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Benjamin Peirce's *Linear Associative Algebra* (1870): New Light on its Preparation and 'Publication'

In fond memory of Max H. Fisch (1900–95)

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1. Content

In 1870 Benjamin Peirce (1809–80) published in lithographic form a book of 153 pages with the above title, in which he classified a wide range of algebras by their defining properties.[1] It came to influence later developments of algebras, especially in the USA. It has been studied historically to some extent,[2] and I shall note the main features here; but the main purpose of this paper is to draw upon little-known manuscripts to describe the manner of its preparation and the circumstances behind its unusual initial appearance.

During the nineteenth century ordinary algebra was joined by a steadily increasing collection of other algebras, often satisfying some different laws: differential operators, functional equations, algebraic logic, probability algebra, substitutions and group theory, determinants, invariants, matrices, and especially for Peirce the quaternions of W. R. Hamilton (1805–65).[3] These last had been introduced in 1843 as a means of handling the equations of mathematical physics by adjoining to the real number system three independent quantities, i, j and k obeying the laws

$$i^2 = j^2 = k^2 = ijk = -1, \quad \text{and } ij = k \text{ and } ji = -k \tag{1}$$

together with permutations among i, j and k, under some means of combination symbolized by concatenation of the symbols; the algebra comprised linear combinations of these quantities under a second means of combination symbolized by '+':

$$q = a + bi + cj + dk \tag{2}$$

for coefficients a, b, c and d.

Peirce, an enthusiast for this algebra from the late 1840s, seems to have taken this work as his main influence by generalizing it enormously to find (hopefully) all

[1] B. Peirce, 'Linear Associative Algebra' (Washington: 1870). On the reprints see notes 8 and 10.

[2] See especially L. Novy, 'Benjamin Peirce's concept of linear algebra', *Acta historiae rerum naturalium necnon technicarum*, Special Issue 7 (1974), 211–30; and H. Pycior, 'Benjamin Peirce's *Linear Associative Algebra*', *Isis*, 70 (1979), 537–51. A more detailed account by Alison Walsh will soon be available in a PhD on the relationships between algebras and logics in both Benjamin Peirce and his son Charles, which is being prepared under my direction. The various other texts are too fleeting or routine to merit citation.

[3] This 'especially' is also true of most historians of algebra, who usually treat only quaternions and the philosophy of common algebra and exclude most of the other algebras mentioned in the text. The rich full story constitutes one of my lectures; unfortunately, I know of no corresponding publication by anyone to cite here.

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'linear' algebras containing between one and six independent quantities satisfying the above and other properties; and the quaternions were one of several examples that he gave of 'quadruple algebras'. The catalogue comprised 2 single algebras, 3 doubles, 5 triples (+2 subcases), 18+3 quadruples, 70+1 quintuples, and 65 sextuples: 163+6 cases in all. It did not include complex numbers; on the contrary, he allowed the coefficients (2) of the units to be complex as well as real numbers, even though factorization and division were not unique.

Unlike Hamilton and his own other researches (as we shall soon see), Peirce did not seek physical applications of the algebras; indeed, he avoided all interpretations of the quantities. In addition, in a novel way for his time of which he sensed the importance, he found results which we would regard as metatheorems about the (im)possibility of an algebra of a given type satisfying certain properties, or the numbers of such algebras when possible.[4]

The 'associative' of Peirce's title, Hamilton's word, referred to the dominant property of these algebras, but where appropriate he also noted the other well-known properties of commutativity and distributivity. This trio may be written

$$a(bc) = (ab)c, ab = ba \text{ and } a(b+c) = ab+ac. \tag{3}$$

For example, quaternions are associative but not commutative. He also proposed two new properties: 'idempotent' (a property introduced by George Boole (1815–64) in connection with logic) and 'nilpotent', respectively

$$d^m = d \text{ and } e^n = 0 \text{ for some positive integers } m \text{ and } n. \tag{4}$$

He created a numbering system for each algebra starting out from these properties ('[1...]' for idempotency, '[2...]' for nilpotency). Both terms have become standard in algebra.

Another feature of Peirce's treatment is worth emphasizing; he laid out the laws for each algebra after the manner of a multiplication table in arithmetic. For example, he presented one of the triple algebras as

	i	j	k
i	i	j	k
j	j	k	0
k	k	0	0

(5)

These tables had been introduced by Arthur Cayley (1821–95) in connection with groups;[5] Peirce both extended and popularized their use with his large collection of algebras.[6]

[4] Peirce stressed the importance of such theorems in his letter covering a copy of the book sent to the diplomat and historian George Bancroft in November 1870. The letter is published in J. Ginsberg, 'A hitherto unpublished letter by Benjamin Peirce', *Scripta mathematica*, 2 (1934), 278–82; it was reprinted in I. B. Cohen (ed.), *Benjamin Peirce: 'Father of Pure Mathematics' in America* (New York: Arno Press, 1980), a reprint collection which is not separately paginated and has no editorial preface.

[5] A. Cayley, 'On the theory of groups, as depending on the symbolic equation $\theta^n = 1$', *Philosophical Magazine*, ser. 4, 7 (1854), 40–7; reprinted in *Collected Mathematical Papers*, vol. 2 (Cambridge: Cambridge University Press, 1889), pp. 123–30.

[6] Cayley himself gives a few examples of the use of arrays before Peirce's time in arts: 130–1, 136 of his 'Report on the progress of the solution of certain special problems of dynamics', *Reports of the British Association for the Advancement of Science*, (1862), 184–252; reprinted in *Collected Mathematical Papers*, vol. 4 (Cambridge: Cambridge University Press, 1891), 513–93.

2. Reception

The importance of Peirce's work was first emphasized by his son Charles Sanders Peirce (1839–1914). Already engaged for some years in uniting and extending the algebras of logic of Boole and De Morgan, he converted the tables into linear combinations of logical relatives in a manner which father praised in a paper of 1875.[7] More importantly, with the agreement of editor Sylvester he had the lithograph reprinted posthumously in 1881 as a long paper in the recently created *American Journal of Mathematics*.[8] He added a headnote hoping that the reprint would help his father's study be recognized as 'a work which may almost be entitled to take rank as the *Principia* of the philosophical study of the laws of algebraical operation'. He added many short footnotes converting the tables in his preferred 'relative forms', two longer footnotes, Benjamin's 1875 paper, and two addenda of his own, one on the general means of finding relative forms and the other on reworking the theory with only real numbers as coefficients in (2).[9] An extracted version of the whole appeared as a book in the following year, with a short list of errata and a new brief preface by Charles indicating that he had thought out his addendum on relative forms only after converting each individual algebra in the short footnotes.[10]

The impact of the work was mostly felt in the USA, and especially around the turn of the century when American mathematics was becoming a significant international force, for new algebras formed quite a speciality.[11] A principal venue for publication was the *Transactions of the American Mathematical Society*, recently founded by E. H. Moore (1862–1932) at the University of Chicago. He took several papers from E. V. Huntington (1874–1952) at Harvard and his own doctoral student Oswald Veblen (1880–1960) as they helped to form model theory.[12] They did not cite Peirce but will certainly have known of this work, and the link between the two theories was soon examined by Moore's colleague L. E. Dickson (1874–1954).[13]

Other Americans studied hypercomplex number systems, generalizations of complex numbers to any finite number n of independent quantities which gave forms similar to Peirce's up to $n = 6$;[14] connections with Peirce were studied in Moore's

[7] B. Peirce, 'On the uses and transformations of linear algebra', *Proceedings of the American Academy of Arts and Sciences*, new ser., 2 (1874–1875), 395–400; reprinted in *American Journal of Mathematics*, 4 (1881), 216–21.

[8] B. Peirce, 'Linear associative algebra with notes and addenda by C. S. Peirce, son of the author', *American Journal of Mathematics*, 4 (1881), 97–229 (including the previous paper). The talented printer was Isaac Friedenwald of Baltimore.

[9] Charles's two principal footnotes and his addenda, respectively pp. 132, 190–4, and pp. 221–9 of note 8, are reprinted in *Writings of Charles S. Peirce*, vol. 4: *1879–1884* (Indianapolis: Indiana University Press, 1989), 311–27.

[10] B. Peirce, *Linear Associative Algebra. New Edition, with Addenda and Notes by C. S. Peirce, Son of the Author* (New York: Van Nostrand, 1882), 133 pages; reprinted in Cohen (note 4). Note the slight change of title from note 8. Charles's new preface is reprinted in *Writings* (ibid.), p. 581.

[11] On the growth of American mathematics at this time, see D. Rowe and K. Parshall, *The emergence of the American Mathematical Research Community* (Providence, RI: American Mathematical Society, 1994), especially ch. 1.

[12] M. Scanlan, 'Who were the American postulate theorists?', *Journal of Symbolic Logic*, 56 (1991), 981–1002.

[13] L. E. Dickson, 'Definition of a linear associative algebra by independent postulates', *Transactions of the American Mathematical Society*, 4 (1903), 1–27; reprinted in *Mathematical Papers*, vol. 2 (New York: Chelsea, 1975), pp. 109–16.

[14] A. Cayley, 'On multiple algebra', *Quarterly Journal of Pure and Applied Mathematics*, 22 (1887), 270–306; reprinted in *Collected Mathematical Papers*, vol. 12 (Cambridge: Cambridge University Press, 1897), 459–89. On the historical context see T. W. Hawkins, 'Hypercomplex numbers, Lie groups, and the creation of group representation theory', *Archive for History of Exact Sciences*, 8 (1972), 243–87.

journal by H. E. Hawkes (1872–1943)[15] and H. Taber (1860–1936).[16] In addition, at Edinburgh University the Scot J. H. M. Wedderburn (1882–1948) progressed from quaternions (his supervisor was P. G. Tait) through hypercomplex number systems to a higher doctorate in 1908 on linear associative algebras, whereupon he promptly moved to Princeton University for the rest of his career.[17]

The mathematician most deeply concerned with Peirce's theory was J. B. Shaw (1866– > 1927). Moore accepted two papers from him in 1903;[18] but when a book-long manuscript arrived in the following year he demurred on grounds of its length and suggested as an alternative outlet the Carnegie Institution,[19] which duly published it in 1907.[20]

Partly through the exertions of son Charles, Peirce's algebras also came to bear upon the development of algebraic logic at Harvard. In a long paper sent to Moore, the philosopher Josiah Royce (1855–1916) reformulated a very general theory of collections due to the English mathematician A. B. Kempe in an algebraically symmetric way,[21] and continued the analysis of symmetric and asymmetric relations for the remaining decade of his life.[22] Later, Peirce's use of multiplication tables was applied to mathematical logic by Royce's colleague H. M. Sheffer (1882–1964) in a manuscript sent around to various friends and colleagues.[23] His procedures differed from the method of truth-tables for this logic, now well known, which also has a Harvard link; it was invented by Bertrand Russell with his pupil Ludwig Wittgenstein around 1914 and first publicized in the spring of that year by Russell at Harvard in a lecture course on logic.[24]

3. Reputation

Peirce's role in all these developments is surprising, in that his lithograph was quite different from his normal publications. Professor of Mathematics at Harvard University from 1833 and Superintendent of the United States Coast Survey from 1867 to 1874, he normally worked in applied mathematics, astronomy and geodesy,

[15] H. E. Hawkes, 'On hypercomplex number systems', *Transactions of the American Mathematical Society*, 3 (1902), 312–30. At Johns Hopkins University editor Frank Morely took Hawkes's 'Estimate of Peirce's "Linear Associative Algebra"', *American Journal of Mathematics*, 24 (1902), 87–95.

[16] H. Taber, 'On hypercomplex number systems. (First [and only] paper)', *American Journal of Mathematics*, 5 (1904), 509–48.

[17] K. H. Parshall, 'Joseph H. M. Wedderburn and the structure theory of algebras', *Archive for History of Exact Sciences*, 32 (1985), 223–349.

[18] J. B. Shaw, 'Theory of linear associative algebra' and 'On nilpotent algebras', *Transactions of the American Mathematical Society*, 4 (1903), 251–87 and 405–22.

[19] See the letters between Moore and Shaw in the University of Chicago Archives, Moore Papers, Box 3, folder 9.

[20] J. B. Shaw, *Synopsis of Linear Associative Algebra. A Report on its Natural Development and Results Reached to the Present Time* (Washington: Carnegie Institution, 1907).

[21] J. Royce, 'The relations of the principles of logic to the foundations of geometry', *Transactions of the American Mathematical Society*, 6 (1905), 353–415; reprinted in *Logical Essays*, edited by D. S. Robinson (Dubuque: Brown, 1971), 379–441. A good summary is given in L. Vercelloni, *Filosofia delle strutture* (Florence: La Nuova Italia Editore, 1989), ch. 5; for Royce's context, see B. Kuklick, *Josiah Royce...* (Indianapolis and New York: Bobbs-Merrill, 1972), especially ch. 10.

[22] See the massive collection of notebooks and some unpublished essays in the Josiah Royce Papers, Harvard University, Pusey Library, especially six boxes of manuscripts on logic, and bound volumes 72–5 *passim*.

[23] H. M. Sheffer, 'General theory of notational relativity' (1921; Sheffer Papers, Harvard University, Pusey Library, box of papers and articles); see also the box of correspondence.

[24] See the notes of Russell's course taken by postgraduate student T. S. Eliot in the Eliot Papers, Harvard University, Houghton Library, file Am. 1691.14.(13), fol. 43.

including the production of several influential textbooks. They gave him an early opportunity for publicity in Europe, when he led a survey team in the summer of 1870 to observe a total eclipse of the sun in Italy. He gave a copy of his book to Charles to present to De Morgan in London in July 1870 on the way out, and in his charming covering letter of introduction of his son he specifically praised De Morgan's work on algebras.[25] On his way home he visited Britain himself, lecturing on his theory to the London Mathematical Society in January 1871 and giving them a copy of the book; presumably he met De Morgan himself then.[26]

Peirce was a major figure in nineteenth-century American science, to an extent which renders astonishing the lack of any substantial historical work on his achievements when so many lesser figures have been treated extensively: possibly his focus upon mathematics has granted him historical leperhood.[27] But this leaves unknown not only an impressive body of publication and teaching and much participation in the professionalization of American science but also a *Nachlass*, hardly known or used by historians. A group of unnumbered boxes contains a very extensive collection of letters, arranged in alphabetical order with a folder for each correspondent—around 500 of them, and including many well-known names in American science and/or the university. The rest comprises eleven numbered boxes containing an unorganized and uncatalogued heap of notebooks, personal documents, scientific notes and files (including some further correspondence)—and drafts of the lithograph.[28]

4. Preparation

Three drafts survive, each one in a large notebook; they once belonged to Peirce's nephew Benjamin Peirce Ellis.[29] None is dated, but the order emerges clearly enough from comparison of the contents. A detailed comparison of them and the book would be worth executing; I gave just a summary overview.

The first one was entitled merely 'Linear Algebra'. After preliminaries on quantity, continuity, dimension and some other topics, 'Book 1. The language of algebra' (the only Book, both here and in all later versions), including basic properties of order and equality, was followed after 12 folios by '*Linear, distributive,*

[25] This letter is transcribed in my 'Peirce between logic and mathematics', in D. Roberts and others (eds), *Studies in the Logic of Charles S. Peirce* (Bloomington: Indiana University Press, 1997), 23–42 (p. 38).

[26] This information is taken from M. Fisch, 'The decisive year [1867] and its early consequence', in *Writings of Charles S. Peirce*, vol. 2: *1867–1871* (Indianapolis: Indiana University Press, 1984), xxi–xxvi (xxxi–xxxiv). Peirce's lectures to the London Mathematical Society are noted in their *Proceedings*, ser. 1, 3 (1869–1871), 220. An early reaction was given by W. Spottiswoode in 'Remarks on some recent generalisations of algebra', *Proceedings*, 4 (1873), 147–64.

[27] The most significant single study to date is still R. C. Archibald (ed.), 'Benjamin Peirce', *American Mathematical Monthly*, 32 (1925), 1–30; reprinted as a pamphlet (Oberlin, Ohio: Mathematical Association of America, 1925). This latter version is reprinted in Cohen (note 4). The obituaries had been gathered together in the 64-page booklet by M. King (ed.), *Benjamin Peirce. A Memorial Collection* (Cambridge, MA: Rand, Avery, 1881). A welcome recent study of Peirce's emergence is provided in E. Hogan, '"A proper spirit is abroad": Peirce, Sylvester, Ward, and American mathematics', *Historia mathematica*, 18 (1991), 158–72.

[28] Peirce Papers, Harvard University, Houghton Library (including material transferred from the Archives of the American Academy of Arts and Science which had been used by Archibald in *ibid.*). It exists in its initial sorting, which was carried out by Max and Ruth Fisch in 1966. Boxes 8, 10 and 11, and some material in other boxes, constitute the *Nachlass* of Peirce's eldest son James Mills (1834–1906), also a professor at Harvard but not of normal family calibre in science (see H. C. Kennedy, 'Towards a biography of James Mills Peirce', *Historia mathematica*, 6 [1979], 195–201).

[29] Peirce Papers, Box 2; Ellis has signed each notebook. A small one containing determinations of some algebras is held in Box 9.

associative algebras' up to some quadruple algebras; a final folio launched 'quintuple algebra' but stopped very quickly. The second draft, named on the inside front cover 'Linear Associative Algebra. Book. 1.', followed the same course and range in its 89 numbered folios but contained far more quadruple algebras. The third draft broadly, and in many places in details, treats the same material; but the 'Investigations of Linear Associative Algebras' contained substantial surveys of quintuple and sextuple algebras. It is often close or even identical to the book, but quite a few variations are evident: usually additions (on sextuple algebras, for example), they also involve changes in underlining and paragraphing. I suspect that yet another version was prepared for the final publication, or maybe the alterations were made on loose sheets which are now lost.[30]

While there are many changes of phrasing in the common parts of the drafts, and the use of tables increases substantially to take over finally many pages in the book, the basic strategy was in place in the first draft: a sequence of numbered sections outlining the basic purpose and the various types of algebra to be explored, followed by the catalogue of algebras (with tables). The basic properties, such as idempotency and nilpotency, were also here.

However, I spotted two connected and very significant changes to the early paragraphs.

5. Philosophy

The first alteration concerns the opening sentence. Knowing that his taxonomy of algebras constituted a *general* mathematical theory of some kind, Peirce sought a comparably over-arching characterization of mathematics, and in the book he came up with a claim of which the endless repetitions by others since have made the book famous: 'Mathematics is the science that draws necessary conclusions'. As one of his apparently few readers who has never been able to understand this assertion, I was very relieved to read in the first draft that 'Mathematics is the science that draws inferences', and in the second draft that 'Mathematics is the science that draws consequences', though the last word was altered to yield the canonical but enigmatic form used in the book. The change is *not* just verbal; he must have realized that the earlier forms were necessary but not sufficient (they are satisfied by other sciences and by the law, for example). So he added the crucial adjective 'necessary'; but what does it denote? Perhaps he was following a tradition in algebra, followed especially be De Morgan, of distinguishing the 'form' of an algebra from its 'matter' (that is, an interpretation or application to a given mathematical and/or physical situation),[31] and claiming that its form would deliver the consequences from the premises. He also always used the active verb 'draws': mathematics is concerned with the *act* of so doing, not the *theory* of doing it, which belongs elsewhere such as with logic. But I suspect that he had hit on such a nice phrase that he did not enquire deeply into its possible meanings.

[30] I did not see in the boxes any sheets of the kind hypothesized here. Some main differences from the third draft occur in sections 27, 33, 49, 51, 57 and 61–4 of the book, and only the titles of sections 67–70 are given in the draft. In a memorandum of 1910 to himself Charles Peirce claimed confidently that the research had been completed for at least a year before lithography was proposed and executed (see R. C. Archibald, 'Benjamin Peirce's Linear Associative Algebra and C. S. Peirce', *American Mathematical Monthly*, 34 [1927], 525–7).

[31] H. Pycior, 'Augustus De Morgan's algebraic work: the three stages', *Isis*, 74 (1983), 211–26; M. Panteki, 'Relationships between algebra, differential equations and logic in England: 1800–1860' (C.N.A.A. [London] doctoral dissertation, 1992), especially chs 3 and 6.

In all drafts and in the book a further conundrum appears a few lines later: Peirce stressed that his definition was subjective, in contrast with the prevailing practice of defining mathematics in some objective way. He always held to this view; in a lecture to the Radical Club in Boston shortly before his death he repeated it and stressed it to be

> wider than the ordinary definitions. It is subjective; they are objective. This will include knowledge in all lines of research. Under this definition mathematics applies to every mode of enquiry.[32]

Perhaps his deep religious faith played a role; since all was done for God, then the subjective had divine access denied to traditional inter-human objectivity.[33] But his line was not adopted by its followers. In particular, while Charles praised the definition itself he rarely referred to father's reading of it:[34] a far finer philosopher, he sought in his own logic and mathematics to keep within human bounds.

The second and related alteration concerns this statement in the second paragraph of the third draft: 'Mathematics, as here defined, belongs to every enquiry; it is even a portion of deductive logic, to the laws of which it is rigidly subject.' This kind of philosophy of mathematics which sites it within (some kind of) logic is now called 'logicism',[35] and is found after Peirce in differing forms in Ernst Schröder (Charles's follower in algebraic logic), Richard Dedekind, Gottlob Frege and Bertrand Russell. But none of them was influenced by this statement, for a *quite opposite* viewpoint appeared in the book: 'Mathematics, as here defined, belongs to every enquiry; moral as well as physical. Even the rules of logic, by which it is rigidly bound could not be deduced without its aid'. Thus mathematics was now helping logic, not the logicistic previous arrangement.

The hand of Charles seems evident here; for in 1908 he told the mathematician and philosopher Cassius Keyser (1862–1947) with pride that

> It was my remonstrances that saved my father from falling into Dedekind's error in making mathematics a branch of logic. He thought over what I said & by a slight change in the phrase made it the absolutely perfect definition that it is. Had he... gone on to characterize mathematics further, by mentioning for example, the precision (!) of its hypotheses—even if they really had been so precise—he would have committed the logical fault of giving a *description* instead of a *definition*.[36]

[32] B. Peirce, 'The impossible in mathematics', in Mrs J. T. Sargent (ed.), *Sketches and Reminiscences of the Radical Club of Chestnut St Boston* (Boston: James R. Osgood, 1880), 376–9 (p. 378).

[33] On Peirce's religiosity see S. R. Peterson, 'Benjamin Peirce: mathematician and philosopher', *Journal of the History of Ideas*, 16 (1955), 89–112; reprinted in Cohen (note 4). Peirce was Christian without formally adopting a creed; rather like De Morgan's description of himself as 'Christian unattached'.

[34] An exception is C. S. Peirce, 'The logic of mathematics in relation to education', *Educational Review*, (1898), 209–16; reprinted in *Collected Papers*, vol. 3 (Cambridge, MA: Harvard University Press, 1933), 346–59 (348). See also the next note.

[35] Our sense of this word was proposed by Rudolf Carnap around 1927, and first published by him (as 'Logizismus') in *Abriss der Logistik, mit besondere Berücksichtigung der Relationstheorie und ihre Anwendungen* (Vienna: Julius Springer, 1929), 2–3, and with more publicity in 'Die logizistische Grundlegung der Mathematik', *Erkenntnis*, 2 (1932), 91–105. In the 1900s and 1910s 'Logizismus' had been used differently, often in connection with phenomenological logic, especially in the writings of Wilhelm Wundt; it had arisen elsewhere, such as in ethics.

[36] C. S. Peirce to C. J. Keyser, 2 April 1908 (Cassius Jackson Keyser Papers, Columbia University, box of catalogued letters), fols 8–9. He went on to discuss difficulties of two other kinds, but made no reference to subjectivity. In the memorandum to himself (note 30) he claimed to have badgered his father into doing this research in the first place.

At that time the most prominent version of logicism was Russell's, which Charles held in high contempt (but with some incomprehension, which was mutual!).[37]

6. Stupefaction

Benjamin Peirce read parts of his erudite researches at various meetings of the National Academy of Sciences, apparently from 1867 onwards. With the country still in mathematical mediocrity, even this august body could not provide him with a suitable audience. (It had been founded only four years earlier, with Peirce as a member.) Eloquent testimony is provided by a surviving copy of a letter[38] which his friend Thomas Hill (1818–91) made on 15 March 1871 and kept in his copy of the book.[39] Hill had been President of Harvard University from 1862 to 1868 (in fact, the predecessor of C. W. Eliot, chemist and former Peirce student), and was about to become an Overseer for three years.

The original letter had been written the previous December by Peirce's assistant at the Geodetic Survey, Julius Erasmus Hilgard (1825–91), and sent to William Adams Richardson (1821–96), a Harvard graduate who followed a distinguished legal and political career, serving as Secretary of the Treasury and Chief Justice of the Court of Claims.[40] Hilgard recalled that:

> Prof. Peirce at one of the sessions of the Academy had just finished, with inspired eye and prophetic manner[,] delivering to a confounded audience one of the most abstruse portions of his researches, and a respectful silence had ensued when Agassiz[41] rose and said:
>
> > I have listened to my friend with great attention and have failed to comprehend a single word of what he has said. If I did not know him to be a man of great mind, if I had not had frequent occasion to feel his power, to admire his judgement and discrimination on ground where our several lines of study touch, in organic morphology, in physics of the globe, I could have imagined that I was listening to the vagaries of a madman, or at best to empty and baseless literal dialectics. But knowing my friend to be not only profound but fruitful in all matters on which I have any judgment, I am forced to the conclusion that there are modes of thought familiar to him, which are inaccessible to me, and I accept in faith not only the logical truth of his investigations, but also their value as means of opening to our comprehension the laws of the universe.
>
> Upon this epilogue there was what the French call a sensation in the hall, and the audience recovered their cheerfulness.

[37] On the huge differences between algebraic and mathematical logics—an extraordinary irony, still poorly appreciated—see my 'Living together and living apart: on the interactions between mathematics and logics from the French Revolution to the First World War', *South African Journal of Philosophy*, 7/2 (1988), 73–82; and my note 25.

[38] Fisch found the copy of the letter in 1970, in the course of his archival investigations of the Peirces, and his transcript (though without indication of provenance) has been available for some years at the Peirce Project, University of Indiana at Indianapolis. I use Fisch's annotations in note 42.

[39] This copy of Peirce's book, dedicated to 'the Reverend Thomas Hill from his ever faithful and ever earnest friend the author', is kept in Pusey Library, Harvard University (HUG 1860 138). Peirce mentioned him in section 4 of the book.

[40] I have not found Hilgard's original letter, or Richardson's original enquiry; neither did Fisch. Hilgard was the Superintendent of the United States Coast and Geodetic Survey (as it was renamed in 1878) from 1881 to 1885.

[41] Louis Agassiz (1807–73), the geologist and naturalist. For clarity of reading I have indented, without quotation marks, his reported statement which now follows.

Hill added his own note to his transcription:

> In copying the above letter of Assistant Hilgard I would add that I think the above meeting was in Hartford in the summer of 1867 when I had the honor to be present by invitation of the Academy,[42] and I think the part which Prof. Peirce read, included page 59.[43]
>
> Rev. Dr. W. H. Furness heard Prof. Peirce read at a meeting of the Am[erican] Assoc[iation] for the Advancement of Science a paper on the transformation of equations, or some kindred topic, and afterward told me 'that he did not understand a single word of it, but that it was one of the most interesting addresses he had ever heard in his life,—interesting merely from the wonderful eloquence of the eye, and enthusiasm of manner in a man whose whole appearance testified to his very high intellectual and moral ability'. Dr. F. 'did not understand what Peirce was saying, but he knew it must be something well worth saying, and of high importance, or it could not inspire such a man with such enthusiasm'.

On thanking Peirce for his copy, Hill had reported on 20 February 1871 that 'I have had a copy on my dressing table for a fortnight and have read it, a page at a time, every night and morning'. But he had been attentive enough to the content to ask if Peirce could give an application of a triple algebra to space and time.[44] Peirce's reply is not extant, but doubtless he told Hill that this reasonable request could not be fulfilled, and that Hamilton's quaternions gave the surprising solution.

7. Publication

The National Academy not only failed to provide Peirce with a proper audience: it could not even publish his work. Let Hilgard tell the consequences; this was the reason for him writing to Richardson in the first place:

> In answer to your inquiry I take pleasure in making the following explanation.
>
> Prof. Peirce's memoir on Linear Associative Algebra had formed the subject of several successive communications to the National Academy of Sciences, and is designed to be printed as part of the Memoirs of that scientific but impecunious body, whenever means for publication become available.
>
> Meantime as that day appears to be yet far off, I proposed to provide for its publication in the present form, printing 100 copies of it by the lithographic method. The work was performed by persons connected with the Coast Survey Office in otherwise unoccupied hours, or in their own leisure time, being in fact in great part a labor of love.
>
> The most laborious part was that of preparing the copy, which was written in lithographic ink on ordinary well sized writing paper. A transfer of these written pages, twelve at a time, was made on a lithographic stone, and 100 copies were printed, after which the transfer was rubbed off, and the next twelve pages laid down on the stone.[45]

[42] The meeting was held from 13 to 16 August in the State House in Hartford, and Hill read a paper.

[43] This page of Peirce's book includes at [612] a form of quaternions in the section on quadruple algebras.

[44] Peirce Papers, correspondence boxes.

[45] [This statement does not seem completely accurate, for the pages of the lithograph are printed recto and verso; sequences of odd- and even-numbered pages must have been applied in tandem to the lithographic stone.]

> The copy was written by a lady who understood not one word of the investigation, but who by great attention succeeded in making a copy far more free from errors than any printers proof ever is,—considering Prof. Peirce's chirography it was a wonderful performance. The few errors that a careful proof reading revealed were corrected before the edition was struck off.

Peirce corroborated Hilgard's role at the close of the lithograph: 'I must conclude this memoir with expressing my thanks to my friend J. E. Hilgard, for the opportunity for issuing these nice lithographic copies, for which I am mainly indebted to this energy and considerable zeal.' The transcribing lady was not mentioned.

Apparently never on sale in the normal sense, Peirce distributed signed copies to a large number of friends and professional colleagues. Many a man would have felt disheartened by this poor form of publication of an obviously important work. Perhaps surprisingly, he did not try to publish in Europe; apart from a few possible publishers in Britain, at least three European mathematical journals took long papers and probably would not have excluded English. Another chance was the Royal Society of London; he had been elected a Foreign Member in June 1852, in his 44th year.[46] But at all events, he expressed only joy in his latest offering to God, in this opening preamble:[47]

> *To my friends*
>
> This work has been the pleasantest mathematical effort of my life. In no other have I seemed to myself to have received so full a reward for my mental labor in the novelty and breadth of the results. I presume that to the uninitiated the formulae will appear cold and cheerless. But let it be remembered that, like other mathematical formulae, they find their origin in the divine source of all geometry. Whether I shall have the satisfaction of taking part in their exposition, or whether that will remain for some more profound expositer, will be seen in the future.

Acknowledgement

For permission to publish the Hill letters I express thanks to the Harvard University Archives and the Houghton Library there; similarly, for Charles Peirce's letter to Keyser, to the Rare Book and Manuscript Library, Columbia University, New York. For comments on my (single) draft, I thank Nathan Houser and Tony Crilly.

46 Royal Society Archives, Certificates 1840–1860, fol. 306.

47 Sadly, Charles omitted this passage from the reprint (note 28) but included it in his preface to the book version (note 10). In December 1916, two years after his death, his widow gave (the remaining?) three copies of the lithograph to Harvard University; all unmarked, they are kept in the Houghton Library.