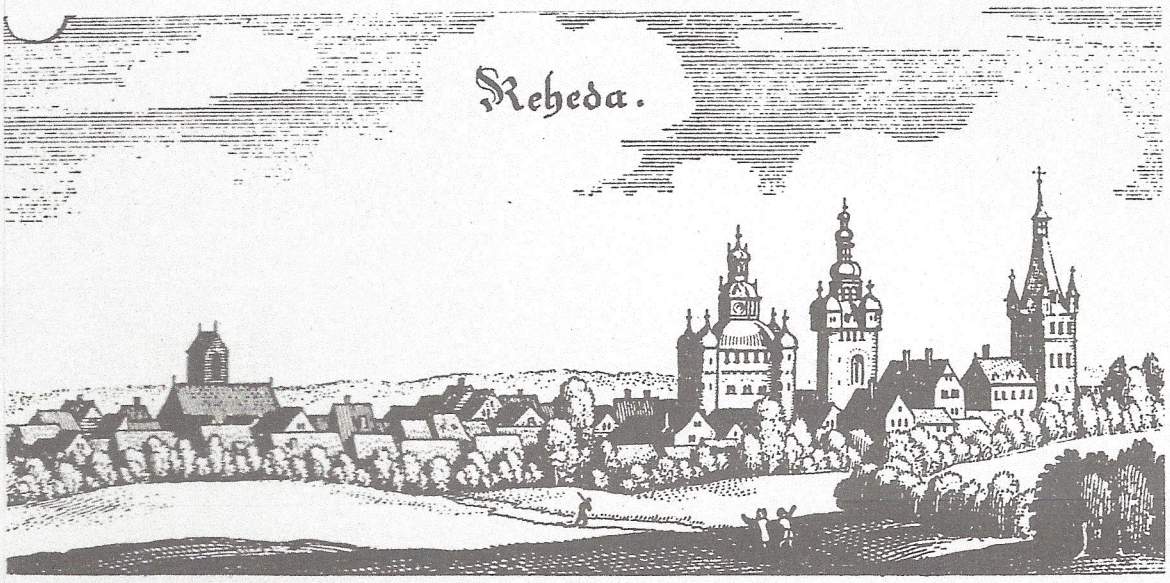
**Arbeiten aus dem**
**Institut für Mathematische Wirtschaftsforschung**

Nr. 25

# Hans W. Gottinger

# Existence of a Utility on a Topological Semigroup

January 1975




Anno 1600 nach Merian

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Schloß Rheda
484 Rheda
Bundesrepublik Deutschland
Federal Republic of Germany

# EXISTENCE OF A UTILITY ON A TOPOLOGICAL SEMIGROUP

HANS W. GOTTINGER

UNIVERSITY OF BIELEFELD

## 1. Introduction

This paper establishes conditions under which a 'measurable' (additive) utility exists where the domain of definition of a utility satisfies mixed algebraic-topological properties inducing a topological semigroup. The various considerations leading to the proof of existence are important in economic theory for at least two reasons:

(1) They lead to a comprehensive treatment of additive utility theory comprising topological and algebraic elements.

(2) They establish a link between additive and expected utility theory to the extent that the same mathematical considerations leading to the derivation of an additive representation are also valid for proving the expected utility theorem.

As regards the first point we would like to recall some well-known facts. The existence of order-preserving (non-additive) utility functions has been proved in general by constructing an isomorphism between an order (preference) system $(X, \precsim)$ and a numerical relational system $(Re, \leq)$. To obtain necessary and sufficient conditions for

embeddability an order topology - in analogy to the natural topology of the real number system - has to be established.
For an additive representation we need some specific algebraic assumptions, in addition to those of the order topology. Both together we find in an ordered topological semigroup.
A construction of additive functions on archimedean ordered semigroups has been accomplished earlier by Krantz et.al. [2], however, although there is some natural connection with their results, our construction is somewhat different and the proof techniques are entirely novel. Also results obtained in the book by L. Fuchs [3] are relevant. As indicated in (2), we wish to arrive at a representation of expected utility as a natural consequence of the embedding process. As pointed out in [2] p. 394 axiomatizations of an order (preference) system leading directly to a representation of utility (combining properties of additive and expected utility) 'are not entirely satisfactory', and it is suggested to look for axioms 'that are formulated in terms of primitives only'.
To my knowledge, the idea of using additive utility theory for deriving expected utility is due to Arrow [1], his proof is only given for the discrete case and lacks the elegance of measure theory.

## 2. Preliminaries

Let $X$ be the set of sure prospects. A binary (preference) relation $\prec$ on $X$ is said to be a complete order (or simple order) if it is transitive, antisymmetric and connected.[1] Subsets of $X$ could be identified as intervals, denoted by $(a,b)$, $[a,b)$, $(a,b]$ and $[a,b]$ with obvious properties. An interval is open if it is either of the form $(a,b)$, $I(a) = \{x : x \prec a\}$, or $I(x) = \{a : a \prec x\}$.

Let $X$ and $Y$ be two sets endowed with a simple order. Then a mapping $f : X \to Y$ is called monotone (or order-preserving) if $f(x) \prec f(y) \Leftrightarrow x \prec y$.

Let the intervals $[a,b)$ generate a topology $\mathcal{T}_1$, and the open intervals generate a topology $\mathcal{T}$, which is referred to as the interval- or order topology. This topology is a Hausdorff topology. In general, $\mathcal{T}_1$ is finer than $\mathcal{T}$. The collection of $I(x)$ generates a $\sigma$-algebra, the elements of which may be called Borel sets. Every finite measure $\mu$ defined on this collection is a Borel measure, it is called normed if $\mu(X) = 1$. Let $J = [0,1]$ be the real unit interval. The following result is well-known:

Let $X$ be a simply $\prec$-ordered set. The mapping $H : X \to J$ with representation $H(x) = \mu(I(x))$ has the following properties:

(i) $H$ is weakly monotone, i.e. $x \prec y \Rightarrow H(x) \prec H(y)$

(ii) If $x \in X$ has a countable neighborhood basis w.r.t. $\mathcal{T}_1$ then $H$ in $X$ is continuous w.r.t. $\mathcal{T}_1$.

(iii) If there exists a countable sequence in $X$, (unbounded above or below) then $\lim H(x) = 1$ or $0$ whenever $x$ increases or decreases indefinitely.

(i) directly follows from the monotony property of $\mu$. Let $a$ be a point with a countable neighborhood basis given $\mathcal{T}_1$. Then there

exists a weakly monotone sequence of points $x_n$ converging to a, given $\mathfrak{T}_1$. Then we have $\mu(I(a)) = \lim \mu(I(x_n))$ (see [4], § 9), and therefore (ii) holds. Likewise one proves (iii).

If $H : X \to J$ is a mapping from a simply ordered set X into J satisfying the conditions (i), (ii), (iii), then a normed Borel measure $\mu$ is defined and represented by $\mu(I(x)) = H(x,\mu)$ and $\mu(X) = 1$. A proof of this statement can be obtained from standard results in measure theory ([4], Ch. II). Let then $\mathcal{M}$ be a set of normed Borel measures on a simply ordered set X. For every $\mu \in \mathcal{M}$ let $H(x,\mu) = \mu(I(x))$. Furthermore let $\mu \prec \nu$, $\mu, \nu \in \mathcal{M}$ if and only if (briefly iff) $H(x,\mu) \leq H(x,\nu)$ for all $x \in X$ and strict inequality exists at least on one open set. Then, it is said, a sequence $\mu_n$ converges to $\mu \in \mathcal{M}$, $\mu = \lim \mu_n$, if $H(\cdot, \mu_n)$ converges pointwise to $H(\cdot,\mu)$. Construct a monotone and continuous mapping $f : X \to J$ and suppose Borel measures $\mu$ exist in $\mathcal{M}$. If one sets $E_f(\mu) = \int f(x)\, d\mu(x)$ then $\mu \to E_f(\mu)$ is a monotone mapping and $E_f(\lim \mu_n) = \lim E_f(\mu_n)$ holds for every convergent sequence $\mu_n$ in $\mathcal{M}$. This conclusion uses Lebesgue's theorem of bounded convergence (see [4], p. 110).

From a measurement-theoretic point of view most of utility theory is concerned with the problem of embedding isomorphically a $\prec$-simply ordered set X, under preservation of its structure, into the set of real numbers. Necessary and sufficient conditions are known for achieving this, most of these conditions require a topological structure on X in order to prove the existence of a continuous, order-preserving utility function.

In order to find a representation by additive utilities usually a qualitative independence assumption is used for the order system $(X, \prec)$. To deal effectively with this kind of assumption we will argue here that a natural way consists of enriching the structure of X.

This enrichment is achieved by defining an algebraic concatenation operation such that $\circ(x,y) = xy$ or simply $(x,y) \longrightarrow xy$. Here $\circ : X \times X \longrightarrow X$ is a continuous function such that (i) X is a topological space (Hausdorff) and (ii) $\circ$ is associative. This turns X into a topological semigroup. Then we could pose the embedding problem for the enriched structure, in particular we want to know under which necessary and sufficient conditions there exists a monotone homeomorphism (topological isomorphism) $f : X \longrightarrow Re$ such that $f(xy) = f(x) + f(y)$. This form would naturally represent 'additive utility', where the ordered space on which utility is defined is a topological space, the operation of the semigroup being continuous in the topology of this space.

The concept of 'homeomorphism' is used in accordance with N. Bourbaki [5], i.e. a topological isomorphism f is a monotone, homeomorphic mapping from a topological space onto another topological space. Also, in what follows other concepts such as 'limit', 'isolated point', 'quotient space', 'compactness' etc. are used in the Bourbakian terminology.

## 3. Construction of the Embedding

In this section let us first generally assume that every nonempty bounded subset of X has an upper limit. X is then locally compact w.r.t. the order topology. The concatenation operation in X should satisfy the following requirements:

**Definition 1:** (i) $(x,y) \longrightarrow xy$ is continuous

(ii) $x \prec y \Longrightarrow vx \prec vy$ for all $v \in X$.

(iii) every element $x \in X$ generates a semigroup.

(iii) assures that concatenation is associative, in particular that it is possible to compute copies according to associativity. We assume that $x$, $x \cdot x = x^2$, $x \cdot x \cdot x = x^3$, ... are perfect copies of a standard element $x$, in the sense that if $x \cdot x = x^2$ and $f$ is a function then $f(a \cdot a) = 2\, f(a)$. The sequence $x, x^2, x^3, \ldots$ could be considered as a standard sequence according to [2].

Lemma 1: Def. 1, (i) - (iii) implies the archimedean property, i.e. if $x \prec x^2$ then $x \prec x^2 \prec x^3 \prec \ldots$ and this sequence is unbounded from above. Likewise, if $x^2 \prec x$ then $x \succ x^2 \succ x^3 \succ \ldots$ is unbounded from below.

Proof. From $x \prec x^2$, due to (ii), we get by complete induction $x \prec x^2 \prec x^3 \prec \ldots$ . Suppose this sequence is bounded from above. Then, by our construction, $\lim x^n = v$ exists. Because of (i), inconjunction with (iii), we have $vx^2 = \lim x^{n+2} = \lim x^{n+1} = vx$. This contradicts (ii), by assumption $x \prec x^2$. Q.E.D.

Lemma 2: Suppose (ii) of Def. 1 and (iv) are satisfied:

(iv) there is a neutral element $e$ such that $ve = v$ for all $v$.

Then the sets $X_+ = \{v : e \precsim v\}$ and $X_- = \{v : v \precsim e\}$ form subsemigroups of $X$.

Proof: Suppose $e \precsim x, y$, then $e \precsim x = xe \precsim xy$. Q.E.D.

The next lemma is extremely useful and forms a partial result for our main theorems.

Lemma 3: Let conditions (i) - (iv) be satisfied. Let $e$ be the smallest element, not being isolated. Then there exists an isomorphism $f$ from the semigroup $Re_+$ of all non-negative real numbers onto $X$ such that $f(0) = e$.

Proof.

1. Let U be a compact, non-trivial interval $[e,a]$ in X. For every $x \in (e,a]$ there exists a smallest natural number n(x) such that $a \prec x^{n(x)+1}$, by Lemma 1. For some real number r let $\langle r \rangle$ be that particular integer for which $0 = r - \langle r \rangle \prec 1$. (In general, there always exist integers d(x), $0 \leq d(x) \leq 1$, so that $\langle (r+s)n(x) \rangle = \langle rn(x) \rangle + \langle sn(x) \rangle + d(x)$ for real s,r and every $x \in U \setminus \{e\}$. Define $x^{d(x)} = e$ if $d(x) = 0$. Clearly, for every $x \in U$ there is defined a mapping from J into U whenever we set $f_x(r) = e$ for $x = e$ and $f_x(r) = x^{\langle rn(x) \rangle}$. $U \setminus \{e\}$ is a $\prec$-directed set, hence the $f_x$ form a net on the compact product space $U^J$. This net must contain a converging subnet $\{f_{x(i)} : i \in M\}$ associated to a directed index set M. Let the limit be denoted by g. Then $g(0) = e$ and $g(1) = a$, since $x(i) \to e$ implying $\lim x(i)^{n(x(i))} = \lim x(i)^{n(x(i))+1}$, the limit equals a, by definition of n(x). For all $0 \leq r,s,r+s \leq 1$ we have $f_x(r+s) \sim f_x(r)\, f_x(s)^{d(x)}$, therefore by forming limits in the subnet we get $g(r+s) \sim g(r)g(s)$.

2. It remains to be shown that g is continuous at 0. Let C be the intersection of all compact sets $\overline{g((0,r))}$ for all $0 < r \leq 1$. By continuity of concatenation one computes $CC \subset C$. Since every element of U, distinct from e, does have a copy not in U, C will only contain the element e. Therefore, the assertion is proved. By the functional property of g, g is continuous from above in every point of $[0,1)$.

Now let $0 < r \leq 1$, the net of $g(r') \in U$, $r' < r$ contains converging subnets since U is compact. Pick up a particular one, say $\{g(r'(i))\}$, and let it have the limit v. Since r-r'(i) converges to zero, we get $g(r) = g(r'(i))g(r-r'(i)) = \lim g(r'(i)) \lim g(r-r'(i)) = ve = v$.

Therefore, all converging subnets have the same limit $g(r)$, this shows continuity of $g$ at $r$ from below.

Let $q$ be a rational number in terms of $p/2^n$, $p,n$ non-negative integers. If $q = p/2^n = p'/2^{n'}$ with $n' \leq n$ then $g(1/2^n)^p = g(1/2^n)^{p'2^{n-n'}} = g(1/2^{n'})^{p'}$.

Now let us define $h(q) = g(1/2^n)^p$.

For doing this we have exploited condition (iii), and we can easily see that for two non-negative dyadic rational numbers $q$ and $q'$ the relation $h(q+q') = h(q)h(q')$ holds. Obviously, $h$ and $g$ coincide on the set of dyadic rational numbers between 0 and 1.

For an arbitrary non-negative $r$ we define $f(r) = h(\langle r \rangle)g(r-\langle r \rangle)$.

If $n$ is a natural number then by approximation from above $\lim f(r) = h(n)\lim g(r-n) = h(n)e = h(n)$ and by approximation from below $\lim f(r) = h(n-1)\lim g(r-n-1) = h(n-1)g(1) = h(n-1)h(1) = h(n)$.

Hence $f$ is an everywhere continuous function on $Re_+$ which coincides with $g$ on the dense subsemigroup of all non-negative dyadic rational numbers. Therefore $f$ is a continuous (algebraic) homomorphism. By the general properties of this homomorphism and Lemma 1 it follows that $f$ is unique. Let $f(Re_+)$ be a connected subset of $X$, hence an interval, since this subset is not bounded from above $f(Re_+) = X$ holds. By monotony $f$ can be extended continuously on the one-point compactifications of $Re_+$ and $X$.

Therefore $f$ is a homeomorphism. Q.E.D.

Lemma 4: Let (i) - (iv) be satisfied. Let $e$ be the smallest element being isolated. Moreover, let the following condition be satisfied:

(v) if $x \prec y$ there exists a $d$ such that $xd \sim y$.

Then there exists a monotone mapping $f$ from the set $Z_+$ of all non-negative integers onto $X_+$ so that $f(xy) = f(x)+f(y)$ for all $x,y \in Z_+$. If $ex = x$ is assumed then $f$ is an isomorphism.

Proof. Denote the lower bound of $X \setminus \{e\}$ by a, by assumption $e \prec a$. Define $f(0) = e$, $f(n) = a^n$ for $n = 1,2,\ldots$ Now let $x \in X$. By Lemma 1 there is a smallest natural number n so that $a^{n-1} \prec x \precsim a^n$. Then there exists a v such that $a^{n-1}v \sim x$, if there were $a \prec v$ then $a^n = a^{n-1}a \prec a^{n-1}v$, contradicting the assumption.
Hence $e \prec v \precsim a$. Since a is the lower bound of $X \setminus \{e\}$ we must have $v \sim a$ and $x \sim a^{n+1}$. Q.E.D.

Now let us sharpen condition (iii).

Definition 2.
(iii)' Every element x belongs to a semigroup $(X,\cdot)$. If $xy \sim e$ then also x and y belong to $(X,\cdot)$.

Lemma 5. Let conditions (i), (ii), (iii)', (iv),(v) be satisfied, suppose e is not the smallest element of X and $X_+$ is discrete. Then X is isomorphic to the additive group of all integers.

Proof. Suppose $xy \sim e$, then $(yx)^n \sim y(xy)^{n-1}x \sim yx$. Hence, because of Lemma 1 and (ii) it follows that $yx \sim e$. Therefore x generates a group $(X,\cdot)$. By Lemma 4 there exists an isomorphism from $Z_+$ onto $X_+$, hence $X_-$ cannot be connected, since otherwise $f(1)$ would be in the non-trivial, connected space $f(1)X_-$, but $f(1)$ is isolated. Because of Lemma 3 the upper bound a of the set $X_- \setminus \{e\}$ is smaller than e. Then there exists an a' so that $aa' \sim e$ because of (iv). Also, because of (ii), $e \prec a'$ since $a' \precsim e$ would imply $e \sim aa' \precsim ae \sim a$. We have $a \sim ae \prec af(1)$, consequently $e \precsim af(1)$ since there is no element between a and e, i.e. $aa' \precsim af(1)$, therefore $a' \precsim f(1)$. Since $f(1)$ is the smallest element next to e, $a' = f(1)$ holds. Now define $g(0) = e$ and $g(-n) = a^n$ for $n = 1,2,\ldots$ Let $x \prec e$. Then there exists a largest natural number n so that $g(-n-1) \prec x \precsim g(-n)$. According to (iv) there exists a v such that $x = g(-n-1)v$, and because

of $e \prec v$ we must even have $v = f(m)$ for some $m > 1$. Since according to (iii)' $g(-1)$ and $f(1)$ belong to a group we have

$$g(-n-1)f(m) = \begin{cases} g(-n-1+m), & \text{if } m < n+1 \\ f(m-n-1), & \text{otherwise.} \end{cases}$$

In any case $g(-n-1)f(m) \succsim g(-n)$. Hence for $m = 1$, $x = g(-n)$ holds. The mapping $h$ with $h(n) \sim f(n)$, $n = 0,1,2,\ldots$ and $h(n) \sim g(n)$, $n = -1,-2,-\ldots$ is the appropriate isomorphism. Q.E.D.

Lemma 6. Let conditions (i), (ii), (iii)', (iv) be satisfied, and let e be not the smallest element that is not isolated in X. Then X is isomorphic to the additive group of all real numbers.

Proof. If e is not isolated from above then there exists an isomorphism f from $Re_+$ onto $X_+$, by Lemma 3; if e ist not isolated from below then there exists an isomorphism g from (the additive semigroup of all non-positive real numbers) $Re_-$ onto $X_-$. Because of (ii) the sets $af(Re_+)$ and $ag(Re_-)$ are non-trivial intervals without gaps (since they are connected), and they contain a as a lower or upper bound, respectively.

Now suppose e is isolated from below but not from above. Then the set $X \setminus \{e\}$ has a largest element a and $[a,e]$ is a gap (contradicting the fact tha a (and by non-triviality of $af(Re_+)$) also e must be contained in $af(Re_+)$. An analogous conclusion holds if e is isolated from above but not from below. Therefore e is not isolated from both sides and both f and g exist. Now let $s < 0$ be such that $0 \prec g(s)f(1)$, this is possible by the assumption of continuity (i). So e is contained in the interval $g(s)f([0,1])$. Hence there exists s' so that $g(s)f(s') = e$. By appropriate norming of f and g it is always possible to obtain $g(-1)f(1) = e$.

Now let $0 \prec t \prec 1$. Then there exists t' such that $g(t')f(t) = e$, since we have $g(-1)f(t) \prec e$ on the one hand, and $e \prec g(0)f(t) = ef(t) \sim f(t)$. By (iii)' everly element $f(1/2^n)$ generates a cyclical subgroup in X. For the element $g(t')$ with $g(t')f(1/2^n) = e$ we have $e = (g(t')f(1/2^n))^{2^n} = g(t')^{2^n} f(1)$. Therefore $2^n t' = -1$ or $t' = -1/2^n$. The union of all subgroups generated by $f(1/2^n)$ is a chain of groups and therefore again a group. Furthermore, this group is dense in X since it is dense in both $X_+$ and $X_-$. The mapping h defined by

$$h(r) = \begin{cases} f(r) & \text{if } 0 \preceq r \\ g(r) & \text{if } r \prec 0 \end{cases}$$

is the appropriate isomorphism. Q.E.D.

## 4. Main Results

Let us summarize the results obtained so far..

Theorem 1. Let X be a simply ordered set in which every non-empty set has an upper bound. Let a binary operation ('concatenation') be defined on X satisfying the following natural conditions:

(i) $(x,y) \to xy$ is continuous

(ii) $x \prec y$ implies $vx \prec vy$ for all v.

(iii) Every $x \in X$ belongs to a semigroup

(iv) There exists a neutral element e such that $ve \sim v$ for all v.

(1) If e is minimal and not isolated then X is isomorphic to the additive semigroup of all non-negative real numbers.

(2) If e is not minimal and not isolated and suppose the following additional condition is satisfied:

(v) If $xy \sim e$ then x and y belong to a group.

Then X is isomorphic to the additive group of all real numbers.

(3) Finally, if e is isolated and suppose the following condition, in addition to (i) - (iv), is satisfied:

(vi) If $x \prec y$ then there exists a d such that $xd \sim y$.

If e is minimal and isolated then it follows from (i) - (iv) and (vi) that: $X \setminus \{e\}$ is isomorphic to the semigroup of all positive integers; if e is not minimal but isolated then it follows from (i) - (vi) that $X \setminus \{e\}$ is isomorphic to the additive group of all integers.

Remark

It is natural to raise the question to which extent it is really necessary to postulate the existence of a neutral element for the derivation of the previous results. In order to give some more specific statements we have to strengthen the weak associativity postulate as expressed in (iii) by a much stronger requirement. This requirement also follows from natural considerations in L. Fuchs [3] p.227, where it is proved that an archimedean, naturally ordered semigroup is commutative.

Definition 3.

(iii)" $(X, \cdot)$ is a commutative semigroup. The embedding property of a regular commutative semigroup into a group is reflected by the following

Lemma 7. Let X be a locally compact simply ordered semigroup satisfying (ii). Then there exists a simply ordered locally compact topological abelian group A containing a subsemigroup $X'$ of X isomorphic to X such that $A = X'X'^{-1}$. If every bounded non-empty subset of X has an upper limit then also A possesses this property.

Proof. 1. Denote by H the direct product $X \times X$ of X. A relation Q on H between $(x,x')$ and $(y,y')$ exists if and only if $xy' \sim x'y$. One can easily verify, on the basis of (ii) and (iii)" that Q is an equivalence relation, and even a congruence relation, e.g. $(x,x')Q(y,y')$ implies $(x,x')(z,z')Q(y,y')(z,z')$. By defining $Q((x,x')Q(y,y')) = Q(xy,x'y')$ on $A = H/Q$ (the quotient space) a semigroup multiplication is defined.

2. Denote $Q(x,x)$ by 1 and set $\left[Q(x,x')\right]^{-1} = Q(x'x)$ so that $vv^{-1} = 1$ for all $v \in A$, hence A is an abelian group. Let a and b be distinct elements of X, then $(ax,a)Q(bx,b)$ holds for all $x \in X$. Therefore, the mapping $f : x \to Q(ax,a)$ is an algebraic homomorphism from X onto a subsemigroup $X'$ of A. The relation $f(x) \sim f(y)$ is equivalent to $a \times a \sim a^2 y$, from which $x \sim y$ can be derived, by (ii) and (iii)", i.e. X is algebraically isomorphic to $X'$. Let $Q(x,x')$ be an arbitrary element of A then $Q(x,x') = Q(ax,a)\left[Q(ax',a)\right]^{-1}$, hence $A = X'X'^{-1}$.

3. Let us now turn to the order and topological properties of A. One gets a simple order on A by establishing $(x,x') \prec (y,y')$ on H iff $xy' \prec x'y$ and this being equivalent to $Q(x,x') \prec Q(y,y')$ on A.
It can be shown that the order topology on A coincides with the quotient topology on $A/Q = A'$. Let $U \subset A'$ be open w.r.t. the quotient topology, i.e. if $Q^{-1}(U)$ is open in A, being the union of open intervals in A. Then $Q^{-1}(U)$ is open in $H = X \times X$, i.e. it is the union of sets $X_1 \times X_2$ such that $X_1 = \{x : a \prec x \prec b\}$, and $X_2 = \{y : c \prec y \prec d\}$. Therefore we have $xc \prec by$ and $ay \prec xd$, i.e. $Q(b,c) \prec Q(x,y) \prec Q(a,d)$. Consequently, U is the union of open intervals, therefore, open w.r.t. the order-topology.
Conversely, let U be open w.r.t. the order-topology. Without loss of generality let U be the open interval between $Q(u,u')$ and $Q(v,v')$.

Then $(x,x')$ is in $Q^{-1}(U)$ iff $ux' \prec u'x$ and $xv' \prec x'v$. The mapping $g : (x,x') \to (xv',x'v)$ of $X \times X$ into itself is continuous. Since a set $D = \{(z,z') : z \prec z'\}$ is open in $X \times X$, also the set $g^{-1}(D)$ in $g(X \times X) = \{(x,x') : xv' \prec x'v\}$ is open. Likewise for the set $\{(x,x') : ux' \prec u'x\}$. From this we conclude that $Q^{-1}(U)$ is open, so U is open in the quotient topology.

4. Furthermore, it follows from previous derivations that the 'onto'mapping $Q : X \times X \to A$ is open since the open sets $X_1 \times X_2$ will be mapped onto open intervals in A. If the $X_i$ are compact intervals then the map is a compact interval. Therefore, A is locally compact.

5. Now we prove that the mapping $(u,v) \to (u,v^{-1})$ from $A \times A$ into itself is continuous. Let m be the continuous mapping from $[(x,x'),(y,y')]$ onto $(xy',x'y)$. The mapping $(u,v) \to uv^{-1}$ from $A \times A$ into A is denoted by $\bar{m}$.

Then the following diagram is commutative:

$$\begin{array}{ccc} H \times H & \xrightarrow{(Q,Q)} & A \times A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \bar{m}} \\ H & \xrightarrow{Q} & A \end{array}$$

Set $(Q,Q)(u,v) = (Q(u),Q(v))$. By continuity of m and Q and $(Q,Q)$ being open it follows that $\bar{m}$ is continuous.

6. Summarizing, we have proved that A is a simply ordered, locally compact abelian topological group. The mapping $f : x \to Q(ax,a)$ from X into A is monotonely continuous, by its uniqueness property it is a monotone homeomorphism since X is locally compact. Hence $X'$ is isomorphic to X in every possible sense: order-theoretic, algebraic and topologcal. Because of $A = X'X'^{-1}$ every bounded, non-empty set in A has an upper limit whenever this is true in $X'$.

Lemma 8. Given the group A of Lemma 7. Let K be the set of all $x \in A$ for which the set of idempotents $x^n$ is bounded above by some $a(x) \in A$. Then we have either $K = \{1\}$ or K is an open subgroup of A. The first case holds iff A is archimedean and $X' \cap K \subseteq \{1\}$. The factorgroup A/K is also archimedean.

Proof. If the set of copies of x and y is bounded from above by a(x) and a(y), respectively, then the set of copies of $xy^{-1}$ is bounded from above by $a(x)a(y)^{-1}$, hence K is a subgroup. First, let $1 \prec v \in K$. If there is no element between 1 and v then A is discrete and trivially K is open. If $1 \precsim u \precsim v$ then the set if idempotents of u is bounded by a(v). Hence K is an interval and therefore open. Now let $Q(xa,a) \in K \cap X'$ and let $1 \leq Q(x^n a,a) \leq Q(r,s)$ for n = 1,2,... i.e. $x^n as \leq ar$ and $x^n s \leq r$. Then there exists $v = \lim x^n s$ and we have xv = v. But $(xv,v)Q(xa,a)$, hence Q(xa,a) = 1. By defintion of K it follows that no non-trivial cyclical group is bounded in A/K.

Lemma 9. Suppose X contains an unbounded, connected interval. Then A is connected, $K = \{1\}$, and A is isomorphic to the additive group of all real numbers, Re.

Proof. Let $I$ be an unbounded, connected interval in $X'$. Without loss of generality let $I$ be unbounded from above and contained in $A_+$. For every $i \in I$ and $g \in A_+$ we have $i \leq ig$ with $ig \in I$. In particular, $II \subset I$. Because of connectedness, every bounded subset in $I$ has an upper limit. Because of condition (ii) the set of all positive copies of elements from $I$, being contained in $I$, is bounded. Let $i \in I$. Then $\{i^{-n} : n = 1,2,...\}$ is unbounded from below. The set $\bigcup i^{-n}I$ is unbounded from below and from above, and as a union

of an increasing sequence of connected sets it is connected. Therefore A is connected. Then A cannot contain a proper open subgroup and K is equal to $\{1\}$. Hence A is archimedean and connected and therefore is isomorphic to Re.

Therefore we have proved the following theorem.

Theorem 2. Let X be a simply ordered set being locally compact w.r.t. the order topology. An everywhere defined concatenation on X, $(x,y) \to xy$ satisfies the following conditions:

(i) $(x,y) \to xy$ is continuous,

(ii) $x \prec y$ implies $vx \prec vy$ for all v,

(ii)" $x(yz) \sim (xy)z$ and $xy \sim yx$ for all x,y,z.

If X contains an unbounded connected subset then X is isomorphic to a closed subsemigroup of the additive group of all real numbers. If every non-empty bounded subset of X has an upper limit then X is ismorphic to a closed subsemigroup of the additive group of all real numbers.

Let N be the semigroup of all natural numbers under the operation of addition. Then the lexiographically ordered product $N \times Re$ forms a locally compact, abelian and ordered semigroup which is not isomorphic to any closed subsemigroup of Re. Incidentally, these closed subsemigroups of Re may be extremely complicated.

Definition 4. Let a 'concatenation' be defined on X satisfying condition (i). Moreover, let $\mu$ and $\nu$ be Borel measures on X with respect to which all open sets of X are measurable. Clearly, this is the case if X has a countable basis of open sets. Hence in the product space $X \times X$ the set of all (u,v) with $uv \prec x$ is measurable w.r.t. the product measure $\mu \times \nu$. Define the mapping H, as in Sec. 2, with

the indicated properties. Designate the Borel measure associated to H with representation

$H(x) = \mu \times \nu( \{(u,v) : \mu\nu < x\} )$ by $\mu\nu$, the convolution of $\mu$ and $\nu$.

Theorem 3. Let $f : X \to Re$ be an isomorphism of X with the additive group of real numbers and set $H_0(f(x),\mu) = \mu(A(x))$ for every Borel measure on X. Then $H_0(r,\mu\nu) = \int H_0(r-s,\mu)dH_0(s,\nu)$.

The set of all Borel measures on X, $\mathcal{M}$ becomes a commutative semigroup - by convolution. The relation $\mu < \nu$ on $\mathcal{M}$ implies $\lambda\mu \leq \lambda\nu$ and even $\lambda\mu < \lambda\nu$ if $H(\cdot,\lambda)$ is monotone and continuous. Then we have

$E_f(\mu) + E_f(\nu) = E_f(\mu\nu)$.

Remark. By assumption , see Sec. 2, for every Borel measure $\mu$ in $\mathcal{M}$ f is a random variable in the usual sense and $E_f(\mu)$ is its expected value. The corresponding distribution function $r \to H_0(r,\mu)$ is defined by $H_0(r,\mu) = \mu\{x : f(x) < r\}$ and $E_f(\mu)$ equals the Stieltjes-integral $\int r dF_0(r,\mu)$ and $H(x,\mu) = H_0(f(x),\mu)$.

Because of $H_0(r,\mu\nu) = \mu\nu( \{(x,y) : f(x) + f(y) < r\} )$ the distribution function $H_0(\cdot,\mu\nu)$ of the sum of both is determined. Since $\mu < \nu$ implies $H_0(r,\mu) < H_0(r,\nu)$ for all r, we conclude $\int H_0(r-s,\mu)dH_0(s,\lambda) \leq \int H_0(r-s,\nu)dH_0(s,\lambda)$ and if $H_0(\cdot,\lambda)$ is continuous and monotone, strict inequality must hold.

## 5. Conclusions

We have proved that there exists an order system $(X, \leq)$ which is embedded in the numerical system $(Re_+, \leq)$ by a an isomorphism f, a utility, that satisfies the functional equation $f(xy) = f(x) + f(y)$.

But we could also identify f as a random variable so that f(xy) designates a gamble or a von N.M. lottery. Given some Borel measure $\mu$ the expected value is represented by $E_f(\mu)$, and correspondingly we can prove that for any two measures $\mu, \nu \in \mathcal{M}$ : $E_f(\mu \nu) = E_f(\mu) + E_f(\nu)$. Therefore, let f(xy) be represented by $E_f(\mu \nu)$, we easily see that the additive utility representation implies the expected utility representation.

Abstract:

EXISTENCE OF A UTILITY ON A TOPOLOGICAL SEMIGROUP

This paper presents a comprehensive mathematical framework in which a unified treatment of additive and expected utility can be given. For achieving this, elaborate structural assumptions, characterizing a simply ordered, topological semigroup, have to be established in order to construct an isomorphism with the additive group of real numbers.

This construction establishes a link between additive and expected utility theory to the extent that the same mathematical considerations leading to the derivation of an additive representation are also valid for proving the expected utility theorem.

References


/1/ K.J. Arrow, Essaysin the Theory of Risk-Bearing, North Holland: Amsterdam 1970 (Chapter 2).

/2/ D. Krantz, R.D. Luce, P. Suppes, A. Tversky, Foundations of Measurement. Vol. I, Academic Press: New York 1971.

/3/ L. Fuchs, Teilweise geordnete algebraische Strukturen, Vandenhoeck & Ruprecht, Göttingen, Zürich 1972.

/4/ P.R. Halmos, Measure Theory, van Nostrand: New York 1950.

/5/ N. Bourbaki, Elements of Mathematics: General Topology (Transl. from French), Addison-Wesley: Reading (Mass). 1968.

Footnote

1) For two elements $x,y \in X$ we define $x \sim y$ if not $x \prec y$ and not $y \prec x$. Moreover, it is assumed throughout that equality implies the relation '$\sim$'.